FIXED POINTS, PERIODIC POINTS, AND COIN-TOSSING SEQUENCES FOR MAPPINGS DEFINED ON TWO-DIMENSIONAL CELLS

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We propose, in the general setting of topological spaces, a definition of two-dimensional oriented cell and consider maps which possess a property of stretching along the paths with respect to oriented cells. For these maps, we prove some theorems on the existence of fixed points, periodic points, and sequences of iterates which are chaotic in a suitable manner. Our results, motivated by the study of the Poincaré map associated to some nonlinear Hill's equations, extend and improve some recent work. The proofs are elementary in the sense that only well-known properties of planar sets and maps and a two-dimensional equivalent version of the Brouwer fixed point theorem are used.

1. Introduction and basic settings

1.1. A motivation from the theory of ODEs. This paper deals with the study of fixed points and periodic points, as well as with the investigation of chaotic dynamics (in a sense that will be described later) for continuous maps defined on generalized rectangles of a Hausdorff topological space X.

Motivated by the study of the Poincaré map associated to some classes of planar ordinary differential systems, like equation

$$\dot{x} = y, \qquad \dot{y} = -w(t)g(x) \tag{1.1}$$

which, in turn, corresponds to the nonlinear scalar Hill equation

$$\ddot{x} + w(t)g(x) = 0, \tag{1.2}$$

we introduced in [42] the concept of a map stretching a two-dimensional *oriented cell* $\widetilde{\mathcal{A}}$ into another oriented cell $\widetilde{\mathcal{B}}$. Formally, an oriented cell $\widetilde{\mathcal{R}}$ was defined in [42] as a pair $(\mathcal{R},\mathcal{R}^-)$, with $\mathcal{R} \subseteq \mathbb{R}^2$ being the homeomorphic image of a rectangle and with the set $\mathcal{R}^- \subseteq \partial\mathcal{R}$ playing a role which may remind us (but in a very weak sense) of that of an exit set in the Conley-Ważewski theory [11, 55, 56]. The stretching definition was then

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thought in order to take into account the orientation of the cells which are involved. In detail, for each cell $\mathcal{A}$, $\mathcal{B}$, we select two disjoint arcs of its boundary and then consider their union which we denote by $\mathcal{A}^-$ (for the cell $\mathcal{A}$) and by $\mathcal{B}^-$ (for the cell $\mathcal{B}$), respectively. A continuous map ψ defined on $\mathcal{A}$ is said to *stretch the oriented cell* $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *to the oriented cell* $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *along the paths*, if for each path $\sigma \subseteq \mathcal{A}$ intersecting both the sides of $\mathcal{A}^-$ there is a subpath $\gamma \subseteq \sigma$ such that $\psi(\gamma) \subseteq \mathcal{B}$ and $\psi(\gamma)$ intersects both the sides of $\mathcal{B}^-$. In this case, we write $\psi : \widetilde{\mathcal{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathcal{B}}$. To be more precise, we should also mention the fact that, in general, the map ψ will not be defined in the whole cell $\mathcal{A}$. For instance, thinking in terms of the Poincaré map associated to (1.2), we may have blowup phenomena which prevent the solutions to be globally defined (e.g., when $w < 0$ and $g(x) \sim |x|^{\alpha-1}x$ at infinity, with $\alpha > 1$, see [4, 6, 8, 28]). However, we can go round this obstacle by suitably modifying the stretching definition and introducing an appropriate compactness condition. With the aim of shortening our presentation in this introductory part of the paper, we ignore for the moment this fact that will be discussed in Section 3 and so we proceed further by assuming the simplified case in which $\mathcal{A} \subseteq D_\psi$ (D_ψ being the domain of ψ).

In [42], taking advantage of some previous technical lemmas developed in [40] (see also [37]) concerning the nonlinear Hill equation (1.2), with $g(x)$ a function having a superlinear growth at infinity and with $w(t)$ a sign-changing weight, we interpreted the results in [40] in terms of the Poincaré map ϕ associated to system (1.1) in order to show that we can find a conical shell

$$\mathcal{W} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ r^2 \leq x^2 + y^2 \leq R^2\} \tag{1.3}$$

and its opposite $-\mathcal{W}$ with respect to the origin such that

$$\phi : \pm(\mathcal{W}, \mathcal{W}^-) \Leftarrow\!\!\!\rightsquigarrow \pm(\mathcal{W}, \mathcal{W}^-), \tag{1.4}$$

(under all the possible four combinations of "+" and "−," and using the convention that $+\mathcal{W} = \mathcal{W}$). Next, as a consequence of (1.4), we proved that for every two-sided sequence of symbols $(s_k)_{k\in\mathbb{Z}}$ with $s_k \in \{-,+\}$, there is a corresponding "coin-tossing" sequence of points $(w_k)_{k\in\mathbb{Z}}$ such that

$$z_k \in s_k\mathcal{W}, \quad z_{k+1} = \psi(z_k), \quad \forall k \in \mathbb{Z}. \tag{1.5}$$

Via a fixed point theorem for planar maps satisfying the stretching property, we also proved (see [42]) that if $(s_k)_{k\in\mathbb{Z}}$ is a periodic sequence of symbols, then the z_k's can be chosen to form a periodic sequence as well. Using the results in [40], the consequence in terms of the nonlinear Hill equation was that, given an equation like

$$\ddot{x} + w(t)|x|^{\alpha-1}x = 0, \quad \alpha > 1, \tag{1.6}$$

with $w : \mathbb{R} \to \mathbb{R}$ a sufficiently regular T-periodic function such that, for some t_0 and $\tau \in]0,T[$,

$$w(t) > 0 \quad \text{on }]t_0, t_0 + \tau[, \qquad w(t) < 0 \quad \text{on }]t_0 + \tau, t_0 + T[, \tag{1.7}$$

then the following property holds: *for every two-sided sequence* $(s_k)_{k\in\mathbb{Z}}$ *with* $s_k \in \{0,1\}$, *there are at least two solutions* $x(\cdot)$ *of (1.6) having exactly* s_k *zeros in the interval* $]t_0 + kT + \tau, t_0 + (k+1)T[$ (if we have a solution $x(\cdot)$ with some oscillatory properties, also $-x(\cdot)$ is a solution with the same zeros). Actually, for (1.6), there are many other solutions with the same properties (even infinitely many!). In fact, we can also prove that there are solutions with exactly $s_k \in \{0,1\}$ zeros in the interval $]t_0 + kT + \tau, t_0 + (k+1)T[$ where $w < 0$ and with a large number of zeros in the interval $]t_0 + kT, t_0 + kT + \tau[$ where $w > 0$. For the precise statements of the corresponding theorems, see [40, 42] (with respect to chaotic-like solutions) and [38, 42] (for results about periodic solutions). We also refer to the pioneering works of Butler [5, 7] on the existence of infinitely many solutions to (1.2) and to Terracini and Verzini who in [54] first showed, using a variational approach, the existence of complex oscillatory properties for the solutions of (1.6). Recent studies about the chaotic dynamics associated to (1.2) in the superlinear case are also included in [9]. Further applications of our approach to (1.2) under different conditions on $g(x)$ can be found in the forthcoming papers [12, 39, 43]. We refer to [41, 43] for a survey of some recent results on this topic.

To conclude this part of the introduction and also recalling a similar observation in [43], we mention the work of Kennedy and Yorke [20] on the topology of stirred fluids, in order to call the reader's attention to the interesting analogies between the Poincaré operator associated to (1.1) with a sign-changing weight and the maps considered in [20] as a result of compositions between a compression-expansion of the fluid along two different directions and a stir-rotation mapping which provides a suitable twist to the fluid (cf. [20, page 210, Figures 10-11]). See also [19] for related results.

The aim of this paper is addressed toward two different, but related, directions. On the one side, we plan to extend our results in [42] to a more general setting (actually, to the case of stretching maps between oriented cells in the general Hausdorff topological spaces). In this way, we may better understand some properties which were devised in [42], having in mind essentially only the case of the Poincaré map associated to planar ODEs, and, consequently now, thanks to a more general setting, to make such properties more suitable with respect to other possible applications (not necessarily to ODEs). On the other hand, after a refinement of our stretching definition, we are able to improve a corresponding fixed point theorem of [42]. Indeed, here we do not require (as in [42, Theorem 2.1]) that $\mathcal{A} = \mathcal{B}$ and we can prove that an intersection condition on the two cells will be sufficient (see Theorem 3.14 below). We also show, by means of a counterexample in Section 3.3, that a technical hypothesis of compactness in the generalized stretching condition cannot be avoided. This makes our results, in some sense, sharp.

