HIGHER-ORDER NIELSEN NUMBERS

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Suppose X, Y are manifolds, $f,g : X \to Y$ are maps. The well-known coincidence problem studies the coincidence set $C = \{x : f(x) = g(x)\}$. The number $m = \dim X - \dim Y$ is called the codimension of the problem. More general is the preimage problem. For a map $f : X \to Z$ and a submanifold Y of Z, it studies the preimage set $C = \{x : f(x) \in Y\}$, and the codimension is $m = \dim X + \dim Y - \dim Z$. In case of codimension 0, the classical Nielsen number $N(f,Y)$ is a lower estimate of the number of points in C changing under homotopies of f, and for an arbitrary codimension, of the number of components of C. We extend this theory to take into account other topological characteristics of C. *The goal is to find a "lower estimate" of the bordism group $\Omega_p(C)$ of C.* The answer is the Nielsen group $S_p(f,Y)$ defined as follows. In the classical definition, the Nielsen equivalence of points of C based on paths is replaced with an equivalence of singular submanifolds of C based on bordisms. We let $S'_p(f,Y) = \Omega_p(C)/\sim_N$, then the Nielsen group of order p is the part of $S'_p(f,Y)$ preserved under homotopies of f. The Nielsen number $N_p(F,Y)$ of order p is the rank of this group (then $N(f,Y) = N_0(f,Y)$). These numbers are new obstructions to removability of coincidences and preimages. Some examples and computations are provided.

1. Introduction

Suppose X, Y are smooth orientable compact manifolds, $\dim X = n+m$, $\dim Y = n$, $m \geq 0$ the *codimension*, $f,g : X \to Y$ are maps, the *coincidence set*

$$C = \mathrm{Coin}(f,g) = \{x \in X : f(x) = g(x)\} \tag{1.1}$$

is a compact subset of $X \setminus \partial X$.

Consider the coincidence problem: "what can be said about the coincidence set C of (f,g)?" One of the main tools is the Lefschetz number $L(f,g)$ defined as the alternating sum of traces of a certain endomorphism on the homology group of Y. The famous Lefschetz coincidence theorem provides a sufficient condition for the existence of

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coincidences (codimension $m = 0$): $L(f,g) \neq 0 \Rightarrow C = \mathrm{Coin}(f,g) \neq \emptyset$, see [1, Section VI.14], and [31, Chapter 7].

Now, what else can be said about the coincidence set? As C changes under homotopies of f and g, a reasonable approach is to try to minimize the "size" of C. In case of zero codimension, C is discrete and we simply minimize the number of points in C. The result is the Nielsen number. It is defined as follows. Two points $p,q \in C$ belong to the same Nielsen class if (1) there is a path s in X between p and q; (2) fs and gs are homotopic relative to the endpoints. A Nielsen class is called essential if it cannot be removed by a homotopy of f, g (alternatively, a Nielsen class is algebraically essential if its coincidence index is nonzero [2]). Then the Nielsen number $N(f,g)$ is the number of essential Nielsen classes. It is a lower estimate of the number of points in C. In case of positive codimension $N(f,g)$ still makes sense as a lower estimate of the number of components of C [32]. However, only for $m = 0$, the Nielsen number is known to be a *sharp* estimate, that is, there are maps f', g' compactly homotopic to f, g such that $C' = \mathrm{Coin}(f',g')$ consists of exactly $N(f,g)$ path components (Wecken property). This minimization is achieved by removing inessential classes through homotopies of f, g.

The Nielsen theory for codimension $m = 0$ is well developed, for the fixed point and the root problems [3, 21, 22], and for the coincidence problem [4]. However, for $m > 0$, the vanishing of the coincidence index does not guarantee that the Nielsen class can be removed. Some progress has been made for codimension $m = 1$. In this case, the secondary obstruction to the removability of a coincidence set was considered by Fuller [13] for Y simply connected. Hatcher and Quinn [18] showed that the obstruction to a higher-dimensional Whitney lemma lies in a certain framed bordism group. Based on this result, necessary and sufficient conditions of the removability of a Nielsen class were studied by Dimovski and Geoghegan [9] and Dimovski [8] for parametrized fixed point theory, that is, when $f : Y \times I \to Y$ is the projection. The results of [9] were generalized by Jezierski [20] for the coincidence problem $f,g : X \to Y$, where X, Y are open subsets of Euclidean spaces or Y is parallelizable. Geoghegan and Nicas [14] developed a parametrized Nielsen theory based on Hochschild homology. For some $m > 1$, sufficient conditions of the local removability are provided in [28]. Necessary conditions of the global removability for arbitrary codimension are considered by Gonçalves, Jezierski, and Wong [33, Section 5] with N a torus and M a nilmanifold.

In these papers, higher-order Nielsen numbers are not explicitly defined (except for [8], see the comment in the end of the paper). However, they all contribute to the problem of finding the lower estimate of the number of components of C. We extend these results to take into account other topological characteristics of C. In the spirit of the classical Nielsen theory, *our goal is to find "lower estimates" of the bordism groups* $\Omega_*(C)$.

The crucial motivation for our approach is the removability results for codimension 1 due to Dimovski and Geoghegan [9] and Jezierski [20]. Consider [20, Theorem 5.3]. Assume that codimension $m = 1$, $n \geq 4$, X, Y are open subsets of Euclidean spaces. Suppose A is a Nielsen class. Then if f, g are transversal, A is the union of disjoint circles. Define the Pontriagin-Thom (PT) map as the composition

$$\mathbf{S}^{n+1} \simeq \mathbf{R}^{n+1} \cup \{*\} \longrightarrow \mathbf{R}^{n+1}/\left(\mathbf{R}^{n+1}\backslash\nu\right) \simeq \overline{\nu}/\partial\nu \xrightarrow{f-g} \mathbf{R}^{n}/\left(\mathbf{R}^{n}\backslash D\right) \simeq \mathbf{S}^{n}, \tag{1.2}$$

where ν is a normal bundle of A, $D \subset \mathbf{R}^n$ is a ball centered at 0 satisfying $(f-g)(\partial\nu) \subset \mathbf{R}^n \backslash D$. It is an element of $\pi_n(\mathbf{S}^{n-1}) = \mathbf{Z}_2$. Then A can be removed if and only if the following conditions are satisfied:

(W1) $A = \partial S$, *where* S *is an orientable connected surface*, $f|_S \sim g|_S \operatorname{rel} A$ (*the surface condition*);

(W2) *the PT map is trivial* (*the* $\mathbf{Z}_2$*-condition*).

Earlier, Dimovski and Geoghegan [9] considered a similar pair of conditions (not independent though) in their Theorem 1.1 and compared them to the codimension 0 case. They write: "… the role of 'being in the same fixed point class' is played by the surface condition (i), while that of the fixed point index is played by the natural orientation. The $\mathbf{Z}_2$-obstruction is a new feature …." One can use the first observation to define the Nielsen equivalence on the set of 1-submanifolds of C (here A is Nielsen equivalent to the empty set). However, we will see that the PT map has to serve as the index of the Nielsen class. The index will be defined in the traditional way but with respect to an arbitrary homology theory h_*. Indeed, in the above situation, it is an element of the stable homotopy group $\pi^S_{n+1}(\mathbf{S}^n) = \mathbf{Z}_2$.

More generally, we define the Nielsen equivalence on the set $M_m(C)$ of all closed singular m-manifolds in $C = \operatorname{Coin}(f,g)$. Two singular m-manifolds $p : P \to C$ and $q : Q \to C$ belong to the same *Nielsen class*, $p \sim_N q$, if

(1) ip and iq are bordant, where $i : C \to N$ is the inclusion, that is, there is a map $F : W \to N$ extending $ip \sqcup iq$ such that W is a bordism between P and Q;

(2) fF and gF are homotopic relative to fp, fq.

Then $S'_m(f,g) = M_m(C)/\sim_N$ is the group of Nielsen classes. Let $S^a_m(f,g)$ be the group of algebraically essential Nielsen classes, that is, the ones with nontrivial index. Then the (algebraic) Nielsen number of order m is the rank of $S^a_m(f,g)$ (these numbers are new obstructions to removability of coincidences). In light of this definition, Jezierski's theorem can be thought of as a Wecken type theorem for $m = 1$.

