FIXED POINT VARIATIONAL SOLUTIONS FOR UNIFORMLY CONTINUOUS PSEUDOCONTRACTIONS IN BANACH SPACES

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Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a uniformly continuous pseudocontraction. If $f : K \to K$ is any contraction map on K and if every nonempty closed convex and bounded subset of K has the fixed point property for nonexpansive self-mappings, then it is shown, under appropriate conditions on the sequences of real numbers $\{\alpha_n\}$, $\{\mu_n\}$, that the iteration process $z_1 \in K$, $z_{n+1} = \mu_n(\alpha_n T z_n + (1-\alpha_n)z_n) + (1-\mu_n)f(z_n)$, $n \in \mathbb{N}$, strongly converges to the fixed point of T, which is the unique solution of some variational inequality, provided that K is bounded.

1. Introduction

Let E be a real Banach space with dual E^* and K a nonempty closed convex subset of E. Let $J : E \to 2^{E^*}$ denote the *normalized duality mapping* defined by $J(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|,\ x \in E\}$ where $\langle \cdot,\cdot\rangle$ denotes the *generalized duality pairing*. Following Morales [6], a mapping T with domain $D(T)$ and range $\mathcal{R}(T)$ in E is called *strongly pseudocontractive* if for some constant $k < 1$ and $\forall x, y \in D(T)$,

$$(\lambda - k)\|x - y\| \le \big\|(\lambda I - T)(x) - (\lambda I - T)(y)\big\| \tag{1.1}$$

for all $\lambda > k$; while T is called a *pseudocontraction* if (1.1) holds for $k = 1$. The mapping T is called *Lipschitz* if there exists $L \ge 0$ such that $\|Tx - Ty\| \le L\|x - y\|$, $\forall x, y \in D(T)$. The mapping T is called *nonexpansive* if $L = 1$ and is called a *(strict) contraction* if $L < 1$. Every nonexpansive mapping is a pseudocontraction. The converse is not true. The example, $T(x) = 1 - x^{2/3}$, $0 \le x \le 1$, is a continuous pseudocontraction which is not nonexpansive. It follows from a result of Kato [3] that T is pseudocontractive if and only if there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \le \|x - y\|^2$, $\forall x, y \in D(T)$.

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In [9], Schu introduced the iterative process (1.2) below and proved the following theorem.

Theorem 1.1 [9, Theorem 2.4, page 113]. *Let K be a nonempty, closed convex, and bounded subset of a Hilbert space H; let $T : K \to K$ be a Lipschitz pseudocontractive map with Lipschitz constant $L \geq 0$; $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1)$ with $\lim_{n\to\infty}\lambda_n = 1$; $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$ with $\lim_{n\to\infty}\alpha_n = 0$ such that $(\{\alpha_n\},\{\mu_n\})$ has property (A), $\{(1-\mu_n)(1-\lambda_n)^{-1}\}$ is bounded, and $\lim_{n\to\infty}(1-\mu_n)/\alpha_n = 0$, where $k_n := (1+\alpha_n^2(1+L)^2)^{1/2}$ and $\mu_n := \lambda_n/k_n$, for all $n \in \mathbb{N}$; fix an arbitrary point $w \in K$, and define that for all $n \in \mathbb{N}$,*

$$z_{n+1} := \mu_{n+1}(\alpha_n T z_n + (1-\alpha_n)z_n) + (1-\mu_{n+1})w. \tag{1.2}$$

Then $\{z_n\}_n$ converges strongly to the unique fixed point of T closest to w.

Here the pair of sequences $(\{\alpha_n\}_n, \{\mu_n\}_n) \subset (0,\infty)\times(0,1)$ is said to have *property* (A) if and only if the following conditions hold.

(i)′ $\{\alpha_n\}_n$ is decreasing;
(ii)′ $\{\mu_n\}_n$ is strictly increasing;
(iii)′ There exists a strictly increasing sequence $\{\beta_n\}_n \subset \mathbb{N}$ such that
  (a)′ $\lim_n(\alpha_n - \alpha_{n+\beta_n})/(1-\mu_n) = 0$;
  (b)′ $\lim_n(1-\mu_{n+\beta_n})(1-\mu_n)^{-1} = 1$;
  (c)′ $\lim_n \beta_n(1-\mu_n) = \infty$.