A further aspect that we briefly consider is the following. As we already noticed in [43, 42], it appears that there are strong connections between our approach and some preceding results of Kennedy and Yorke [21] and Kennedy, Koçak, and Yorke in [18] about topological horseshoes. Now we show how we may enter in the framework of

[18, 21] (with the advantage of having available for our case some tools already developed in [18, 21]) and which are the main differences. To summarize here our interpretation, we recall that in [21] the authors consider a continuous map $f : X \supseteq Q \to X$, where Q is a locally connected and compact subset of a separable metric space X. The set Q is assumed to contain two (nonempty) disjoint and compact subsets end_0 and end_1 such that each component of Q intersects both end_0 and end_1. A *connection* $\Gamma \subseteq Q$ is a continuum which intersects both end_0 and end_1, while a *preconnection* $\gamma \subseteq Q$ is a continuum for which $f(\gamma)$ is a connection. Furthermore, a *crossing number* k is defined as the largest number such that every connection contains at least k mutually disjoint preconnections. Then, under the above hypotheses on X, Q, and f and assuming also that

$$k \geq 2, \tag{1.8}$$

there exists a closed invariant set $Q_I \subseteq Q$ for which $f|_{Q_I}$ is semiconjugated to a one-sided shift on k symbols (cf. [21, Theorem 1]). Now, if we have an oriented cell $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$, we can consider the two components of $\mathcal{A}^-$ as the subsets end_0 and end_1 for the set $Q = \mathcal{A}$ and thus we may read the situation $\psi : \widetilde{\mathcal{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathcal{A}}$ as a particular case of a crossing number

$$k \geq 1. \tag{1.9}$$

This makes clear that, from some point of view, our setting is only a particular case of that considered in [21] (and also in [18]), but, due to the restricted situation considered by us, we have the possibility to obtain some more information (e.g., the existence of fixed points or periodic points) that is not provided in [18, 21]. In [18], the authors suggested studying the problem of a crossing number $k = 1$. In fact, in [18, Section 7], they wrote: "we have generalized the notion of horseshoe maps in this paper, but further generalizations could be possible if the case $k = 1$ was better understood." We hope that our results in Section 3 may be regarded as a possible contribution in this direction.

In Section 4, we discuss how to consider in our setting the case $k \geq 2$ and obtain a theorem about coin-tossing dynamics on k-symbols for ψ along its iterates. Applying our fixed point theorem, we also prove that every periodic sequence of symbols is actually realized by some periodic point of the map ψ (see [31, 32, 52, 53, 60, 61, 62] for other papers in which a similar definition of chaos is considered). We stress the fact that, besides our stretching condition $\psi : \widetilde{\mathcal{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathcal{B}}$, only an assumption about the intersection of $\mathcal{A}$ and $\mathcal{B}$ is required. Such an assumption turns out to be particularly simple to express when ψ is a homeomorphism, just looking at the manner in which $\mathcal{A}$ and $\mathcal{B}$ intersect each other.

As a last remark, we notice that all our results are obtained using only elementary properties either from the theory of compact connected sets [1, 25, 57] or from the topology of the Euclidean plane [17, 33]. The only more sophisticated tools will be the Brouwer fixed point theorem in dimension two and the Jordan-Shoenflies theorem. Even if we have to pay the price for the limitation in using simple tools by the fact that, at this stage, the applications of our theorems are confined to a two-dimensional setting, nevertheless we think that our approach may have a "pedagogical" interest too, since it shows a way to obtain fixed points, periodic points, and chaotic-type dynamics using only elementary properties.

1.2. Main definitions. In the plane $\mathbb{R}^2$ endowed with the Euclidean norm $\|\cdot\|$, we consider the unit square $\mathcal{Q} = [0,1]^2$ and its vertical sides (edges)

$$\mathcal{Q}_l^- = \{0\} \times [0,1], \qquad \mathcal{Q}_r^- = \{1\} \times [0,1], \tag{1.10}$$

and horizontal sides (edges)

$$\mathcal{Q}_b^+ = [0,1] \times \{0\}, \qquad \mathcal{Q}_t^+ = [0,1] \times \{1\}. \tag{1.11}$$

We also define the sets

$$\mathcal{Q}^- = \mathcal{Q}_l^- \cup \mathcal{Q}_r^-, \qquad \mathcal{Q}^+ = \mathcal{Q}_b^+ \cup \mathcal{Q}_t^+ \tag{1.12}$$

and call the pair $\widetilde{\mathcal{Q}} = (\mathcal{Q}, \mathcal{Q}^-)$ the standard two-dimensional oriented cell.

Throughout the paper, X will be a Hausdorff topological space. By a *continuum* of X we mean a compact and connected subset of X. Among the continua of X, we will consider also the *paths* and *arcs* which are the continuous and the homeomorphic images of the unit interval $[0,1]$, respectively. A subset $\mathcal{R} \subseteq X$ is called a *two-dimensional cell* (or simply a *cell* when no confusion may arise) if there is a homeomorphism h of $\mathcal{Q} \subseteq \mathbb{R}^2$ onto $\mathcal{R} \subseteq X$. Clearly, $\mathcal{R}$, as a topological space, inherits the topological properties of $\mathcal{Q}$, so that it is a compact, connected, simply connected, and metrizable space and the compact subsets of $\mathcal{R}$ are those subsets of $\mathcal{R}$ which are closed relatively to $\mathcal{R}$, or, that is the same since X is a Hausdorff space, the closed sets of X which are contained in $\mathcal{R}$.

We denote $\partial\mathcal{R} \subseteq \mathcal{R} = h(\partial\mathcal{Q})$ and call it the *contour* of $\mathcal{R}$. Note that if $\mathcal{R}$ is a cell, then its contour is determined independently of h. In particular, $\partial\mathcal{R}$ is a homeomorphic image of the unit circumference $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and then it is a simple closed curve (a Jordan curve).

Definition 1.1. An *oriented cell* is a pair $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$, where

$$\mathcal{R}^- = \mathcal{R}_0^- \cup \mathcal{R}_1^- \subseteq \partial\mathcal{R} \tag{1.13}$$

is the union of two disjoint (compact) arcs. The closure in $\partial\mathcal{R}$ of the set $\mathcal{R} \setminus \mathcal{R}^-$ is the disjoint union of two arcs too. We denote this closure by $\mathcal{R}^+$ and its two components by $\mathcal{R}_0^+$ and $\mathcal{R}_1^+$.

If $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ is an oriented cell with $h : \mathcal{Q} \to \mathcal{R}$ a homeomorphism defining $\mathcal{R}$, we have that $h^{-1}(\mathcal{R}_0^-)$ and $h^{-1}(\mathcal{R}_1^-)$ are two disjoint arcs of $\partial\mathcal{Q}$. As a consequence of the Jordan-Shoenflies theorem (see [17, 33]), it is not difficult to see that there is a homeomorphism $h_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h_1(\mathcal{Q}) = Q$, $h_1(\partial\mathcal{Q}) = \partial\mathcal{Q}$, and $h_1(h^{-1}(\mathcal{R}_0^-)) = \mathcal{Q}_l^-$, $h_1(h^{-1}(\mathcal{R}_1^-)) = \mathcal{Q}_r^-$. Hence, $h_0 = h \circ h_1^{-1} : \mathbb{R}^2 \supseteq \mathcal{Q} \to \mathcal{R} \subseteq X$ is a homeomorphism with $h_0(\mathcal{Q}) = \mathcal{R}$, $h_0(\partial\mathcal{Q}) = \partial\mathcal{R}$, $h_0(\mathcal{Q}_l^-) = \mathcal{R}_0^-$, and $h_0(\mathcal{Q}_r^-) = \mathcal{R}_1^-$. If we like, we can take h_0 so that $h_0(\mathcal{Q}_b^+) = \mathcal{R}_0^+$ and $h_0(\mathcal{Q}_t^+) = \mathcal{R}_1^+$. As a consequence of this fact, for any oriented cell

$\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$, there is a homeomorphism $q : \mathbb{R}^2 \supseteq \mathfrak{Q} \to \mathcal{R} \subseteq X$ having the same properties listed above for h_0. To indicate the occurrence of this situation, we will write $q : \widetilde{\mathfrak{Q}} \diamond\!\!\to \widetilde{\mathcal{R}}$.

Extending a little this definition, we will write $q : \widetilde{\mathcal{P}} \diamond\!\!\to \widetilde{\mathcal{R}}$, also when $\mathcal{P} = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ is a planar rectangle with $\mathcal{P}^-$ equal to the union of two opposite (closed) sides and $\mathcal{P}^+$ equal to the union of the other two (closed) sides and $q : \mathbb{R}^2 \supseteq \mathcal{P} \to \mathcal{R} \subseteq X$ is a homeomorphism with $q(\mathcal{P}) = \mathcal{R}$, $q(\partial\mathcal{P}) = \partial\mathcal{R}$, mapping the left side of $\mathcal{P}$ onto $\mathcal{R}_0^-$ and the right side of $\mathcal{P}$ onto $\mathcal{R}_1^-$ and, similarly, mapping the lower and the upper sides of $\mathcal{P}$ onto $\mathcal{R}^+$.