The most immediate applications of the coincidence theory for positive codimension lie in control theory. A *dynamical system* on a manifold M is determined by a map $f : M \to M$. Then the next state $f(x)$ depends only on the current one, $x \in M$. In case of a *control system*, the next state $f(x,u)$ depends not only on the current one, $x \in M$, but also on the input, $u \in U$. Suppose we have a fiber bundle given by the bundle projection $U \to N \xrightarrow{g} M$ and a map $f : N \to M$. Here N is the state-input space, U is the space of inputs, and M is the space of states of the system. Then the equilibrium set of the system $C = \{x \in M : f(x,u) = x\}$ is the coincidence set of the pair (f,g). A *continuous control system* [25, page 16] is defined as a commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{h} & TM \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_M} & \\ M & & \end{array} \tag{1.3}$$

where N is a fiber bundle over M. Then the equilibrium set $C = \{(x,u) \in N : h(x,u) = x\}$ of the system is the preimage of M under h.

Instead of the coincidence problem, throughout the rest of the paper we apply the approach outlined above to the Nielsen theory for the so-called *preimage problem* considered by Dobreńko and Kucharski [10]. Suppose X, Y, Z are connected CW-complexes, $Y \subset Z$, $f : X \to Z$ is a map. The problem studies the set $C = f^{-1}(Y)$ and can be easily specialized to the fixed point problem if we put $Z = X \times X$, $Y = d(X)$, $f = (\mathrm{Id}, g)$, to the root problem if Y is a point, and to the coincidence problem if $Z = Y \times Y$, Y is the diagonal of Z, $f = (F, G)$ (see [23]).

Suppose X, Y, Z are smooth manifolds and f is transversal to Y. Then under the restriction $\dim X + \dim Y = \dim Z$, the preimage $C = f^{-1}(Y)$ of Y under f is discrete. The Nielsen number $N(f, Y)$ is the sharp lower estimate of the least number of points in $g^{-1}(Y)$ for all maps g homotopic to f [10, Theorem 3.4], that is, $N(f, Y) \le \#g^{-1}(Y)$ for all $g \sim f$. If we omit the above restriction, C is an r-manifold [1, Theorem II.15.2, page 114], where

$$r = \dim X + \dim Y - \dim Z. \tag{1.4}$$

The setup. X, Y, Z are connected CW-complexes, $Y \subset Z$,

$$\dim X = n + m, \qquad \dim Y = n, \qquad \dim Z = n + k, \tag{1.5}$$

$f : X \to Z$ is a map, the *preimage set* $C = f^{-1}(Y)$, the *codimension* of the problem is

$$r = n + m - k, \tag{1.6}$$

and $j : C \to X$ is the inclusion.

The paper is organized as follows. Just as for the coincidence problem, we define the Nielsen equivalence of singular q-manifolds in C and the group of Nielsen classes $S'_q(f) = M_q(C)/\sim_N = \Omega_q(C)/\sim_N$, where Ω_* is the orientable bordism group (Section 2). Next, we identify the part of $S'_q(f)$ preserved under homotopies of f. The result is the Nielsen group $S_q(f)$, the group of topologically essential classes (Section 3). As we have described above, *the Nielsen group is a subgroup of a quotient group of* $\Omega_q(C)$ *and, in this sense, its "lower estimate."*

Proposition 1.1. *$S_*(f)$ is homotopy invariant.*

The Nielsen number of order p, $p = 0, 1, 2, \ldots$, is defined as $N_p(f, Y) = \operatorname{rank} S_p(f, Y)$. Clearly, the classical Nielsen number is equal to $N_0(f)$.

Proposition 1.2. *$N_p(f) \le \operatorname{rank} \Omega_p(g^{-1}(Y))$ if $f \sim g$.*

In Section 4, we discuss the naturality of the Nielsen group. In particular, we obtain the following.

Proposition 1.3. *Given $Z, Y \subset Z$. Then S_* is a functor from the category of preimage problems as pairs (X, f), $f : X \to Z$, with morphisms as maps $k : X \to U$ satisfying $gk = f$, to the category of graded abelian groups.*

For the manifold case, there is an alternative approach to essentiality. In Section 5, the "preimage index" is defined simply as $I_f = f_* : \Omega_*(C) \to \Omega_*(Y)$. It is a homomorphism

on $S'_*(f)$ and the group of algebraically essential Nielsen classes is defined as $S^a_*(f,Y) = S'_*(f,Y)/\ker I_f$. We show that every algebraically essential class is topologically essential. In Section 6, we consider the traditional index $\mathrm{Ind}_f(P)$ of an isolated subset P of C in terms of a generalized homology h_*. It is defined in the usual way as the composition

$$h_*(X,X\setminus U) \xleftarrow{\simeq} h_*(V,V\setminus U) \xrightarrow{f_*} h_*(Z,Z\setminus Y), \tag{1.7}$$

where $V \subset \overline{V} \subset U$ are neighborhoods of P. Then we show how it is related to I_f. In Section 7, we consider some examples of computations of these groups, especially in the setting of the PT construction.

In Sections 8 and 9, based on Jezierski's theorem, we prove the following Wecken type theorem for codimension 1.

Proposition 1.4. *Under conditions of Jezierski's theorem, f,g are homotopic to f',g' such that*

$$S_p(f,g) \simeq \Omega_p(\mathrm{Coin}(f',g')), \quad p = 0,1. \tag{1.8}$$

To motivate our definitions, in the beginning of each section, we will review the relevant part of Nielsen theory for the preimage problem following Dobreńko and Kucharski [10] and McCord [23].

All manifolds are assumed to be orientable and compact.

2. Nielsen classes

In Nielsen theory, two points $x_0,x_1 \in C = f^{-1}(Y)$ belong to the same Nielsen class, $x_0 \sim x_1$, if

(1) there is a path $\alpha : I \to X$ such that $\alpha(i) = x_i$;
(2) there is a path $\beta : I \to Y$ such that $\beta(i) = f(x_i)$;
(3) $f\alpha$ and β are homotopic relative to $\{0,1\}$.

This is an equivalence relation partitioning C into a finite number of Nielsen classes. However, since we want Nielsen classes to form a group, we should think of x_0, x_1 as singular 0-manifolds in C (a singular p-manifold in M is a map $s : N \to M$, where N is a p-manifold). Then conditions (1) and (2) express the fact that x_0, x_1 are bordant in X, and $f(x_0)$, $f(x_1)$ are bordant in Y.

Recall [6, 29] that two orientable compact closed p-manifolds N_0, N_1 are called *bordant* if there is a *bordism* between them, that is, an orientable compact $(p+1)$-manifold W such that $\partial W = N_0 \sqcup -N_1$. Two singular orientable compact closed manifolds $s_i : N_i \to M$, $i = 0,1$, are *bordant*, $s_0 \sim_b s_1$, if there is a map $h : W \to M$ extending $s_0 \sqcup s_1$, where W is a bordism between N_0 and N_1.

Let $M_p(A,B)$ denote the set of all singular orientable compact closed p-manifolds $s : (N,\partial N) \to (A,B)$.

Definition 2.1. Two singular p-manifolds $s_0, s_1 \in M_p(C)$ in C, that is, maps $s_i : S_i \to C$, $i = 0, 1$, are *Nielsen equivalent*, $s_0 \sim_N s_1$, if

(1) js_0, js_1 are bordant in X via a map $H : W \to X$ extending $s_0 \sqcup s_1$ such that W is a bordism between S_0 and S_1;
(2) fs_0, fs_1 are bordant in Y via a map $G : W \to Y$ extending $fs_0 \sqcup fs_1$;
(3) fH and G are homotopic relative to $S_0 \sqcup S_1$.

We denote the Nielsen class of $s \in M_p(C)$ by $[s]_N$ or simply $[s]$.

Proposition 2.2. *$\sim_N$ is an equivalence relation on $M_p(C)$.*

Definition 2.3. The group of Nielsen classes of order p, $S'_p(f, Y)$, or simply $S'_p(f)$, is defined as

$$S'_p(f, Y) = M_p(C)/\sim_N . \tag{2.1}$$

The group of Nielsen classes for the coincidence problem will be denoted by $S'_p(f, g)$.