The first iterative process of this nature was introduced by Halpern [2]: for any fixed $w \in K$ and arbitrary $z_0 \in K$,

$$z_{n+1} = \mu_n T z_n + (1-\mu_n)w, \quad n = 0,1,2,\ldots, \tag{1.3}$$

where $\{\mu_n\}$ is a sequence in $(0,1)$ with $\lim_{n\to\infty}\mu_n = 1$.

In [8], Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces, where he proved the following theorem.

Theorem 1.2 [8, Theorem 2.2, page 48]. *Let H be a Hilbert space, let $T : K \to K$ be a nonexpansive self-mapping of a nonempty closed convex subset K of H, and let $f : K \to K$ be a contraction. With an initial $z_0 \in K$, define the sequence $\{z_n\}$ by*

$$z_{n+1} = \frac{1}{1+\epsilon_n} T z_n + \frac{\epsilon_n}{1+\epsilon_n} f(z_n). \tag{1.4}$$

Supposed that $\lim_{n\to\infty}\epsilon_n = 0$, $\sum_{n=1}^{\infty}\epsilon_n = \infty$, and $\lim_{n\to\infty}|1/\epsilon_{n+1} - 1/\epsilon_n| = 0$. Then $\{z_n\}$ converges strongly to the unique solution of the variational inequality:

$$\text{find } \widetilde{x} \in F(T) \text{ such that } \langle (I-f)\widetilde{x}, \widetilde{x} - x\rangle \leq 0, \quad \forall x \in F(T), \tag{1.5}$$

(i.e., the unique solution of the operator $\mathrm{Proj}_{F(T)} \circ f$*).*

Xu [12] extended Theorem 1.2 to the more general *uniformly smooth* Banach spaces. If Π_K denotes the set of all contractions on K, he proved the following theorem.

Theorem 1.3 [12, Theorem 4.2, page 289]. *Let E be a uniformly smooth Banach space, K a closed convex subset of E, and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_K$. Assume that $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$ *or* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{z_n\}$ generated by $z_0 \in K$,

$$z_{n+1} := \alpha_n f(z_n) + (1-\alpha_n)Tz_n, \quad n = 0,1,2,\dots, \tag{1.6}$$

converges strongly to $Q(f)$, where $Q : \Pi_K \to F(T)$ is defined by $Q(f) := \sigma - \lim_{t\to 0} x_t$, with x_t satisfying

$$x_t = tTx_t + (1-t)f(x_t). \tag{1.7}$$

Let K be a nonempty closed convex and bounded subset of a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Further to Theorems 1.2 and 1.3, the purpose of this paper is to use the following iteration process: $z_1 \in K$,

$$z_{n+1} = \mu_n(\alpha_n Tz_n + (1-\alpha_n)z_n) + (1-\mu_n)f(z_n), \quad n \in \mathbb{N}, \tag{1.8}$$

where $\{\mu_n\}_n$, $\{\alpha_n\}_n$ are sequences in $(0,1)$ and $f : K \to K$ is a contraction map, to approximate the fixed point of a uniformly continuous pseudocontraction, which solves some variational inequality. If the map f is a constant map then we recover the iteration process (1.2) from (1.8).

2. Preliminaries

Let E be a real normed linear space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm and E is called *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$. E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$.

The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\right\}, \quad \tau > 0. \tag{2.2}$$

E is equivalently said to be smooth if $\rho_E(\tau) > 0\ \forall \tau > 0$. Every uniformly smooth Banach space is a reflexive Banach space with a uniformly Gâteaux differentiable norm. An example given in [7] illustrates that this inclusion is proper.

Let E be a linear space and let K be a subset of E. Then, for any $x \in K$, the set $I_K(x) = \{x + \lambda(z-x) : z \in K,\ \lambda \geq 1\}$ is called the *inward set* of x. A mapping $T : K \to E$ is said to satisfy the *inward condition* if $Tx \in I_K(x)$ for each $x \in K$, and is said to satisfy the *weakly inward condition* if $Tx \in cl[I_K(x)]$, the closure of $I_K(x)$, for each $x \in K$.

We will let LIM_n be a Banach limit. Recall that $\operatorname{LIM}_n \in (\ell^\infty)^*$ such that $\|\operatorname{LIM}_n\| = 1$, $\liminf_{n\to\infty} a_n \leq \operatorname{LIM}_n a_n \leq \limsup_{n\to\infty} a_n$, and $\operatorname{LIM}_n a_n = \operatorname{LIM}_n a_{n+1}$ for all $\{a_n\}_n \in \ell^\infty$.