If $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ is an oriented cell of X and $\phi : X \supseteq \mathcal{R} \to \phi(\mathcal{R}) \subseteq X$ is a *homeomorphism* of $\mathcal{R}$ onto its image $\phi(\mathcal{R})$, we have that $\phi(\mathcal{R})$ is a two-dimensional cell with $\partial\phi(\mathcal{R}) = \phi(\partial\mathcal{R})$. In this case, if we set

$$\phi(\mathcal{R})^- := \phi(\mathcal{R}^-), \tag{1.14}$$

we can define, in a canonical way, the oriented cell $\phi(\widetilde{\mathcal{R}})$ as

$$\phi(\widetilde{\mathcal{R}}) := \widetilde{\phi(\mathcal{R})} = (\phi(\mathcal{R}), \phi(\mathcal{R}^-)). \tag{1.15}$$

Remark 1.2. Our definition of oriented cell $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ fits with that of $(1,1)$-*window* considered recently by Gidea and Robinson in [15]. More precisely, given $q : \widetilde{\mathfrak{Q}} \diamond\!\!\to \widetilde{\mathcal{R}}$, we have that $(\mathcal{R}, q)$ is a $(1,1)$-window according to [15, page 56]. In [15, Section 5], the authors apply an extension of the method of *correctly aligned windows* (see also [13, 32, 62, 63]) to the existence of symbolic dynamics for higher-dimensional systems and hence [15] deals with the case of (u,s)-windows with u and s possibly greater than one, as well. We point out, however, that our definition of a map stretching an oriented cell to another along the paths (see Section 3, Definition 3.1 below) requires fairly less conditions than the corresponding definition of a window forward correctly aligned with another window under a map, as considered in [15, Definition 5.2]. We refer to [64] for a recent and general treatment of such an approach and to [3, 59] for further applications of Zgliczyński's method.

The next definition is crucial for our applications.

Let $\widetilde{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$ and $\widetilde{\mathcal{N}} = (\mathcal{N}, \mathcal{N}^-)$ be two oriented cells in X.

Definition 1.3. (see Figure 1.1) $\widetilde{\mathcal{M}}$ is said to be a *horizontal slice* of $\widetilde{\mathcal{N}}$, in symbols:

$$\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}, \tag{1.16}$$

if

$$\mathcal{M} \subseteq \mathcal{N} \tag{1.17}$$

and, either

$$\mathcal{M}_0^- \subseteq \mathcal{N}_0^-, \qquad \mathcal{M}_1^- \subseteq \mathcal{N}_1^-, \tag{1.18}$$

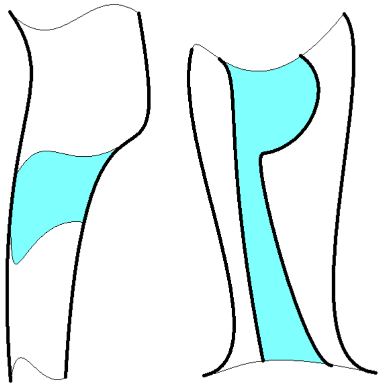

Figure 1.1. Examples of $\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{N}}$ (the left and the right figures, respectively). The painted areas represent $\widetilde{\mathcal{M}}$ as embedded in $\widetilde{\mathcal{N}}$. The $[\cdot]^-$-sets for the oriented cells $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ are indicated with a bold line.

or

$$\mathcal{M}_0^- \subseteq \mathcal{N}_1^-, \qquad \mathcal{M}_1^- \subseteq \mathcal{N}_0^-. \tag{1.19}$$

Similarly, $\widetilde{\mathcal{M}}$ is said to be a *vertical slice* of $\widetilde{\mathcal{N}}$, in symbols:

$$\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{N}}, \tag{1.20}$$

if

$$\mathcal{M} \subseteq \mathcal{N} \tag{1.21}$$

and, either

$$\mathcal{M}_0^+ \subseteq \mathcal{N}_0^+, \qquad \mathcal{M}_1^+ \subseteq \mathcal{N}_1^+, \tag{1.22}$$

or

$$\mathcal{M}_0^+ \subseteq \mathcal{N}_1^+, \qquad \mathcal{M}_1^+ \subseteq \mathcal{N}_0^+. \tag{1.23}$$

Remark 1.4. From the definition, it is clear that

$$(\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}) \wedge (\widetilde{\mathcal{N}} \subseteq_h \widetilde{\mathcal{M}}) \Longrightarrow \widetilde{\mathcal{M}} = \widetilde{\mathcal{N}} \tag{1.24}$$

and also that

$$(\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{N}}) \wedge (\widetilde{\mathcal{N}} \subseteq_v \widetilde{\mathcal{M}}) \Longrightarrow \widetilde{\mathcal{M}} = \widetilde{\mathcal{N}}. \tag{1.25}$$

On the other hand, perhaps, more interesting is the fact that

$$(\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}) \wedge (\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{N}}) \Longrightarrow \widetilde{\mathcal{M}} = \widetilde{\mathcal{N}}. \tag{1.26}$$

The proof is omitted and is left to the reader. Observe also that from $(\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}) \wedge (\widetilde{\mathcal{N}} \subseteq_v \widetilde{\mathcal{M}})$ we can only obviously deduce that $\mathcal{M} = \mathcal{N}$, but, in general, we cannot conclude that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{N}}$.

2. A key lemma

The main result of this section is Lemma 2.3 where we rephrase in terms of oriented cells a classical theorem of plane topology. Our result is strongly related to the fact (already applied by Hastings in [16, page 131]) that if a closed set separates the plane, then some component of this set separates the plane too [36]. Analogous results were applied by Conley [10] and Butler [5] (in a more or less explicit form) in some papers dealing with ordinary differential equations. A proof of a variant of Lemma 2.3 was given in [46]. See also [43] for a proof which follows the argument in [46], using a two-dimensional version of the Alexander addition theorem as presented in [49, page 82]. Here we provide a different proof which reduces the statement of the lemma to an equivalent form of the Brouwer fixed point theorem, namely, the Poincaré-Miranda theorem [23, 26] that we recall here for the reader's convenience in the two-dimensional case.

Theorem 2.1 (Poincaré-Miranda theorem). *Let $(f,g) : \Xi = [-a_1,a_1] \times [-a_2,a_2] \to \mathbb{R}^2$ be a continuous vector field such that $f(-a_1,y) \le 0 \le f(a_1,y)$, for each $|y| \le a_2$ and $g(x,-a_2) \le 0 \le g(x,a_2)$, for each $|x| \le a_1$. Then, there exists $(x_0,y_0) \in \Xi$ such that $f(x_0,y_0) = 0$ and $g(x_0,y_0) = 0$.*

Proof. Consider the situation in which $a_1 = a_2 = 1$ for $\Xi = [-1,1]^2$ (the general case easily follows via an elementary change of variables) and define the function $\eta : \mathbb{R} \to \mathbb{R}$,

$$\eta(s) = \min\{1, \max\{-1, s\}\}. \tag{2.1}$$

Consider now the continuous map

$$\phi : \Xi \longrightarrow \Xi, \quad \phi(x,y) = \big(\eta(x - f(x,y)), \eta(y - g(x,y))\big) \tag{2.2}$$

which has a fixed point (x_0,y_0) by the Brouwer fixed point theorem. It is not difficult to check that (x_0,y_0) is actually a zero of the vector field (f,g). The same proof extends to the N-dimensional case ($N > 2$). Conversely, it is straightforward to obtain a proof of the Brouwer fixed point theorem for the rectangle via the Poincaré-Miranda theorem. In fact, if $\phi : \Xi \to \Xi$, then $I - \phi$ satisfies the assumptions of Theorem 2.1. □

Remark 2.2. The Poincaré-Miranda theorem was first announced by Poincaré in 1883 [44] and published in 1884 [45], with reference to a proof using the Kronecker's index [27]. In the two-dimensional case, Poincaré also proposed a heuristic argument which reads as follows (cf. [27]). The "curve" $g = 0$ starts at some point of $x = -a_1$ and ends at some point of $x = a_1$ and, in the same manner, the "curve" $f = 0$ starts at some point of $y = -a_2$ and ends at some point of $y = a_2$. Hence, the two "curves" meet at some point of the square Ξ. The name of Miranda is associated to this theorem for his proof (1940) [29] of the equivalence to the Brouwer fixed point theorem (see also [26]). For different proofs of the Poincaré-Miranda theorem in the N-dimensional case, see [23, 30, 47, 48].