In contrast to the classical Nielsen theory, the elements of Nielsen classes are not points but sets of points. Even in the case of $p = 0$, one has more to deal with. For example, suppose $C = \{x, y\}$ and $x \sim_N y$. The elements of $S'_0(f)$ are $[\{x, y\}] = [\{x\}]$, $[\{-x, -y\}] = [\{-x\}] = -[\{x\}]$, $[\{x\} \cup \{-y\}] = [\varnothing]$, $[\{x\} \cup \{y\}] = [\{x\} \cup \{x\}] = [2\{x\}] = 2[\{x\}]$, and so forth.

Another example. Suppose $X = Z = \mathbf{S}^2$, Y is the equator of Z, f a map of degree 2 such that $C = f^{-1}(Y)$ is the union of two circles C_1 and C_2 around the poles. Then $S'_1(f) = \mathbf{Z}$ generated by $C_1 \sqcup C_2$. A similar construction applies to $X = Z = \mathbf{S}^n$, $Y = \mathbf{S}^{n-1}$, $n \geq 2$, then $S'_{n-1}(f) = \mathbf{Z}$ is generated by the union of two copies of $\mathbf{S}^{n-1}$.

Let $M^h_p(A, B)$ denote the semigroup of all homotopy classes, relative to boundary, of maps $s \in M_p(A, B)$. Consider the commutative diagram

$$\begin{array}{ccccc}
M^h_{p+1}(X, C) & \xrightarrow{\delta} & M^h_p(C) & \xrightarrow{j_*} & M^h_p(X) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
M^h_{p+1}(Z, Y) & \xrightarrow{\delta} & M^h_p(Y) & \xrightarrow{k_*} & M^h_p(Z) \\
\uparrow{\scriptstyle I_*} & \nearrow{\scriptstyle \delta} & & & \\
M^h_{p+1}(Y, Y) & & & &
\end{array} \tag{2.2}$$

where δ is the boundary map, I is the inclusion. Then we have an alternative way to define the group of Nielsen classes:

$$S'_p(f, Y) = M^h_p(C)/\delta\left(f_*^{-1}(\operatorname{Im} I_*)\right). \tag{2.3}$$

Let $\Omega_p(A, B)$ denote the group of bordism classes in $M_p(A, B)$ with $\sqcup$ as addition. Then Ω_* is a generalized homology [6, 29].

Proposition 2.4. *If $s_0 \sim_N s_1 \sim_b s_2$, then $s_0 \sim_N s_2$. Therefore, $\sim_N$ is an equivalence relation on $\Omega_*(C)$.*

Proposition 2.5. *If $s_0 \sim_N s_1$, $t_0 \sim_N t_1$, then $s_0 \sqcup t_0 \sim_N s_1 \sqcup t_1$. Therefore, $\sim_N$ is preserved under the operation of $\Omega_*(C)$. Thus $S'_*(f,Y) = \Omega_*(C)/\sim_N$ is a group.*

Next we discuss the naturality of this group.

Definition 2.6. Suppose another preimage problem $f' : X' \to Z' \supset Y'$ is connected to the first by maps $k : X \to X'$ and $h : Z \to Z'$ such that $f'k = hf$ and $h(Y) \subset Y'$ (see the diagram in Proposition 2.8). Then we define the *map induced by k and h*,

$$k'_* : S'_*(f,Y) \longrightarrow S'_*(f',Y'), \tag{2.4}$$

by $k'_*([s]_N) = [ks]_N$.

Proposition 2.7. *k'_* is well defined.*

Proof. Let $C' = f'^{-1}(Y')$. If $x \in C$, then $f(x) = y \in Y$. Let $x' = k(x)$ and $y' = h(y) \in h(Y) \subset Y'$. Then by assumption $g(x') = y'$, so $x' \in C'$. Therefore, the following diagram commutes:

$$\begin{array}{ccc} (X,C) & \xrightarrow{k} & (X',C') \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ (Z,Y) & \xrightarrow{h} & (Z',Y') \end{array} \tag{2.5}$$

The second preimage problem has a diagram analogous to (2.2). Together they provide two opposite faces of a 3-dimensional diagram with other faces supplied by the diagram above. The diagram commutes. Therefore, for each $s \in M_p(C)$, $s \sim_N \varnothing \Rightarrow ks \sim_N \varnothing$. □

Proposition 2.8. *Suppose the following diagram for three preimage problems commutes:*

$$\begin{array}{ccccc} Y & \xrightarrow{h} & Y' & \xrightarrow{l} & Y'' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{h} & Z' & \xrightarrow{l} & Z'' \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f''} \\ X & \xrightarrow{k} & X' & \xrightarrow{j} & X'' \end{array} \tag{2.6}$$

Then $j'_*k'_* = (jk)'_* : S'_*(f,Y) \to S'_*(f'',Y'')$.

Proof. From the definition, $(jk)'_*([s]_N) = [jks]_N$ and $j'_*k'_*([s]_N) = j'_*([ks]_N) = [jks]_N$. □

Proposition 2.9. $(\mathrm{Id}_X)'_* = \mathrm{Id}_{S'_*(f,Y)}$.

Corollary 2.10. *If $\mathcal{P}$ is the category of preimage problems as quadruples (X,Z,Y,f), $Y \subset Z$, $f : X \to Z$, with morphisms as pairs of maps (k,h) satisfying Definition 2.6, then S'_* is a functor from $\mathcal{P}$ to $\mathbf{Ab}_*$, the graded abelian groups.*

3. Topologically essential Nielsen classes

In the classical theory, a Nielsen class is called essential if it cannot be removed by a homotopy. More precisely, suppose $F : I \times X \to Z$ is a homotopy of f, then the t-section $N_t = \{x \in X : (t,x) \in N\}$, $0 \le t \le 1$, of the Nielsen class N of F is a Nielsen class of $f_t = F(t,\cdot)$ or is empty [10, Corollary 1.5]. Next, we say that the Nielsen classes N_0, N_1 of f_0, f_1, respectively, are in the F-Nielsen relation if there is a Nielsen class N of F such that N_0, N_1 are the 0- and 1-sections of N. This establishes an "equivalence" relation between some Nielsen classes of f_0 and some Nielsen classes of f_1. Given a Nielsen class N_0 of f_0, if for any homotopy there is a Nielsen class of f_1 corresponding to N_0, then N_0 is called essential. In our theory, the F-Nielsen relation takes a simple form of two homomorphisms from $S'_*(f_0)$, $S'_*(f_1)$ to $S'_*(F)$.

Suppose $F : I \times X \to Z$ is a homotopy, $f_t(\cdot) = F(t,\cdot) : X \to Z$, and let $i_t : X \to \{t\} \times X \to I \times X$ be the inclusions. Since $f_t = Fi_t$, the homomorphism $i'_{t*} : S'_*(f_t) \to S'_*(F)$ is well defined for each $t \in [0,1]$ (Proposition 2.7). The following result is crucial.

Theorem 3.1. *Suppose $F : I \times X \to Z$ is a homotopy of f, $F|_{\{0\}\times X} = f$. Suppose $i : X \to \{0\} \times X \to I \times X$ is the inclusion. Then $i'_* : S'_*(f) \to S'_*(F)$ is injective.*

Proof (cf. [10, Lemma 1.4]*).* Suppose $v \in M_p(f^{-1}(Y))$, $v : M \to f^{-1}(Y)$, where M is a p-manifold. Then $u = iv = \{0\} \times v \in M_p(F^{-1}(Y))$, so that $u : M \to F^{-1}(Y) \supset \{0\} \times f^{-1}(Y)$. Suppose $[u]_N = 0$ in $S'_p(F)$, then there is a $U \in M_{p+1}(I \times X, F^{-1}(Y))$, $U : (W,\partial W) \to (I \times X, F^{-1}(Y))$, such that $M = \partial W$, $U|_M = u$, and $FU : (W,\partial W) \to (Z,Y)$ is homotopic relative to $M = \partial W$ to a $G \in M_{p+1}(Y,Y)$. Then $U = (P,V)$, where $P : W \to I$, $P|_M = \{0\}$, and $V : W \to X$, $V|_M = v$. Define a homotopy $H : I \times W \to Z$ by

$$H(s,x) = F\big((1-s)P(x), V(x)\big). \tag{3.1}$$

Then $H(0,x) = F(P(x),V(x)) = FU(x)$, $H(1,x) = F(0,V(x)) = fV(x)$. Suppose $x \in M$. Then first, $H(s,x) = F((1-s)\cdot 0, v(x)) = F(0,v(x)) = f(v(x))$; second, $FU(x) = Fu(x) = Fiv(x) = fv(x)$; third, $fV(x) = fv(x)$. Thus FU and fV are homotopic relative to M. Therefore, fV is homotopic to G relative to M. We have proven that if $[u]_N = i'_*[v]_N = 0$ in $S'_p(F)$, then $[v]_N = 0$ in $S'_p(f)$. Therefore, $\ker i'_* = \{0\}$. □

Thus the Nielsen classes of a map are included in the Nielsen classes of its homotopy.