The *modulus of uniform continuity*, $\delta(\epsilon)$, of T is defined for all $\epsilon > 0$ by

$$\delta(\epsilon) = \sup\{\lambda : \|x-y\| < \lambda \Longrightarrow \|Tx - Ty\| < \epsilon\} \tag{2.3}$$

and $\delta(0) = 0$. By [4, Proposition 3], $\delta(\epsilon)$ is nondecreasing, $0 \leq \delta(\epsilon) \leq \infty$, and $\delta(\|Tx - Ty\|) \leq \|x-y\|$, for all $x, y \in E$. Furthermore, [4, Propositions 1 and 2] assert that the function

$$\phi(t) = \sup\{s : \delta(s) \leq t\} \tag{2.4}$$

called the *pseudo-inverse* of δ is nondecreasing and right continuous, $0 \leq \phi(t) \leq \infty$ for $t \geq 0$ and $\|Tx - Ty\| \leq \phi(\|x-y\|)$ $\forall x, y \in E$.

The following lemmas will be needed in the sequel. Lemma 2.1 is well known, (see, e.g., [7]). The proof of Lemma 2.2 can be deduced from [11, Lemma 2.5].

Lemma 2.1. *Let E be an arbitrary real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{2.5}$$

for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$.

Lemma 2.2. *Let $\{a_n\}_n$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n, \quad n \in \mathbb{N}, \tag{2.6}$$

where $\{\alpha_n\}_n \subset [0,1]$, $\{\beta_n\}_n \subset [0,1]$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3, Proposition 2.4, and Lemma 2.5 that follow appear in [10]. For completeness, we present also their proofs.

Lemma 2.3. *Let E be a Banach space. Suppose K is a nonempty closed convex subset of E and $T : K \to E$ is a continuous pseudocontraction satisfying the weakly inward condition. Then for each contraction map $f : K \to K$, with contraction constant $\alpha \in [0,1)$, there exists a unique continuous path $t \to x_t \in K$, $t \in [0,1)$ satisfying*

$$x_t = tTx_t + (1-t)f(x_t). \tag{2.7}$$

Proof. Let $f : K \to K$ be a contraction map with constant $\alpha \in [0,1)$. Then, for each $t \in [0,1)$, the mapping $T_t^f : K \to E$ defined by $T_t^f(x) = tTx + (1-t)f(x)$ is a continuous strong pseudocontraction with constant $t + (1-t)\alpha \in [0,1)$, which satisfies the weakly inward condition. By [1, Corollary 1], T_t^f has a unique fixed point $x_t \in K$, that is,

$$x_t = tTx_t + (1-t)f(x_t). \tag{2.8}$$

To prove the continuity of the path, we follow the same line of arguments as in [7]. Let $t_0 \in [0,1)$. Then for all $j(x_t - x_{t_0}) \in J(x_t - x_{t_0})$,

$$\begin{aligned} \|x_t - x_{t_0}\|^2 &= t\langle Tx_t - Tx_{t_0}, j(x_t - x_{t_0})\rangle + (1-t)\langle f(x_t) - f(x_{t_0}), j(x_t - x_{t_0})\rangle \\ &\quad + (t - t_0)\langle Tx_{t_0} - f(x_{t_0}), j(x_t - x_{t_0})\rangle \\ &\le (t + (1-t)\alpha)\|x_t - x_{t_0}\|^2 + |t - t_0|\,\|Tx_{t_0} - f(x_{t_0})\|\,\|x_t - x_{t_0}\|, \end{aligned} \tag{2.9}$$

so that $\|x_t - x_{t_0}\| \le (|t - t_0|/(1-t)(1-\alpha))\|Tx_{t_0} - f(x_{t_0})\|$. Hence the proof. □

Proposition 2.4. *Let E be a Banach space and let K be a nonempty closed convex subset of E. Let the mapping $T : K \to E$ be a pseudocontraction such that for each contraction map, $f : K \to K$ with contraction constant $\alpha \in [0,1)$, the equation*

$$x = tTx + (1-t)f(x) \tag{2.10}$$

has a solution x_t for every $t \in [0,1)$. Then the following hold.