Lemma 2.3. *Let* $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *be an oriented cell in* X *and suppose that* $\mathcal{S} \subseteq \mathcal{R}$ *is a compact set such that* $\sigma \cap \mathcal{S} \neq \emptyset$, *for each path* σ *contained in* $\mathcal{R}$ *and joining* $\mathcal{R}_0^-$ *to* $\mathcal{R}_1^-$. *Then* $\mathcal{S}$ *contains a continuum* $\mathcal{C}$ *joining* $\mathcal{R}_0^+$ *to* $\mathcal{R}_1^+$.

Proof. We consider the vertical strip

$$\mathcal{V} = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1\} = [-1,1] \times \mathbb{R}, \tag{2.3}$$

bounded by the vertical lines

$$V_{-1} = \{-1\} \times \mathbb{R}, \qquad V_1 = \{1\} \times \mathbb{R}. \tag{2.4}$$

In $\mathcal{V}$, we consider the horizontal lines

$$\begin{gathered} H_{-1} = [-1,1] \times \{-1\}, \qquad H_{-1/2} = [-1,1] \times \left\{\frac{-1}{2}\right\}, \\ H_{1/2} = [-1,1] \times \left\{\frac{1}{2}\right\}, \qquad H_1 = [-1,1] \times \{1\} \end{gathered} \tag{2.5}$$

and the rectangle $\mathcal{P} = [-1,1] \times [-1/2, 1/2] \subseteq \mathbb{R}^2$. We also define

$$\begin{gathered} \mathcal{P}_0^- = H_{-1/2}, \qquad \mathcal{P}_1^- = H_{1/2}, \\ \mathcal{P}_0^+ = V_{-1} \cap \mathcal{P}, \qquad \mathcal{P}_1^+ = V_1 \cap \mathcal{P}. \end{gathered} \tag{2.6}$$

Finally, set

$$\mathcal{P}^- = \mathcal{P}_0^- \cup \mathcal{P}_1^-, \qquad \mathcal{P}^+ = \mathcal{P}_0^+ \cup \mathcal{P}_1^+, \tag{2.7}$$

and take a homeomorphism $q : \widetilde{\mathcal{P}} \diamond\!\!\rightarrow \widetilde{\mathcal{R}}$. With these positions, the compact set

$$\mathcal{T} = q^{-1}(\mathcal{S}) \subseteq \mathcal{P} \tag{2.8}$$

satisfies the following path-intersection property:

(P1) $\mathcal{T} \cap \sigma \neq \emptyset$, for each path σ contained in $\mathcal{P}$ and joining $H_{-1/2}$ to $H_{1/2}$.

Clearly, we have also the following property satisfied:

(P2) $\mathcal{T}$ is a compact subset of $\mathcal{P}$ such that $\mathcal{T} \cap \sigma \neq \emptyset$, for each path σ contained in $\mathcal{V}$ and joining H_{-1} to H_1.

Consider now the set $A = \mathcal{V} \setminus \mathcal{T}$ which is open in $\mathcal{V}$ and locally arcwise-connected. From (P2), we have that A is not connected and the segments H_{-1} and H_1 belong to different components of A. Hence, there are two open disjoint sets A_{-1} and A_1 with $A = A_{-1} \cup A_1$ and such that $H_{-1} \subseteq A_{-1}$, $H_1 \subseteq A_1$. Next, proceeding like in [46], we define the function

$$\begin{gathered} w(x,y) = \begin{cases} -1, & \text{if } (x,y) \in A_{-1}, \\ 0, & \text{if } (x,y) \in \mathcal{T}, \\ 1, & \text{if } (x,y) \in A_1, \end{cases} \\ g(x,y) := w(x,y)\,\mathrm{dist}\,((x,y);\mathcal{T}). \end{gathered} \tag{2.9}$$

The map $g : \mathbb{R}^2 \supseteq \mathcal{V} \to \mathbb{R}$ is continuous and satisfies the following properties:

$$\begin{aligned} g(x,y) &< 0, \quad \text{for } (x,y) \in A_{-1}, \\ g(x,y) &= 0, \quad \text{for } (x,y) \in \mathcal{T}, \\ g(x,y) &> 0, \quad \text{for } (x,y) \in A_1 \end{aligned} \tag{2.10}$$

and, in particular,

$$\begin{aligned} g(x,y) &< 0, \quad \forall (x,y) \in H_{-1}, \\ g(x,y) &> 0, \quad \forall (x,y) \in H_1. \end{aligned} \tag{2.11}$$

Assume now, by contradiction, that $\mathcal{T}$ does not contain a continuum joining V_{-1} to V_1. Hence, by the Whyburn lemma (cf. [25, chapter V], [57]), it follows that the nonempty disjoint compact sets $V_{-1} \cap \mathcal{T}$ and $V_1 \cap \mathcal{T}$ are separated in $\mathcal{T}$, that is, there are closed subsets $F_{-1} \supseteq V_{-1} \cap \mathcal{T}$ and $F_1 \supseteq V_1 \cap \mathcal{T}$ with $F_{-1} \cap F_1 = \emptyset$, and $F_{-1} \cup F_1 = \mathcal{T}$.

Finally, on the square $\Xi = [-1,1]^2$ we consider the compact disjoint sets

$$\hat{F}_{-1} = \Xi \cap (F_{-1} \cup V_{-1}), \qquad \hat{F}_1 = \Xi \cap (F_1 \cup V_1) \tag{2.12}$$

and define the continuous function

$$f(x,y) := \operatorname{dist}\big((x,y); \hat{F}_{-1}\big) - \operatorname{dist}\big((x,y); \hat{F}_1\big). \tag{2.13}$$

By the definition of f, we have that

$$f(x,y) < 0, \quad \forall (x,y) \in V_{-1} \cap \Xi, \qquad f(x,y) > 0, \quad \forall (x,y) \in V_1 \cap \Xi. \tag{2.14}$$

The continuous vector field

$$(f,g) : \Xi \longrightarrow \mathbb{R}^2 \tag{2.15}$$

satisfies the assumptions of the Poincaré-Miranda theorem. In fact, by (2.14), $f < 0$ on the left side of Ξ, $f > 0$ on the right side of Ξ and, by (2.11), $g < 0$ on the lower side of Ξ and $g > 0$ on the upper side of Ξ. Therefore, there is at least a point $(x_0, y_0) \in \Xi$ such that

$$f(x_0, y_0) = 0, \qquad g(x_0, y_0) = 0. \tag{2.16}$$

The second condition in (2.16) then implies that $(x_0, y_0) \in \mathcal{T} \subseteq \hat{F}_{-1} \cup \hat{F}_1$. On the other hand, $f(x_0, y_0) < 0$ if $(x_0, y_0) \in \hat{F}_{-1}$ and $f(x_0, y_0) > 0$ if $(x_0, y_0) \in \hat{F}_1$. This gives a contradiction to the first condition in (2.16) and concludes the proof. □

Remark 2.4. We have just given a proof of Lemma 2.3 using the Poincaré-Miranda theorem. Conversely, it is easy now, following exactly the argument proposed by Poincaré in [45] and recalled in Remark 2.2, to obtain a proof of the two-dimensional version of the Poincaré-Miranda theorem using Lemma 2.3. In fact, consider a continuous vector field $(f,g) : \Xi \to \mathbb{R}^2$ as in Theorem 2.1. Using Lemma 2.3, we have that the compact set $\{(x,y) \in \Xi : g(x,y) = 0\}$ contains a continuum $\mathcal{C}_1$ connecting $x = -a_1$ to $x = a_1$ and, by

Table 3.1

For every	there is a(an)
path	path
path	arc
path	continuum
arc	arc
arc	continuum
continuum	continuum
$\sigma \subseteq \mathcal{A}$ with $\sigma \cap \mathcal{A}_0^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_1^- \neq \emptyset$	$\gamma \subseteq \sigma \cap \mathcal{K}$ with $\psi(\gamma) \cap \mathcal{B}_0^- \neq \emptyset$, and $\psi(\gamma) \cap \mathcal{B}_1^- \neq \emptyset$

the same reason, also the set $\{(x,y) \in \Xi : f(x,y) = 0\}$ contains a continuum $\mathcal{C}_2$ connecting $y = -a_2$ to $y = a_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ (see, e.g., [35, Lemma 3] for a proof), we have guaranteed the existence of a zero for the vector field (f,g). See also [46] for still another proof of Theorem 2.1, using Lemma 2.3. Finally, we observe that Lemma 2.3 seems to be also connected to the Hex theorem [14] which claims the impossibility of a draw in the Hex game.

Remark 2.5. We call the reader's attention also to an interesting remark by Easton [13, page 113] where the separation property of Lemma 2.3 is interpreted in the cohomological setting.

3. Mappings with a stretching property and their fixed points

3.1. Main definition and some equivalent formulations. Let $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ be two oriented cells in the Hausdorff topological space X, let $\psi : X \supseteq D_\psi \to X$ be a continuous map, and let $\mathcal{D} \subseteq D_\psi \cap \mathcal{A}$.