This theorem generalizes both the fact that the intersection of a Nielsen class of F with $\{0\} \times X$ is a Nielsen class of f_0 [10, Corollary 1.5], for codimension 0, and the fact that (W1) is homotopy invariant [20, Lemma 4.2], for codimension 1 (see Section 8).

Now the following are monomorphisms:

$$S'_*(f_0) \xrightarrow{i'_{0*}} S'_*(F) \xleftarrow{i'_{1*}} S'_*(f_1). \tag{3.2}$$

Let

$$M_*^F = \operatorname{Im} i'_{0*} \cap \operatorname{Im} i'_{1*}. \tag{3.3}$$

Then M_*^F is isomorphic to some subgroups of $S'_*(F)$, $S'_*(f_0)$, $S'_*(f_1)$ (as a subgroup of $S'_*(f_0)$, M_*^F should be understood as the set of Nielsen classes of f_0 preserved by F). Now we say that a class $s_0 \in S'_*(f_0)$ of f_0 is *F-related* to a class $s_1 \in S'_*(f_1)$ of f_1 if there is $s \in S'_*(F)$ such that $i'_{0*}(s_0) = s = i'_{1*}(s_1)$. Then $s_1 = i'^{-1}_{1*} i'_{0*}(s_0)$ if defined, otherwise we can set $s_1 = 0$. Thus some classes cannot be reduced to zero by a homotopy and we call them (*topologically*) *essential Nielsen classes*. Together (plus zero) they form a group as follows.

Definition 3.2. The group of (*topologically*) *essential Nielsen classes* is defined as

$$S_*(f,Y) = \bigcap \{M_*^F : F \text{ is a homotopy of } f\} \subset S'_*(f,Y). \tag{3.4}$$

($S_p(f,Y)$ can also be called the *Nielsen group of order* p, while $S'_p(f,Y)$ the *pre-Nielsen group*.)

If $f \sim g$, then $S_*(f) \simeq S_*(g)$. Therefore, we have the following.

Theorem 3.3. *$S_*(f)$ is homotopy invariant. Moreover, for any g homotopic to f, there is a monomorphism $S_*(f) \to S'_*(g)$.*

Now $S_*(f)$ is a subgroup of $S'_p(f)$, which is a quotient of $\Omega_*(f^{-1}(Y))$. In this sense, $S_*(f)$ is a "lower estimate" of $\Omega_*(g^{-1}(Y))$ for any g homotopic to f.

Definition 3.4. The Nielsen number of order p, $p = 0,1,2,\ldots$, is defined as

$$N_p(f,Y) = \operatorname{rank} S_p(f,Y). \tag{3.5}$$

The *Nielsen number* for the coincidence problem is denoted by $N_p(f,g)$.

Corollary 3.5. *Suppose $f \sim g$. Then*

$$N_*(f) \le \operatorname{rank} \Omega_*\left(g^{-1}(Y)\right). \tag{3.6}$$

Clearly, $N_0(f)$ is equal to the classical Nielsen number and provides a lower estimate of the number of path components of $f^{-1}(Y)$.

It is easy to verify that this theory is still valid if the oriented bordism Ω_* is replaced with the unoriented bordism, or the framed bordism (see examples in Section 7), or bordism with coefficients. In fact, a similar theory for an arbitrary homology theory is valid because every homology theory can be constructed as a bordism theory with respect to manifolds with singularities [5].

4. Naturality of $S_*(f)$

Under the conditions of Definition 2.6, the homomorphism $k_* : S_*(f) \to S_*(g)$ can be defined as a restriction of k'_* and the analogues of Propositions 2.7, 2.8, and 2.9 hold. We simplify the situation in comparison to Section 2 by assuming that Z and $Y \subset Z$ are fixed.

Definition 4.1. Suppose another preimage problem $g : U \to Z$ is connected to the first by a map $k : X \to U$ such that $gk = f$. Then the *homomorphism induced by* k,

$$k_* : S_*(f) \longrightarrow S_*(g), \tag{4.1}$$

is defined as the restriction of $k'_* : S'_*(f) \to S'_*(g)$ on $S_*(f) \subset S'_*(f)$.

Proposition 4.2. *k_* is well defined.*

Proof. For convenience let $f = f_0$, $g = g_0$, $k = k_0$. Suppose G is a homotopy between g_0 and g_1, K between k_0 and k_1. Let $F = GK$, then F is a homotopy between f_0 and f_1. Let $L(t,x) = (t, K(t,x))$. Then we have a commutative diagram:

$$\begin{array}{ccccc} U & \xrightarrow{j_1} & I \times U & \xleftarrow{j_0} & U \\ \uparrow{\scriptstyle k_1} & & \uparrow{\scriptstyle L} & & \uparrow{\scriptstyle k_0} \\ X & \xrightarrow{i_1} & I \times X & \xleftarrow{i_0} & X \end{array} \tag{4.2}$$

where $i_s : X \to \{s\} \times X \to I \times X$ and $j_s : U \to \{s\} \times U \to I \times U$, $s = 0, 1$, are the inclusions. Further, if we add a vertex Z to this diagram, we have a commutative pyramid with the other edges provided by f_0, f_1, g_0, g_1, G, F. Then by naturality of the map induced on S'_* (Proposition 2.8), we have another commutative diagram:

$$\begin{array}{ccccc} S'_*(g_1) & \xrightarrow{j'_{1*}} & S'_*(G) & \xleftarrow{j'_{0*}} & S'_*(g_0) \\ \uparrow{\scriptstyle k'_{1*}} & & \uparrow{\scriptstyle L'_*} & & \uparrow{\scriptstyle k'_{0*}} \\ S'_*(f_1) & \xrightarrow{i'_{1*}} & S'_*(F) & \xleftarrow{i'_{0*}} & S'_*(f_0) \end{array} \tag{4.3}$$

Here the horizontal arrows are injective (Theorem 3.1). Therefore, the restriction $k'_{0*} = k'_{1*} = L'_* : M^F_* \to M^G_*$ is well defined. This conclusion is true for all G, K, so that the restriction $k'_{0*} : \cap_{F=GK} M^F_* \to \cap_G M^G_*$ is well defined. Since $S_*(f)$ is a subset of the former and the latter is $S_*(g)$, the statement follows. □

Proposition 4.3. *Suppose the following diagram for three preimage problems commutes:*

$$\begin{array}{ccccc} & & Z & & \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle f'} & \overset{f''}{\nwarrow} & \\ X & \xrightarrow{k} & X' & \xrightarrow{j} & X'' \end{array} \tag{4.4}$$

Then $j_*k_* = (jk)_* : S_*(f) \to S_*(f'')$.

Proposition 4.4. $(\mathrm{Id}_X)_* = \mathrm{Id}_{S_*(f)}$.

Corollary 4.5. *Given $Z, Y \subset Z$. If $\mathcal{P}(Z,Y)$ is the category of preimage problems as pairs (X, f), $f : X \to Z$, with morphisms as maps $k : X \to U$ satisfying $gk = f$, then S_* is a functor from $\mathcal{P}(Z,Y)$ to $\mathbf{Ab}_*$ (cf.* [21, Chapter 3]*).*

Corollary 4.6. *If k is a homotopy invariance, $gk = f$, then $S_*(f) = S_*(g)$, that is, the preimage theory $f : X \to Z \supset Y$ is "homotopy invariant" (cf.* [16]*) with respect to X.*

5. The bordism index as a homomorphism on $S'_*(f)$

In the classical Nielsen theory, the coincidence index provides an algebraic count of coincidence points. It satisfies the usual properties: (1) homotopy invariance: the index is invariant under homotopies of f, g; (2) additivity: the index over a union of disjoint sets is equal to the sum of the indices over these sets; (3) existence of coincidences: if the index is nonzero, then there is a coincidence; (4) normalization: the index is equal to the Lefschetz number; (5) removability: if the index is zero, then a coincidence can be (locally or globally) removed by a homotopy. From the point of view of our approach, the additivity property means that we associate an integer to every 0-class, that is, we have a homomorphism $S'_0(f) \to \mathbf{Z} = \Omega_0(Y)$.

Definition 5.1. The index of $[s]_N \in S'_q(f)$, where $s \in \Omega_q(C)$, is defined as

$$I_f(s) = f_*(s), \tag{5.1}$$

where $f_* : \Omega_q(C) \to \Omega_q(Y)$.

Proposition 5.2. *The index is well defined as a homomorphism*

$$I_f : S'_*(f) \longrightarrow \Omega_*(Y). \tag{5.2}$$

Proof. Suppose $s \sim_N \varnothing$, then by definition, $s \sim_b \varnothing$ in X via some H and $fs \sim_b \varnothing$ in Z via some G homotopic to fH. Therefore, $f_*(s) = 0$ in $\Omega_q(Y)$, so $I_f([s]_N) = 0$. □

Of course, $I_f(z) \neq 0 \Rightarrow z \neq 0$.

Suppose $F : I \times X \to Z$ is a homotopy. As before, let $f_t(\cdot) = F(t, \cdot) : X \to Z$, and let $i_t : X \to \{t\} \times X \to I \times X$ be the inclusions.

Theorem 5.3. *Suppose $z_0 \in S'_q(f_0)$ and $i'_{0*}(z_0) = z \in S'_q(F)$. Then $I_{f_0}(z_0) = I_F(z)$.*

Proof. Let $C = F^{-1}(Y)$, $C_0 = f_0^{-1}(Y)$. Now if $z = [s]_N$, $s \in \Omega_q(C)$ and $z_0 = [s_0]_N$, $s_0 \in \Omega_q(C_0)$, then $i_{0*}(s_0) \sim_N s$. Therefore, $I_F(z) = F_*(s) = F_* i_{0*}(s_0) = f_{0*}(s_0) = I_{f_0}(z_0)$. □

Corollary 5.4. *If $z_0 \in S'_q(f_0)$, $z_1 \in S'_q(f_1)$ are F-related, then $I_{f_0}(z_0) = I_{f_1}(z_1)$. Thus the index I_f is preserved under homotopies.*

In the classical theory Nielsen classes are sets and the algebraically essential classes are the ones with nonzero index. Similarly, we call $z \in S'_q(f)$ *algebraically essential* if $I_f(z) \neq 0$.

Corollary 5.5. *Every algebraically essential class is topologically essential, that is, z cannot be reduced by a homotopy to the zero p-class, and, therefore, z cannot be "removed" by a homotopy.*

We define the *group of algebraically essential Nielsen classes* as

$$S^a_*(f, Y) = S'_*(f, Y)/\ker I_f. \tag{5.3}$$

Suppose another preimage problem $g : U \to Z$ is connected to the first by a map $k : X \to U$ such that $gk = f$. Then just like in the previous section, we define the *homomorphism induced by k*,

$$k_*^a : S_*^a(f) \longrightarrow S_*^a(g), \tag{5.4}$$

as a restriction of k'_*. Moreover, similar properties are satisfied. Thus we have the following.

Corollary 5.6. *Given $Z, Y \subset Z$. If $\mathcal{P}(Z,Y)$ is the category of preimage problems as pairs (X,f), $f : X \to Z$, with morphisms as maps $k : X \to U$ satisfying $gk = f$, then S_*^a is a functor from $\mathcal{P}(Z,Y)$ to $\mathbf{Ab}_*$.*

6. The index of an isolated set of preimages

From now on, we assume that X, Y, Z are smooth orientable compact manifolds, Y is a submanifold of Z.

Suppose $P \subset C$ is an isolated set of preimages. Let $U \subset V$ be neighborhoods of P in X such that $\overline{U} \subset \operatorname{Int} V$ and $V \cap C = P$. In the classical Nielsen theory, the index $\operatorname{Ind}(f,P)$ of P is defined as the image of the generator z of $H_{n+m}(X) \simeq \mathbf{Z}$ under the composition

$$H_{n+m}(X) \longrightarrow H_{n+m}(X, X\setminus U) \xleftarrow{\simeq} H_{n+m}(V, V\setminus U) \xrightarrow{f_*} H_{n+m}(Z, Z\setminus Y). \tag{6.1}$$

Under the restriction $\dim X + \dim Y = \dim Z$, we are in the classical situation: each class a is an isolated set of preimages A and the index a is defined as the index of A.

In case of a nonzero codimension, we can have $H_{n+m}(X) \neq \mathbf{Z}$, therefore it makes sense to replace in the above definition the generator z with an arbitrary element of $H_*(X)$. This turns the index into a graded homomorphism $H_*(X) \to H_*(Z, Z\setminus Y)$ (which is equal to the Lefschetz homomorphism [27] for the coincidence problem). This generality is justified by a number of examples in [15, 27] that show that in order to detect coincidences in case of a nonzero codimension one may need to take into account the whole domain of this homomorphism.

A fixed point index with respect to generalized cohomology was considered by Dold [11]. Another example is [24] where the coincidence index is computed in term of cobordism. In addition, we will see in Section 9 that for a nonzero codimension the index expressed in terms of singular homology may be inadequate for removability (some algebraically inessential classes are essential). Therefore, under the above restrictions, the singular homology H_* should be replaced with a generalized homology h_*.

Definition 6.1. Suppose W is a neighborhood of Y in Z. The *index* $\operatorname{Ind}_f(P; h_*)$, or simply $\operatorname{Ind}_f(P)$, of the set P with respect to h_* is the following homomorphism

$$h_*(X, X\setminus U) \xleftarrow{\simeq} h_*(V, V\setminus U) \xrightarrow{f_*} h_*(W, W\setminus Y). \tag{6.2}$$

The index does not depend on the choice of U, see [31, page 189]. The next theorem is proven similarly to [31, Lemmas 7.1, 7.2, 7.4, pages 190–191], respectively.

Theorem 6.2. (I) *(additivity) If $P = P_1 \cup \cdots \cup P_s$ is a disjoint union of subsets of P such that $P_i = P \cap f^{-1}(Y)$ is compact and $f_i = f|_{P_i}$ for each $i = 1,\ldots,s$, then* $\operatorname{Ind}_f(P) = \sum_{i=1}^{s} \operatorname{Ind}_{f_i}(P_i)$.

(II) *Existence of preimages. If* $\operatorname{Ind}_f(P) \neq 0$, *then* $P \neq \emptyset$.

(III) *Homotopy invariance. Suppose $f_t : X \to Z$, $0 \leq t \leq 1$, is a homotopy and $D = \cup_t f_t^{-1}(Y)$ is a compact subset of X. Then*

$$\operatorname{Ind}_{f_0}(P) = \operatorname{Ind}_{f_1}(P). \tag{6.3}$$

Now we show how the indices I_f and Ind_f are related to each other.

Definition 6.3. Let $R_p(f)$ be the set of all elements of $M_p(C)$ represented as sums of finite collections of connected singular manifolds such that the sum of any subcollection is not Nielsen equivalent to the empty set. Then *the image of* $z \in S'_p(f)$ is defined as

$$\operatorname{Im} z = \cup \{ \operatorname{Im} s : s \in R_p(f),\ s \in z \}. \tag{6.4}$$

In particular, if $P \in \Omega_0(M)$ is a subset of C and $z = [P]_N$, then $\operatorname{Im} z = P$.