Definition 3.1. $(\mathcal{D}, \psi)$ is said to *stretch* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths*, in symbols:

$$(\mathcal{D}, \psi) : \widetilde{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{B}}, \tag{3.1}$$

if there is a compact set $\mathcal{K} \subseteq \mathcal{D}$ such that the following conditions are satisfied:

(H1) $\psi(\mathcal{K}) \subseteq \mathcal{B}$,

(H2) for every path $\sigma \subseteq \mathcal{A}$ with $\sigma \cap \mathcal{A}_0^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_1^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma \cap \mathcal{K}$ with $\psi(\gamma) \cap \mathcal{B}_0^- \neq \emptyset$ and $\psi(\gamma) \cap \mathcal{B}_1^- \neq \emptyset$.

Remark 3.2. This definition is slightly more general than the corresponding one proposed in [42], where we assumed the properness of $(\mathcal{D}, \psi)$ on the bounded sets.

We also observe that there are various equivalent means to express the condition (H2). They are listed in Table 3.1.

The interested reader is invited to provide a proof of this claimed equivalence.

We notice that $(\mathcal{D}, \psi) : \widetilde{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{B}}$ *does not imply* that $\psi(\mathcal{D}) \subseteq \mathcal{B}$. However, we do have $\psi(\mathcal{K}) \subseteq \mathcal{B}$.

The compact set $\mathcal{K}$ plays a crucial role in our stretching definition (see also the corresponding Section 3.3). In some applications, a natural choice of $\mathcal{K}$ is explicitly known in advance, in others, only its existence (in some implicit manner) will be guaranteed. In what follows, to put in evidence the presence of $\mathcal{K}$ in the stretching condition, we will write sometimes $(\mathcal{D},\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$ instead of $(\mathcal{D},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$. Note also that $(\mathcal{D},\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$ implies that $(\mathcal{D}',\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$ for each $\mathcal{D}'$ with $\mathcal{K} \subseteq \mathcal{D}' \subseteq \mathcal{A}$. Therefore, if $(\mathcal{D},\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$, then $(\mathcal{K},\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$ and we can write simply $(\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$ to avoid redundancy. Finally, when $\mathcal{D} = D_\psi \cap \mathcal{A}$ (e.g., when $\mathcal{D} = \mathcal{A} \subseteq D_\psi$), we also write $\psi : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$ for $(D_\psi \cap \mathcal{A},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$.

Remark 3.3. Using Lemma 2.3, it is easy to see that if $(\mathcal{D},\mathcal{K},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$, then there is a continuum $\mathcal{C} \subseteq \mathcal{K}$, joining $\mathcal{A}_0^+$ to $\mathcal{A}_1^+$.

A simple case in which we have the stretching property satisfied is given in the following lemma.

Lemma 3.4. *Let $\mathcal{D} \subseteq \mathcal{A}$ and suppose that there is a compact set $\mathcal{H} \subseteq \mathcal{D}$ such that*

(H3) *for any path $\sigma \subseteq \mathcal{A}$ such that $\sigma \cap \mathcal{A}_0^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_1^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma \cap \mathcal{H}$ such that $\psi(\gamma) \subseteq \mathcal{B}$ and $\psi(\gamma) \cap \mathcal{B}_0^- \neq \emptyset$, $\psi(\gamma) \cap \mathcal{B}_1^- \neq \emptyset$.*

Then $(\mathcal{D},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$.

Proof. The set

$$\mathcal{K} = \{x \in \mathcal{H} : \psi(x) \in \mathcal{B}\} \tag{3.2}$$

is closed in $\mathcal{H}$ and thus, compact. By definition of $\mathcal{H}_1$, we have that $\psi(\mathcal{K}) \subseteq \mathcal{B}$ and therefore (H1) is satisfied. Now it is easy to check that (by our choice of $\mathcal{K}$), (H3) implies (H2). This ends the proof. □

Actually, the condition expressed in Lemma 3.4 is equivalent to Definition 3.1. The proof of Lemma 3.4 also suggests the following consequence.

Corollary 3.5. *Let $\mathcal{D} \subseteq \mathcal{A}$, and suppose that $\psi^{-1}(\mathcal{B}) \cap \mathcal{D}$ is compact and*

(H4) *for any path $\sigma \subseteq \mathcal{A}$ such that $\sigma \cap \mathcal{A}_0^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_1^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma \cap \mathcal{D}$ such that $\psi(\gamma) \subseteq \mathcal{B}$ and $\psi(\gamma) \cap \mathcal{B}_0^- \neq \emptyset$, $\psi(\gamma) \cap \mathcal{B}_1^- \neq \emptyset$.*

Then $(\mathcal{D},\psi) : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$.

Of course, an analogous table like that of Remark 3.2 can be considered with respect to conditions (H3) and (H4).

We also observe that, according to Lemma 3.4 (for $\mathcal{D} = \mathcal{H} = \mathcal{A}$), when $\mathcal{A} \subseteq D_\psi$, then $\psi : \widetilde{\mathcal{A}} \mathrel{\lhd\!\!\rightsquigarrow} \widetilde{\mathcal{B}}$, provided that for any path $\sigma \subseteq \mathcal{A}$ such that $\sigma \cap \mathcal{A}_0^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_1^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma$ such that $\psi(\gamma) \subseteq \mathcal{B}$ and $\psi(\gamma) \cap \mathcal{B}_0^- \neq \emptyset$, $\psi(\gamma) \cap \mathcal{B}_1^- \neq \emptyset$.

Other elementary observations are contained in the next results.

Lemma 3.6. *Suppose that* $(\mathscr{D}_1,\mathscr{K}_1,\psi_1):\widetilde{\mathscr{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{B}}$ *and* $(\mathscr{D}_2,\mathscr{K}_2,\psi_2):\widetilde{\mathscr{B}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{C}}$, *then* $(\mathscr{D}_{1,2}, \mathscr{K}_{1,2},\psi_2\circ\psi_1):\widetilde{\mathscr{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{C}}$, *with*

$$\mathscr{D}_{1,2} := \{z\in\mathscr{D}_1 : \psi_1(z)\in\mathscr{D}_2\}, \qquad \mathscr{K}_{1,2} := \{z\in\mathscr{K}_1 : \psi_1(z)\in\mathscr{K}_2\}. \tag{3.3}$$

Lemma 3.7. *If* $(\mathscr{D},\mathscr{K},\psi):\widetilde{\mathscr{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{B}}$, *then, for every* $\widetilde{\mathscr{M}} \subseteq_h \widetilde{\mathscr{A}}$ *and every* $\widetilde{\mathscr{N}} \subseteq_v \widetilde{\mathscr{B}}$, *it follows that* $(\mathscr{D}\cap\mathscr{M},\mathscr{K}\cap\mathscr{M},\psi):\widetilde{\mathscr{M}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{N}}$.

Proof. Both results easily follow from the definition. □

Remark 3.8. Let $\widetilde{\mathscr{A}} = (\mathscr{A},\mathscr{A}^-)$ be an oriented cell in the Hausdorff space X and let $\psi : X \supseteq \mathscr{A} \to X$ be a homeomorphism of $\mathscr{A}$ onto its image $\psi(\mathscr{A})$. In this case, as already explained in the introduction, we can define a structure of oriented cell $\psi(\widetilde{\mathscr{A}})$ for $\psi(\mathscr{A})$ by setting

$$\psi(\widetilde{\mathscr{A}}) := \widetilde{\psi(\mathscr{A})} = (\psi(\mathscr{A}),\psi(\mathscr{A})^-), \quad \text{with } \psi(\mathscr{A})^- = \psi(\mathscr{A}^-). \tag{3.4}$$

By Definition 3.1, it is clear that in this case we have $\psi:\widetilde{\mathscr{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\psi(\mathscr{A})}$. More generally, if ψ (only continuous and not necessarily a homeomorphism) is defined on $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}} = (\mathscr{B},\mathscr{B}^-)$ is another cell in X such that

$$\psi(\mathscr{A}_0^-) \subseteq \mathscr{B}_0^-, \qquad \psi(\mathscr{A}_1^-) \subseteq \mathscr{B}_1^- \tag{3.5}$$

or

$$\psi(\mathscr{A}_0^-) \subseteq \mathscr{B}_1^-, \qquad \psi(\mathscr{A}_1^-) \subseteq \mathscr{B}_0^-, \tag{3.6}$$

then, $\psi:\widetilde{\mathscr{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{B}}$.

3.2. The fixed point property

Theorem 3.9. *Let* $\widetilde{\mathscr{R}} = (\mathscr{R},\mathscr{R}^-)$ *be an oriented cell in* X. *If* $(\mathscr{D},\mathscr{K},\psi):\widetilde{\mathscr{R}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{R}}$, *then there is* $w\in\mathscr{K}$ *such that* $\psi(w) = w$.