Suppose f is transversal to Y. Then C is an r-submanifold of X.

Proposition 6.4. *$\operatorname{Im} z$ is an isolated subset of C and therefore an r-submanifold of X.*

Proposition 6.5. *If $z = 0 \in S'_*(f)$, then* $\operatorname{Im} z = \emptyset$.

Clearly if $\operatorname{Ind}_f(\operatorname{Im} z) \neq 0$, then $\operatorname{Im} z \neq \emptyset$. However, this does not imply that z is essential. The case of $p = 0$ is an exception. For convenience we restate the following familiar result.

Proposition 6.6. *If $P \in S'_0(f)$ and $\operatorname{Ind}_f(P; h_*) \neq 0$, where h_* is an arbitrary homology theory, then P is essential.*

The relation between the essentiality of the class and its index is more subtle when $p > 0$.

If T is a tubular neighborhood of a submanifold M, then $\varphi_M : \Omega_{q+k}(T, T\backslash M) \to \Omega_q(M)$ is the Thom isomorphism [30, page 309], [12, page 321].

Suppose $z \in S'_p(f)$, $z = [s]$, where $s \in \Omega_p(M)$. Let $P = \operatorname{Im} z$, then it is an r-submanifold of C. Let T and T' be tubular neighborhoods of C and P, respectively, such that T' is an isolated subset of T. Then the inclusion $i : T' \to T$ is a bundle map. Suppose $s = i_*(s')$ for some $s' \in \Omega_p(P)$. From the naturality of the Thom isomorphism, we have the commutativity of the following diagram:

$$\begin{array}{ccccc}
\Omega_p(P) & \xrightarrow{i_*} & \Omega_p(C) & \xrightarrow{f_*} & \Omega_p(Y) \\
\uparrow{\scriptstyle \varphi_P} & & \uparrow{\scriptstyle \varphi_C} & & \uparrow{\scriptstyle \varphi_Y} \\
\Omega_{p+k}(T', T'\backslash P) & \xrightarrow{i_*} & \Omega_{p+k}(T, T\backslash C) & \xrightarrow{f_*} & \Omega_{p+k}(W, W\backslash Y)
\end{array} \tag{6.5}$$

where W is a tubular neighborhood of Y. Then $I_f(z) = f_*(s) = f_* i_*(s')$. Therefore,

$$\varphi_Y^{-1} I_f(z) = \varphi_Y^{-1} f_* i_*(s') = f_* i_* \varphi_P^{-1}(s') = \mathrm{Ind}_f(P;\Omega_*)(\varphi_P^{-1}(s')). \tag{6.6}$$

Thus we have proven the following.

Theorem 6.7. *For each $z \in S'_p(f)$, $I_f(z) = \varphi_Y \mathrm{Ind}_f(P;\Omega_*)\varphi_P^{-1}(s')$, where $z = [s]$, $P = \mathrm{Im}\, z$, $s = i_*(s')$.*

The right-hand side can be used for an alternative definition of an algebraically essential class.

Corollary 6.8. *If $P \in \Omega_r(M)$, $z = [P]$, then $\mathrm{Ind}_f(P;\Omega_*)(O_P) = \varphi_Y^{-1} I_f(z)$, where $O_P \in \Omega_{n+m}(T', T' \setminus P)$ is the fundamental class of P.*

Moreover, if $\mathrm{Ind}_f(P;H_*) \neq 0$, then $P \in S'_0(f)$ is essential. Thus for $r = 0$, we recover the traditional definition of an algebraically essential class.

7. Some examples

Nielsen numbers are hard to compute. Nielsen groups and higher-order Nielsen numbers are no different. Below we consider some special cases when the computation is feasible.

Just as before suppose $X = Z = \mathbf{S}^2$, Y is the equator of Z, f a map of degree 2 such that $C = f^{-1}(Y)$ is the union of two circles C_1 and C_2 around the poles. But there is only one generator of $S'_1(f)$, $C = C_1 \cup C_2$. Also $\mathrm{Ind}_f(C) \neq 0$. Hence, $N_0(f) = 1$. This is in fact a "sharp" estimate of the number of components of C (Wecken property for codimension $r = 1$) because f is homotopic to the suspension g, of the degree 2 map of the equator, so that $g^{-1}(Y)$ is a circle. The same conclusion applies to $X = Z = \mathbf{S}^n$, $Y = \mathbf{S}^{n-1}$, codimension $r = n - 1$, $n \geq 2$.

For more examples of this nature, see [26] and [9, Theorem 1.2 and Section 12].

In [28], we showed that the cohomology coincidence index I^A_{fg} is the only obstruction to removability of an isolated subset A of the coincidence set if any of the three following conditions is satisfied: (1) M is a surface; (2) the fiber of g is acyclic; or (3) the fiber of g is an m-sphere for $m = 4, 5, 12$, and n large. Of course if the homology index $\mathrm{Ind}_{(f,g)}(A;H_*)$ is trivial, then so is I^A_{fg}. Therefore, under these restrictions, an algebraically inessential class can be removed.

Proposition 7.1. *Suppose that if two singular q-manifolds in C are bordant in X, then they are Nielsen equivalent. Then $S'_q(f) \simeq j_*\Omega_q(C)$, where $j : C \to X$ is the inclusion.*

Corollary 7.2. *If under the conditions of the proposition $\Omega_q(X) = 0$, then $S_q(f) = 0$.*

The condition of this proposition is satisfied if we simply assume that f is homotopic to f' with $f'(X) \subset Y$. For the coincidence problem, this result takes the following form.

Theorem 7.3. *If $f, g : X \to Z$ are homotopic, then $S'_q(f,g) \simeq j_*\Omega_q(\mathrm{Coin}(f,g))$. Moreover, $N_q(f,g) \leq \mathrm{rank}\,\Omega_q(X)$.*

Theorem 7.4. *Suppose Y is $(q-1)$-connected, $f^* : H^q(Y;\pi_q(Y)) \to H^q(X;\pi_q(Y))$ is trivial, and Z is $(q+1)$-connected. Then $S'_q(f) \simeq j_*\Omega_q(C)$. Moreover, $N_q(f) \leq \mathrm{rank}\,\Omega_q(X)$.*

Proof. Suppose $s_0, s_1 \in j_*\Omega_q(C)$ are bordant in X via $H : W \to X$, that is, $s_i : S_i \to X$, $\partial W = S_0 \sqcup S_1$, $H|_{S_i} = s_i$. Since Y is $(q-1)$-connected and

$$\delta^* f^* (s_0 \sqcup s_1)^* : H^q(Y; \pi_q(Y)) \longrightarrow H^{q+1}(W, S_0 \sqcup S_1; \pi_q(Y)) \tag{7.1}$$

is trivial, the classical obstruction theory [1, page 497] is applied to prove that the map $f(s_0 \sqcup s_1) : S_0 \sqcup S_1 \to Y$ can be extended to $G : W \to Y$. Further, since Z is $(q+1)$-connected, $[W, Z]_{\mathrm{rel}\, S_0 \sqcup S_1} = 0$. Therefore, G and fH are homotopic relative to $S_0 \sqcup S_1$. Thus, if two singular q-manifolds in C are bordant in X, then they are Nielsen equivalent. Now the theorem follows from the above proposition. □

The relation between the homotopy class of a map and the preimage of a point is direct in the setting of the PT construction [7, page 196]. For the rest of the section we assume that the Nielsen groups $S'_q(f)$, $S_q(f)$ are computed with respect to the *framed* bordism, that is, $S'_q(f)$ is a quotient group of $\Omega^{\mathrm{fr}}_q(C)$.

Let $Y = \{p\}$, $p \in Z = \mathbf{S}^k$, and $r \leq k-2$. Then the conditions of the theorem above are satisfied. Therefore, $S'_r(f) \simeq j_*\Omega^{\mathrm{fr}}_r(C)$. Now, f is homotopic to a map g if and only if $C = f^{-1}(p)$ is framed bordant to $K_g = g^{-1}(p)$ in X. Let $j^g : K_g \to X$ be the inclusion. Then

$$S_r(f) \simeq \bigcap_{g \sim f} j^g_* \Omega^{\mathrm{fr}}_r(K_g) = \bigcap_{K_g \sim_b C} j^g_* \Omega^{\mathrm{fr}}_r(K_g). \tag{7.2}$$

Thus we have proven the following.

Theorem 7.5. *Suppose $Z = \mathbf{S}^k$ and $r = m - k \leq k-2$. Then*

$$S_r(f, \{p\}) = \bigcap_{K \sim_b C} j^K_* \Omega^{\mathrm{fr}}_r(K), \tag{7.3}$$

where $j^K : K \to X$ is the inclusion.

In particular, for codimension 1, $N_1(f, \{p\})$ is equal to the number of circles in $f^{-1}(p)$ not framed bordant to the empty set.

Corollary 7.6. *Suppose $Z = \mathbf{S}^k$ and $r = m - k \leq k-2$. Then $S_r(f, \{p\}) = 0$ if and only if $f^{-1}(p) \sim_b \varnothing$.*

Proof. The right-hand side in the above theorem contains C. □

Nielsen groups can be easily computed for the generators of $[\mathbf{S}^k, \mathbf{S}^m]$, see [7, page 208].

8. Wecken property of order 1, codimension 1

We say that the preimage problem $f : X \to Z \supset Y$ satisfies the *Wecken property of order p* if $S_p(f)$ is "realizable", that is, there is some h homotopic to f such that $S_p(f) \simeq \Omega_p(h^{-1}(Y))$.

Recall that a preimage problem is reduced to the coincidence problem $f,g : X \to Y$ by putting $Z = Y \times Y$, Y the diagonal of Z, $f = (F,G)$, $C = (f,g)^{-1}(Y) = \mathrm{Coin}(f,g)$. Also $\dim Z = 2\dim Y = 2n$, so $k = n$.

Then the definition of Wecken property is in the obvious extension of the one above: the pair (f,g) satisfies the *Wecken property of order* p if $S_p(f,g) \simeq \Omega_p(\mathrm{Coin}(f',g'))$ for some f', g' homotopic to f, g.

Assume that f, g are transversal. Then $C = \mathrm{Coin}(f,g)$ is a 1-submanifold of X. Suppose A is 1-submanifold of C. Recall condition (W1) $A = \partial S$, *where* S *is an orientable connected surface,* $f|_S \sim g|_S \operatorname{rel} A$.

Proposition 8.1. *$A = \partial S$, where S is an orientable (not necessarily connected) surface, $f|_S \sim g|_S \operatorname{rel} A \Leftrightarrow A \sim_N \varnothing$, that is, A belongs to the zero 1-class.*

Proof. Since A is the boundary of S, we have $A \sim_b \varnothing$ via S. Secondly, $f|_S \sim g|_S \operatorname{rel} A$, hence $(f,g)|_S \sim h \operatorname{rel} A$ such that $\mathrm{Im}\, h$ lies in the diagonal of $Z = Y \times Y$. Thus $A \sim_N \varnothing$. □

Consider the following result due to Jezierski [20, Theorem 3.1]. Its proof is based on his 1-parameter Whitney lemma.

Proposition 8.2 (Jezierski). *Suppose $n \geq 4$ and f is smooth. Then g is homotopic to g' such that the pair (f,g') is transversal and each Nielsen class (or even an isolated subset of a Nielsen class* [19]*) of (f,g) is a circle.*

We use this result to prove the following Wecken-type result for codimension 1.

Theorem 8.3. *Suppose $n \geq 4$ and f is smooth. Then the coincidence problem $f,g : X \to Y$ satisfies the Wecken property of order* 1*; specifically, g is homotopic to g' such that*

$$S_1(f,g) \simeq \Omega_1\big(\mathrm{Coin}(f,g')\big). \tag{8.1}$$

Moreover, $N_1(f,g) = \operatorname{rank} \Omega_1(\mathrm{Coin}(f,g'))$ is equal to the number of circles in $\mathrm{Coin}(f,g')$ *not satisfying (W1).*

Proof. Suppose, according to the above proposition, that all 0-classes are circles, $A_1,\ldots,A_s$. Suppose also that $A_1,\ldots,A_t$ satisfy condition (W1) and the rest do not. We view $A_1,\ldots,A_t$ as singular 1-manifolds. Then, first, $A_i \sim_N \varnothing$ for $i = 1,\ldots,t$ according to Proposition 8.1. Hence, for these i, $A_i \in 0 \in S_1'(f,g')$, so they don't concern us. Now, suppose $A_i \sim_N A_j$ for some $i > j > t$ via some surface H. If A_i and A_j were subsets of different components of H then each would satisfy condition (W1). Therefore H can be assumed connected. But, then $A_i \cup A_j$ satisfies condition (W1), and moreover every pair of points $x \in A_i$, $y \in A_j$ is Nielsen equivalent. Therefore, by Proposition 8.2 $A_i \cup A_j$ can be further reduced to a single circle. Hence, we can assume that each A_i, $i = t+1,\ldots,s$, belongs to a different nonzero 1-class. Thus the generators of $S_1'(f,g')$ are $[A_i]_N$, $i = t+1,\ldots,s$. Now the fact that each of these classes is essential follows from the homotopy invariance of (W1) [20, Lemma 4.2]. □

A similar result for the root problem is easy to prove.

THEOREM 8.4. *Suppose $Z = \mathbf{S}^k$ and $1 = m - k \leq k - 2$. Then the root problem $f : X \to Z \ni p$ satisfies the Wecken property of order* 1 *(with respect to framed bordism).*

Proof. Just as above assume that $C = f^{-1}(p)$ is the disjoint union of circles such that $A_1, \ldots, A_t$ are framed bordant to the empty set and $A_{t+1}, \ldots, A_s$ are not. Then C is framed bordant to $K = A_{t+1} \cup \cdots \cup A_s$. Finally, $S_1(f, \{p\}) = \Omega_1^{\mathrm{fr}}(K)$ by Theorem 7.5. □

9. Wecken property of order 0, codimension 1

Suppose A is a 1-submanifold of C. Recall condition (W2) *the PT map is trivial*. The proposition below explains why the PT map should be understood as the coincidence index.

PROPOSITION 9.1. *(W2)* $\Leftrightarrow \mathrm{Ind}_{(f,g)}(A; \pi_*^S) = 0$ *(i.e., A is algebraically inessential with respect to π_*^S).*

Proof. Let $U \subset T$ be tubular neighborhoods A. We state (W2) as follows:

$$\mathrm{PT} : \mathbf{S}^{n+1} \longrightarrow T/\partial T \xrightarrow{f-g} \mathbf{R}^n/(\mathbf{R}^n \backslash 0) \simeq \mathbf{S}^n \text{ is trivial.} \tag{9.1}$$

Since $n \geq 4$, this is equivalent to the following:

$$\mathrm{PT}_* : \pi_*^S(\mathbf{S}^{n+1}) \longrightarrow \pi_*^S(\mathbf{S}^n) \text{ is trivial.} \tag{9.2}$$

Consider the commutative diagram

$$\begin{array}{ccccc} & & T/\partial T & \xrightarrow{f-g} & (\mathbf{R}^n, \mathbf{R}^n \backslash 0) \\ & \nearrow & \downarrow \simeq & & \uparrow \simeq d \\ \mathbf{S}^{n+1} \longrightarrow \mathbf{S}^{n+1}/(\mathbf{S}^{n+1} \backslash U) & & & & \\ & \searrow & (T, T \backslash U) & \xrightarrow{f,g} & (\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n \backslash \Delta) \end{array} \tag{9.3}$$

where Δ is the diagonal and $d(x, y) = x - y$. Now if we apply the stable homotopy functor π_*^S to the diagram, we have PT_* in the upper path and the index of A with respect to $h_* = \pi_*^S$ in the lower. But d is a homotopy equivalence [12, Lemma VII.4.13, page 200], and the statement follows. □

Observe that the stable homotopy index $\mathrm{Ind}_{(f,g)}(A; \pi_*^S)$ is better at detecting essential classes than the traditional index with respect to singular homology. In fact, the latter would not work in the above argument as $\pi_{n+1}^S(\mathbf{S}^n) = \mathbf{Z}_2$ cannot be replaced with $H_{n+1}(\mathbf{S}^n) = 0$. Secondly, all the Nielsen numbers of higher-order in Section 7 would be zero if computed with respect to singular homology.