Proof. The proof is almost the same like that we already presented in [42, Theorem 6] or [43, Theorem 1]. We give some details of it for completeness.

Let $q:\widetilde{\mathscr{Q}} \diamond\!\!\rightarrow \widetilde{\mathscr{R}}$. Setting $\mathscr{F} = q^{-1}(\mathscr{K})$ and $\phi = (\phi_1,\phi_2) = q^{-1}\circ\psi\circ q$, we have that $(\mathscr{F},\phi): \widetilde{\mathscr{Q}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathscr{Q}}$, where we have set $(\mathscr{F},\phi)$ for $(\mathscr{F},\mathscr{F},\phi)$. For $x = (x_1,x_2)\in\mathscr{Q}$, we consider the compact set

$$\mathscr{S} = \{x\in\mathscr{F} : \phi_1(x) = x_1\}. \tag{3.7}$$

Note that $\phi(\mathscr{F}) \subseteq \mathscr{Q}$ (by (H1)). Let $\sigma\subseteq\mathscr{Q}$ be a path such that $\sigma\cap\mathscr{Q}_l^- \neq \emptyset$ and $\sigma\cap\mathscr{Q}_r^- \neq \emptyset$. By the stretching hypothesis, there is a subpath $\gamma\subseteq\sigma\cap\mathscr{F}$ with $\phi(\gamma)\subseteq\mathscr{Q}$ and $\phi(\gamma)\cap\mathscr{Q}_l^- \neq \emptyset$ as well as $\phi(\gamma)\cap\mathscr{Q}_r^- \neq \emptyset$. Therefore, $0 \le \phi_1(x) \le 1$, for all $x\in\gamma$ and, moreover, $\phi_1(y) = 0 \le y_1$ and $\phi_1(z) = 1 \ge z_1$ for some points $y = (y_1,y_2)$ and $z = (z_1,z_2)$ in γ. By the Bolzano theorem, the map $x\mapsto\phi_1(x) - x_1$ vanishes somewhere in γ, that is, there is some point $x\in\gamma\subseteq\sigma\cap\mathscr{F}$ with $\phi_1(x) = x_1$. In this manner, we have proved that any path contained in $\mathcal{Q}$ and joining $\mathcal{Q}_l^-$ to $\mathcal{Q}_r^-$ intersects the set $\mathcal{S}$. As a consequence of Lemma 2.3, we know that $\mathcal{S}$ contains a continuum $\mathcal{C}$ joining $\mathcal{Q}_b^+$ to $\mathcal{Q}_t^+$. From $\mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{F}$ and $\phi(\mathcal{F}) \subseteq \mathcal{Q}$, it also follows that $\phi_2(\mathcal{C}) \subseteq [0,1]$ so that $x_2 - \phi_2(x) \leq 0$ for $x \in \mathcal{C} \cap \mathcal{Q}_b^+$ and $x_2 - \phi_2(x) \geq 0$ for $x \in \mathcal{C} \cap \mathcal{Q}_t^+$. Applying again the Bolzano theorem, we have the existence of a point $z = (z_1, z_2) \in \mathcal{C}$ such that $\phi_2(z) = z_2$ and, as $z \in \mathcal{C} \subseteq \mathcal{S}$, we also know that $\phi_1(z) = z_1$ so that we conclude that $\phi(z) = z$, with $z \in \mathcal{F}$. Clearly, $w = q(z) \in \mathcal{K}$ is a fixed point of ψ. This concludes the proof. □

Remark 3.10. Variants of Theorem 3.9 can be easily obtained by considering the cases in which one component or both components of $\mathcal{R}^-$ degenerate to a point. For example, consider the situation in which we have a two-dimensional cell $\mathcal{R} \subseteq X$ and we select two different points $P_1, P_2 \in \partial\mathcal{R}$. Assume that there is a compact set $\mathcal{H} \subseteq \mathcal{D}$ such that for any path $\sigma \subseteq \mathcal{R}$ with $P_1, P_2 \in \sigma$ there is a path $\gamma \subseteq \sigma \cap \mathcal{H}$ such that $P_1, P_2 \in \psi(\gamma) \subseteq \mathcal{R}$. Then, as a consequence of Theorem 3.9, we can prove the existence of a fixed point of ψ in $\mathcal{H}$. Indeed let $S_1 = q^{-1}(P_1)$ and $S_2 = q^{-1}(P_2)$, where $q : \widetilde{\mathcal{Q}} \diamond\!\!\rightarrow \widetilde{\mathcal{R}}$. It is possible to find a continuous surjection $p : \mathcal{Q} \to \mathcal{Q}$ such that $p(\mathcal{Q}_l^-) = \{S_1\}$, $p(\mathcal{Q}_r^-) = \{S_2\}$ and $p : \mathcal{Q} \setminus (\mathcal{Q}_l^- \cup \mathcal{Q}_r^-) \to \mathcal{Q} \setminus \{S_1, S_2\}$ is bijective. Then, if we set $\phi = q^{-1} \circ \psi \circ q \circ p$, we have that $(\phi, \mathcal{K}) : \mathcal{Q} \mathrel{\triangleleft\!\!\rightsquigarrow} \mathcal{Q}$ for a suitable choice of the compact set $\mathcal{K}$ and Theorem 3.9 applies.

Remark 3.11. We just gave a proof of Theorem 3.9 by an argument based on Lemma 2.3, which, in turn, was proved using the Poincaré-Miranda theorem. On the other hand, the Poincaré-Miranda theorem itself (Theorem 2.1) can be proved via Lemma 2.3 as we showed in Remark 2.4 using Poincaré suggestion. So, it is not a surprise if we can give now a proof of Theorem 2.1 via Theorem 3.9. To this end, we consider the continuous $(f, g) : \Xi = [-a_1, a_1] \times [-a_2, a_2] \to \mathbb{R}^2$ such that $f(-a_1, y) \leq 0 \leq f(a_1, y)$, for each $|y| \leq a_2$, and $g(x, -a_2) \leq 0 \leq g(x, a_2)$, for each $|x| \leq a_1$. We take a standard orientation of Ξ, taking by Ξ^- the union of its left and right sides. Without loss of generality we also assume, like in the proof of Theorem 2.1, that $a_1 = a_2 = 1$ and recall the map $\eta : \mathbb{R} \to \mathbb{R}$, $\eta(s) = \min\{1, \max\{-1, s\}\}$, in order to define ψ as

$$\psi : \Xi \ni (x, y) \longmapsto \big(x + f(x, y), \eta(y - g(x, y))\big). \tag{3.8}$$

Now, it can be easily checked that $\psi : \widetilde{\Xi} \mathrel{\triangleleft\!\!\rightsquigarrow} \widetilde{\Xi}$, and therefore, by Theorem 3.9, we have that there is $(x_0, y_0) \in \Xi$ such that $\psi(x_0, y_0) = (x_0, y_0)$. A direct inspection allows now to verify that $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$ and with this our proof ends. Thus we can conclude that, like the Poincaré-Miranda theorem, also Theorem 3.9 is equivalent to the Brouwer fixed point theorem in dimension $N = 2$.

3.3. The role of the compactness in the definition. We consider an oriented cell which is a rectangle of the plane $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$, with

$$\mathcal{R} = [0,1] \times [-1,1], \qquad \mathcal{R}_0^- = \{0\} \times [-1,1], \qquad \mathcal{R}_1^- = \{1\} \times [-1,1]. \tag{3.9}$$

We consider the subset Γ of $\mathcal{R}$ defined by

$$\begin{aligned}\Gamma = (\{0\}\times[-1,1]) &\cup \left(\bigcup_{n=1}^{\infty}\left\{\frac{1}{n}\right\}\times[-1,1]\right)\\ &\cup \left(\bigcup_{k=1}^{\infty}\left[\frac{1}{2k},\frac{1}{2k-1}\right]\times\{1\}\right)\\ &\cup \left(\bigcup_{k=1}^{\infty}\left[\frac{1}{2k+1},\frac{1}{2k}\right]\times\{-1\}\right)\end{aligned}\tag{3.10}$$

and put

$$\begin{aligned}\mathcal{D} &= ([0,1]\times[-1,1])\setminus\Gamma\\ &= \left(\bigcup_{k=1}^{\infty}\left]\frac{1}{2k},\frac{1}{2k-1}\right[\times[-1,1[\right)\cup\left(\bigcup_{k=1}^{\infty}\left]\frac{1}{2k+1},\frac{1}{2k}\right[\times]-1,1]\right).\end{aligned}\tag{3.11}$$

For $(x,y)\in\mathcal{D}\subseteq\mathcal{R}$, we define the map $\psi:\mathcal{D}\to\mathbb{R}^2$,

$$\psi(x,y)=\begin{cases}\left(\operatorname{cotan}\left(\dfrac{\pi}{x}\right),\dfrac{1}{2}(y+1)\right), & \text{for } \left\lfloor\dfrac{1}{x}\right\rfloor \text{ odd},\\[2ex] \left(\operatorname{cotan}\left(\dfrac{\pi}{x}\right),\dfrac{1}{2}(y-1)\right), & \text{for } \left\lfloor\dfrac{1}{x}\right\rfloor \text{ even},\end{cases}\tag{3.12}$$

where $\lfloor r\rfloor$ is the integer part of the real number r, that is, the greatest integer j such that $j\le r<j+1$.