Recall Jezierski's Wecken type theorem [20, Theorem 5.3].

Proposition 9.2 (Jezierski). *Let $f,g : X \to Y$ be an admissible map between open subsets of $\mathbf{R}^{n+1}$, $\mathbf{R}^n$, respectively, $n \geq 4$. Then there are maps f', g' compactly homotopic to f, g, respectively, such that the Nielsen classes satisfying (W1) and (W2) disappear and the remaining ones become circles.*

Suppose we are left with the circles $A_1, A_2, \ldots, A_t$ each satisfying condition (W1) but not (W2), and $A_{t+1}, A_{t+2}, \ldots, A_s$ satisfying (W2) but not (W1). Then each $A_1, A_2, \ldots, A_t$ is an (algebraically) essential 0-class (Proposition 6.6). Also each $A_{t+1}, A_{t+2}, \ldots, A_s$ is an essential 1-class (Theorem 8.3), therefore an essential 0-class as well. Thus we have proven the following.

Theorem 9.3. *Suppose X, Y are open subsets of $\mathbf{R}^{n+1}$, $\mathbf{R}^n$, respectively, $n \geq 4$. Then there are maps f', g' compactly homotopic to f, g, respectively, such that* $\operatorname{Coin}(f',g')$ *has exactly $N_0(f,g)$ path components, that is, the coincidence problem $f,g : X \to Y$ satisfies the Wecken property of order* 0.

A result of this type is proven by Gonçalves and Wong [17, Theorem 4(iii)] for the root problem in case of an arbitrary codimension. In the terminology of the present paper their theorem reads as follows: if X, Z are nilmanifolds, $p \in Z$, then there is g homotopic to f such that $g^{-1}(p)$ has exactly $N_0(f, \{p\})$ components.

Another codimension 1 Wecken type theorem is given by Dimovski [8] for the parametrized fixed point problem: $F : I \times Y \to Y$. Here F is a PL-map, Y a compact connected n-dimensional PL-manifold contained in $\mathbf{R}^n$, $n \geq 4$. He defines two independent indices of a Nielsen class V, $\operatorname{ind}_1(F,V)$ and $\operatorname{ind}_2(F,V)$, corresponding to conditions (W1), (W2), and then defines a Nielsen number $N(F)$ as the number of Nielsen classes with either $\operatorname{ind}_1(F,V) \neq 0$ or $\operatorname{ind}_2(F,V) \neq 0$. His [8, Theorem 4.4(4)] reads: if F is homotopic to H such that H has only isolated circles of fixed points and isolated fixed points, then the number of fixed points classes of H is bigger than or equal to $N(F)$. However, $N(F)$ is not a lower bound of the number of components of the fixed point set. An examination of the proof of this theorem [8, Theorems 4.1, 4.2] reveals that only *local* homotopies, that is, ones constant outside a neighborhood of the given class, are allowed. (This is the reason why there is an obvious correspondence between Nielsen classes of two homotopic maps and there is no need for such a construction as the one in Section 3 of the present paper.) In fact, $N(F)$ can be larger than the estimate provided in the above theorem—Jezierski [20, Example 6.4] gives an example of a Nielsen class that can be removed by a global homotopy but not by a local one.

References

[1] G. E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1993.

[2] R. B. S. Brooks and R. F. Brown, *A lower bound for the Δ-Nielsen number*, Trans. Amer. Math. Soc. **143** (1969), 555–564.

[3] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Ill, 1971.

[4] R. F. Brown and H. Schirmer, *Nielsen coincidence theory and coincidence-producing maps for manifolds with boundary*, Topology Appl. **46** (1992), no. 1, 65–79.

[5] S. Buoncristiano, C. P. Rourke, and B. J. Sanderson, *A Geometric Approach to Homology Theory*, London Mathematical Society Lecture Note Series, no. 18, Cambridge University Press, Cambridge, 1976.
[6] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., vol. 33, Academic Press, New York, 1964.
[7] J. F. Davis and P. Kirk, *Lecture Notes in Algebraic Topology*, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, Rhode Island, 2001.
[8] D. Dimovski, *One-parameter fixed point indices*, Pacific J. Math. **164** (1994), no. 2, 263–297.
[9] D. Dimovski and R. Geoghegan, *One-parameter fixed point theory*, Forum Math. **2** (1990), no. 2, 125–154.
[10] R. Dobreńko and Z. Kucharski, *On the generalization of the Nielsen number*, Fund. Math. **134** (1990), no. 1, 1–14.
[11] A. Dold, *The fixed point transfer of fibre-preserving maps*, Math. Z. **148** (1976), no. 3, 215–244.
[12] ———, *Lectures on Algebraic Topology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 200, Springer-Verlag, Berlin, 1980.
[13] F. B. Fuller, *The homotopy theory of coincidences*, Ann. of Math. (2) **59** (1954), 219–226.
[14] R. Geoghegan and A. Nicas, *Parametrized Lefschetz-Nielsen fixed point theory and Hochschild homology traces*, Amer. J. Math. **116** (1994), no. 2, 397–446.
[15] R. Geoghegan, A. Nicas, and J. Oprea, *Higher Lefschetz traces and spherical Euler characteristics*, Trans. Amer. Math. Soc. **348** (1996), no. 5, 2039–2062.
[16] R. Geoghegan, A. Nicas, and D. Schütz, *Obstructions to homotopy invariance in parametrized fixed point theory*, Geometry and Topology: Aarhus (1998), Contemp. Math., vol. 258, American Mathematical Society, Rhode Island, 2000, pp. 157–175.
[17] D. L. Gonçalves and P. N.-S. Wong, *Nilmanifolds are Jiang-type spaces for coincidences*, Forum Math. **13** (2001), no. 1, 133–141.
[18] A. Hatcher and F. Quinn, *Bordism invariants of intersections of submanifolds*, Trans. Amer. Math. Soc. **200** (1974), 327–344.
[19] J. Jezierski, Personal communication.
[20] ———, *One codimensional Wecken type theorems*, Forum Math. **5** (1993), no. 5, 421–439.
[21] B. J. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemporary Mathematics, vol. 14, American Mathematical Society, Rhode Island, 1983.
[22] T.-H. Kiang , *The Theory of Fixed Point Classes*, Springer-Verlag, Berlin, 1989.
[23] C. K. McCord, *The three faces of Nielsen: coincidences, intersections and preimages*, Topology Appl. **103** (2000), no. 2, 155–177.
[24] K. Mukherjea, *Coincidence theory for manifolds with boundary*, Topology Appl. **46** (1992), no. 1, 23–39.
[25] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1990.
[26] Y. W. Rong and S. C. Wang, *The preimages of submanifolds*, Math. Proc. Cambridge Philos. Soc. **112** (1992), no. 2, 271–279.
[27] P. Saveliev, *Lefschetz coincidence theory for maps between spaces of different dimensions*, Topology Appl. **116** (2001), no. 1, 137–152.
[28] ———, *Removing coincidences of maps between manifolds of different dimensions*, Topol. Methods Nonlinear Anal. **22** (2003), no. 1, 105–113.
[29] R. E. Stong, *Notes on Cobordism Theory*, Mathematical notes, Princeton University Press, New Jersy, 1968.
[30] R. M. Switzer, *Algebraic Topology—Homotopy and Homology*, Die Grundlehren der mathematischen Wissenschaften, vol. 212, Springer-Verlag, New York, 1975.
[31] J. W. Vick, *Homology Theory. An Introduction to Algebraic Topology*, 2nd ed., Graduate Texts in Mathematics, vol. 145, Springer-Verlag, New York, 1994.

[32] P. Wong, *Reidemeister number, Hirsch rank, coincidences on polycyclic groups and solvmanifolds*, J. reine angew. Math. **524** (2000), 185–204.
[33] D. Gonçalves, J. Jezierski, and P. Wong, *Obstruction theory and coincidences in positive codimension*, preprint, 2002.

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