A simple analysis shows that the following stretching property holds with respect to the pair $(\mathcal{D},\psi)$:

(H5) for every path $\sigma\subseteq\mathcal{R}$ with $\sigma\cap\mathcal{R}_0^-\neq\emptyset$ and $\sigma\cap\mathcal{R}_1^-\neq\emptyset$, there is a path $\gamma\subseteq\sigma\cap\mathcal{D}$ with $\psi(\gamma)\subseteq\mathcal{R}$ and $\psi(\gamma)\cap\mathcal{R}_0^-\neq\emptyset$, $\psi(\gamma)\cap\mathcal{R}_1^-\neq\emptyset$.

It is evident that (H5) is exactly the same like (H4) of Corollary 3.5 with $\mathcal{A}=\mathcal{B}=\mathcal{R}$, however, $(\mathcal{D},\psi):\mathcal{R}\not\Leftarrow\!\!\!\!\rightsquigarrow\mathcal{R}$ according to Definition 3.1 or its equivalent versions (e.g., Lemma 3.4). The failure of the compactness condition in Definition 3.1 has as a consequence the fact that Theorem 3.9 cannot be applied and, in fact, the map $\psi(x,y)$ defined above has no fixed points. Moreover, also the property of Remark 3.3 cannot be invoked here since, in our example, $\mathcal{D}$ does not contain any connected subset $\mathcal{C}$ with $\mathcal{C}\cap\mathcal{R}_0^+\neq\emptyset$ and $\mathcal{C}\cap\mathcal{R}_1^+\neq\emptyset$, in spite of the fact that $\mathcal{D}\cap\mathcal{R}_0^+\neq\emptyset$ and $\mathcal{D}\cap\mathcal{R}_1^+\neq\emptyset$.

3.4. Intersection of cells and the fixed point property. Let $\widetilde{\mathcal{A}}=(\mathcal{A},\mathcal{A}^-)$, $\widetilde{\mathcal{B}}=(\mathcal{B},\mathcal{B}^-)$, and $\widetilde{\mathcal{M}}=(\mathcal{M},\mathcal{M}^-)$ be oriented cells in X.

Definition 3.12. (see Figure 3.1) $\widetilde{\mathcal{B}}$ is said to *cross* $\widetilde{\mathcal{A}}$ *in* $\widetilde{\mathcal{M}}$, in symbols:

$$\widetilde{\mathcal{M}}\in\{\widetilde{\mathcal{A}}\pitchfork\widetilde{\mathcal{B}}\},\tag{3.13}$$

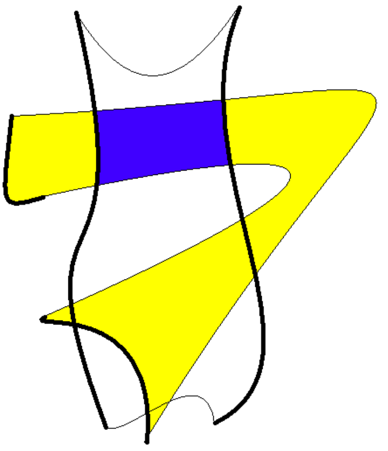

Figure 3.1. Example of oriented cells $\widetilde{\mathcal{R}}$ (white) and $\psi(\widetilde{\mathcal{R}})$ (light color) crossing into a slice $\widetilde{\mathcal{M}}$ (darker color) and thus giving a fixed point in $\mathcal{M}$ for a homeomorphism $\psi : \mathcal{R} \to \psi(\mathcal{R})$. The $[\cdot]^-$-sets are indicated with a bold line. Among the two cells which are the connected components of the intersection $\psi(\mathcal{R}) \cap \mathcal{R}$, only one is suitable to play the role of the $\mathcal{M}$ for the application of Corollary 3.16.

if

$$\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{A}}, \qquad \widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{B}}. \tag{3.14}$$

Remark 3.13. We borrowed the symbol $\pitchfork$ from the case of transversal intersections. However, we point out that even if $\widetilde{\mathcal{M}} \in \{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\}$ holds when $\mathcal{A}$ and $\mathcal{B}$ are manifolds with (piecewise) smooth boundary and $\mathcal{B}$ intersects transversally $\mathcal{A}$ in $\mathcal{M}$ (with the boundary of $\mathcal{M}$ made of the $[\cdot]^-$-sets of $\mathcal{A}$ and the $[\cdot]^+$-sets of $\mathcal{B}$), nevertheless, $\widetilde{\mathcal{B}}$ may cross $\widetilde{\mathcal{A}}$ according to our definition also in cases in which a transversal intersection of the two cells does not occur.

Theorem 3.14. *Let $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ be oriented cells in X. If $(\mathcal{D}, \mathcal{K}, \psi) : \widetilde{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{B}}$ and there is an oriented cell $\widetilde{\mathcal{M}}$ such that $\widetilde{\mathcal{M}} \in \{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\}$, then there exists $w \in \mathcal{K} \cap \mathcal{M}$ such that $\psi(w) = w$.*

Proof. By definition, from $\widetilde{\mathcal{M}} \in \{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\}$, we know that $\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{B}}$. Then, by Lemma 3.7 and the assumption that $(\mathcal{D}, \mathcal{K}, \psi) : \widetilde{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{B}}$, we have that

$$(\mathcal{D} \cap \mathcal{M}, \mathcal{K} \cap \mathcal{M}, \psi) : \widetilde{\mathcal{M}} \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{M}}. \tag{3.15}$$

Hence, we can apply Theorem 3.9 which guarantees the existence of a fixed point w for ψ with $w \in \mathcal{K} \cap \mathcal{M}$. This concludes the proof. □

Remark 3.15. A possible geometric interpretation of Theorem 3.14 may lead to the following description. Consider two cells and a continuous map ψ which deforms the first cell into the second one, by expanding the first cell along the *west-east* direction and contracting it *north-south*, then ψ has at least a fixed point in the intersection of the two cells (provided that such an intersection has a suitable "good" embedding into the two cells). This makes some connection between Theorem 3.14 and various related results of fixed points for maps satisfying an expansion-contraction property, either from the area of ordinary differential equations (e.g., [2, 24]) or from the realm of the applications of the fixed point index, or degree theory, or Lefschetz-type fixed point theorems (see, e.g., [53, 61, 63, 64] for very general results).

A direct application of Theorem 3.14 and of Remark 3.8 gives the following corollary.

Corollary 3.16. *Let* $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *be an oriented cell in* X *and let* $\psi : D_\psi \supseteq \mathcal{R} \to \psi(\mathcal{R}) \subseteq X$ *be a homeomorphism of* $\mathcal{R}$ *onto its image. If* $\psi(\widetilde{\mathcal{R}})$ *crosses* $\widetilde{\mathcal{R}}$ *at some oriented cell* $\widetilde{\mathcal{M}}$, *then* ψ *has at least one fixed point in* $\mathcal{M} \subseteq \mathcal{R} \cap \psi(\mathcal{R})$.

A further development of the above-considered definition of crossing of two cells combined with the stretching property comes now with Section 4.

4. Topological horseshoes and coin-tossing dynamics

In this section, we show a natural application of our approach to *topological horseshoes* (see, e.g., [21]), that is, we prove some features which are common to the classical Smale horseshoe (cf. [34, 50, 51, 58]) under a general topological setting.

Let $\psi : X \supseteq D_\psi \to X$ be a continuous map. Following [22] and modifying a little a corresponding definition considered therein, we say that ψ has a *chaotic dynamics of coin-tossing type on k symbols* if $k \geq 2$ and there is a metrizable space $Z \subseteq X$ and k pairwise disjoint compact sets $W_1, \ldots, W_k \subseteq Z \cap D_\psi$ such that, for each two-sided sequence $(s_n)_{n\in\mathbb{Z}}$ with

$$s_n \in \{1, \ldots, k\}, \quad \forall n \in \mathbb{Z}, \tag{4.1}$$

there is a sequence of points $(z_n)_{n\in\mathbb{Z}}$ with

$$z_n \in W_{s_n}, \quad z_{n+1} = \psi(z_n), \quad \forall n \in \mathbb{Z}. \tag{4.2}$$

Theorem 4.1. *Suppose that* $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *and* $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *are oriented cells in* X. *If* $(\mathcal{D}, \mathcal{K}, \psi) : \widetilde{\mathcal{A}} \multimap\!\!\!\to \widetilde{\mathcal{B}}$ *and there are* $k \geq 2$ *oriented cells* $\widetilde{\mathcal{M}}_1, \ldots, \widetilde{\mathcal{M}}_k$ *such that*

$$\widetilde{\mathcal{M}}_i \in \{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\}, \quad \text{for } i = 1, \ldots, k, \tag{4.3}$$

with

$$\mathcal{M}_i \cap \mathcal{M}_j \cap \mathcal{K} = \emptyset, \quad \forall i \neq j, \text{ with } i, j \in \{1, \ldots, k\}, \tag{4.4}$$

then the following conclusions hold:

(a_1) ψ *has a chaotic dynamics of coin-tossing type on* k *symbols (with respect to the sets* $W_i = \mathcal{K}_i = \mathcal{K} \cap \mathcal{M}_i$*),*

(a_2) *for each one-sided infinite sequence* $\mathbf{s} = (s_0, s_1, \dots, s_n, \dots) \in \{1, \dots, k\}^{\mathbb{N}}$ *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{K}_{s_0}$ *with*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{M}_{s_0})_0^+ \neq \emptyset, \qquad \mathcal{C}^{\mathbf{s}} \cap (\mathcal{M}_{s_0})_1^+ \neq \emptyset, \tag{4.5}$$

such that for each point $w \in \mathcal{K}_{s_0}$, *the sequence*

$$z_{j+1} = \psi(z_j), \quad z_0 = w, \quad \text{for } j = 0, 1, \dots, n, \dots \tag{4.6}$$

satisfies

$$z_j \in \mathcal{K}_{s_j}, \quad \forall j = 0, 1, \dots, n, \dots, \tag{4.7}$$

(a_3) ψ *has a fixed point in each set* $\mathcal{K}_i := \mathcal{M}_i \cap \mathcal{K}$ *and, for each finite sequence* $(s_0, s_1, \dots, s_m) \in \{1, \dots, k\}^{m+1}$, *with* $m \geq 1$, *there is at least one point* $z^* \in \mathcal{K}_{s_0}$ *such that*

$$z_{j+1} = \psi(z_j), \quad z_0 = z^*, \quad \text{for } j = 0, 1, \dots, m \tag{4.8}$$

defines a sequence of points with

$$z_j \in \mathcal{K}_{s_j}, \quad \forall j = 0, 1, \dots, m, \ z_{m+1} = z^*. \tag{4.9}$$

Proof. We begin by claiming that

$$(\mathcal{K}_i, \psi) : \widetilde{\mathcal{B}} \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{B}}, \quad \forall i = 1, \dots, k. \tag{4.10}$$

Indeed, it is immediate to check that $\psi(\mathcal{K}_i) \subseteq \mathcal{B}$ if $\mathcal{K}_i = \mathcal{K} \cap \mathcal{M}_i$. Moreover, let $\sigma \subseteq \mathcal{B}$ be a path with $\sigma \cap \mathcal{B}_0^- \neq \emptyset \neq \sigma \cap \mathcal{B}_1^-$; since $\widetilde{\mathcal{M}}_i \subseteq_v \widetilde{\mathcal{B}}$, it is possible to find a path $\sigma' \subseteq \mathcal{M}_i$ such that $\sigma' \cap (\mathcal{M}_i)_0^- \neq \emptyset \neq \sigma' \cap (\mathcal{M}_i)_1^-$; we have in particular that $\sigma' \cap \mathcal{A}_0^- \neq \emptyset \neq \sigma' \cap \mathcal{A}_1^-$, since $\widetilde{\mathcal{M}}_i \subseteq_h \widetilde{\mathcal{A}}$. By condition (H2) in Definition 3.1 there is a path $\gamma \subseteq \sigma' \cap \mathcal{K} = \sigma' \cap \mathcal{K}_i$ such that $\psi(\gamma) \cap \mathcal{B}_0^- \neq \emptyset \neq \psi(\gamma) \cap \mathcal{B}_1^-$ and our claim is proved.

We are now in position to apply a result already obtained in [43] and statements (a_1), (a_2), and (a_3), respectively, follow from statements (e_1), (e_2), and (e_3) of [43, Theorem 2.4]. The corresponding proof in [43] is performed for cells embedded in $\mathbb{R}^2$, but it is straightforward to check that the same argument works here without any modification. □

Using Remark 3.8 and Theorem 4.1, we obtain the following corollary.

Corollary 4.2. *Let* $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *be an oriented cell in* X *and let* $\psi : D_\psi \supseteq \mathcal{R} \to \psi(\mathcal{R}) \subseteq X$ *be a homeomorphism of* $\mathcal{R}$ *onto its image. If there are* $k \geq 2$ *pairwise disjoint oriented cells* $\widetilde{\mathcal{M}}_1, \dots, \widetilde{\mathcal{M}}_k$ *such that* $\widetilde{\mathcal{M}}_i \in \{\widetilde{\mathcal{R}} \pitchfork \psi(\widetilde{\mathcal{R}})\}$, *for* $i = 1, \dots, k$, *then the same conclusions of Theorem 4.1 hold with respect to* $\mathcal{K} = \mathcal{R}$ *(see Figure 4.1).*

We conclude this paper with the following theorem which is a restatement of [43, Theorem 2.2] for the general setting of topological spaces. The proof is omitted as it does not require any substantial change with respect to the argument presented in [43, Theorem 2.2]. We also refer to [43] for other related results and applications.

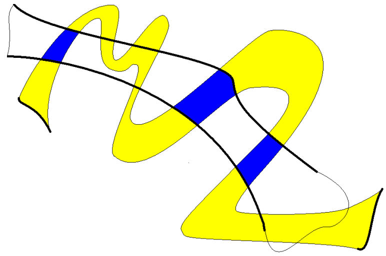

Figure 4.1. Example of oriented cells $\widetilde{\mathcal{R}}$ (white) and $\psi(\widetilde{\mathcal{R}})$ (light color) with crossings into three slices (darker color) and thus giving a coin-tossing dynamics on three symbols for a homeomorphism $\psi : \mathcal{R} \to \psi(\mathcal{R})$. The $[\cdot]^-$-sets are indicated with a bold line. Among the five cells which are the connected components of the intersection $\psi(\mathcal{R}) \cap \mathcal{R}$, only the three painted with darker color are suitable to play the role of the $\mathcal{M}_i$'s for the application of Corollary 4.2.

Theorem 4.3. *Assume that in X there is a (double) sequence of oriented cells $(\widetilde{\mathcal{A}}_i)_{i\in\mathbb{Z}}$ and maps $((\mathcal{D}_i,\psi_i))_{i\in\mathbb{Z}}$, with $\mathcal{D}_i \subseteq \mathcal{A}_i$, such that $(\mathcal{D}_i,\mathcal{K}_i,\psi_i) : \widetilde{\mathcal{A}}_i \Leftarrow\!\!\rightsquigarrow \widetilde{\mathcal{A}}_{i+1}$ for each $i \in \mathbb{Z}$. Then the following conclusions hold:*

(b$_1$) *there is a sequence $(w_i)_{i\in\mathbb{Z}}$ with $w_i \in \mathcal{K}_i \subseteq \mathcal{D}_i$ and $\psi_i(w_i) = w_{i+1}$ for all $i \in \mathbb{Z}$;*

(b_2) *for each $j \in \mathbb{Z}$ there is a compact and connected set $\mathcal{C}_j \subseteq \mathcal{K}_j$ satisfying*

$$\mathcal{C}_j \cap (\mathcal{A}_j)_b^+ \neq \emptyset, \qquad \mathcal{C}_j \cap (\mathcal{A}_j)_t^+ \neq \emptyset \tag{4.11}$$

and such that for each $w \in \mathcal{C}_j$ there is a sequence $(y_\ell)_{\ell\geq j}$, with $y_\ell \in \mathcal{D}_\ell$ and $y_j = w$, $y_{\ell+1} = \psi_\ell(y_\ell)$ for each $\ell \geq j$;

(b$_3$) *if there are integers h, k with $h < k$ such that $\widetilde{\mathcal{A}}_h = \widetilde{\mathcal{A}}_k$, then there is a finite sequence $(z_i)_{h\leq i\leq k}$, with $z_i \in \mathcal{D}_i$ and $\psi_i(z_i) = z_{i+1}$ for each $i = h,\dots,k-1$, such that $z_h = z_k$, that is, z_h is a fixed point of $\psi_{k-1} \circ \cdots \circ \psi_h$.*

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