(i) *If for some $u \in K$, the path $y_t = tTy_t + (1-t)u$ is bounded, then for any contraction map $f : K \to K$, the path $\{x_t\}$ described by (2.7) is bounded.*

(ii) *If T has a fixed point in K, then the path $\{x_t\}$ is bounded.*

(iii) *If $x^* \in F(T)$, then for all $j(x_t - x^*) \in J(x_t - x^*)$,*

$$\langle x_t - f(x_t), j(x_t - x^*)\rangle \le 0. \tag{2.11}$$

(iv) *If $0 \le s \le t < 1$ then*

$$\|x_t - Tx_t\| \le \frac{1+\alpha}{1-\alpha}\|x_s - Tx_s\|. \tag{2.12}$$

Proof. (i) Let the path $\{y_t\}$ given by $y_t = tTy_t + (1-t)u$, for some $u \in K$, be bounded. Then the set $\{f(y_t)\}$ is bounded. Let $j(x_t - y_t) \in J(x_t - y_t)$. From the estimates

$$\begin{aligned} \|x_t - y_t\|^2 &= t\langle Tx_t - Ty_t, j(x_t - y_t)\rangle + (1-t)\langle f(x_t) - u, j(x_t - y_t)\rangle \\ &\le t\|x_t - y_t\|^2 + (1-t)\|f(x_t) - u\|\|x_t - y_t\|, \end{aligned} \tag{2.13}$$

we have that $\|x_t - y_t\| \le \|f(x_t) - u\| \le \alpha\|x_t - y_t\| + \|f(y_t) - u\|$. Thus,

$$\|x_t - y_t\| \le \frac{1}{1-\alpha}\|f(y_t) - u\|. \tag{2.14}$$

Hence, $\{x_t\}$ is bounded.

(ii) Let $x^* \in F(T)$, and let $j(x_t - x^*) \in J(x_t - x^*)$. Then

$$\begin{aligned} \|x_t - x^*\|^2 &= t\langle Tx_t - x^*, j(x_t - x^*)\rangle + (1-t)\langle f(x_t) - x^*, j(x_t - x^*)\rangle \\ &\le t\|x_t - x^*\|^2 + (1-t)\|f(x_t) - x^*\|\|x_t - x^*\| \end{aligned} \tag{2.15}$$

so that $\|x_t - x^*\| \le \|f(x_t) - x^*\| \le \alpha\|x_t - x^*\| + \|f(x^*) - x^*\|$. Thus,

$$||x_t - x^*|| \le \frac{1}{1-\alpha}||f(x^*) - x^*||. \tag{2.16}$$

Hence, $\{x_t\}$ is bounded.

(iii) Let $x^* \in F(T)$, and let $j(x_t - x^*) \in J(x_t - x^*)$. Then

$$\begin{aligned}
&\langle x_t - f(x_t), j(x_t - x^*)\rangle \\
&\quad = t\langle Tx_t - f(x_t), j(x_t - x^*)\rangle = t\langle Tx_t - x^*, j(x_t - x^*)\rangle \\
&\qquad + t\langle x^* - f(x_t), j(x_t - x^*)\rangle \le t\langle x_t - f(x_t), j(x_t - x^*)\rangle.
\end{aligned} \tag{2.17}$$

Thus, $\langle x_t - f(x_t), j(x_t - x^*)\rangle \le 0$.

(iv) Let $0 \le s \le t < 1$. Then

$$\begin{aligned}
||x_t - Tx_t|| &= \frac{1-t}{t}||x_t - f(x_t)|| \\
&\le \frac{1-t}{t}\left[(1+\alpha)||x_t - x_s|| + \frac{s}{1-s}||x_s - Tx_s||\right] \\
&\le \frac{1-t}{t}\left[\frac{(1+\alpha)(t-s)}{(1-\alpha)(1-t)(1-s)} + \frac{s}{1-s}\right]||x_s - Tx_s|| \\
&\le \frac{(1+\alpha)(1-t)}{(1-\alpha)t}\left[\frac{t-s}{(1-t)(1-s)} + \frac{s}{1-s}\right]||x_s - Tx_s|| \\
&= \frac{1+\alpha}{1-\alpha}||x_s - Tx_s||.
\end{aligned} \tag{2.18}$$

□

Lemma 2.5. *Let E be a* reflexive *Banach space with a uniformly Gâteaux differentiable norm, let K be a nonempty closed convex subset of E, let $T : K \to E$ be a continuous pseudocontraction satisfying the weakly inward condition, and let $f : K \to K$ be a contraction map with constant $\alpha \in [0,1)$. Suppose that every nonempty closed convex and bounded subset of K has the fixed point property (f.p.p.) for nonexpansive self-mappings. If there exists $u_0 \in K$ such that the set*

$$B = \{x \in K : Tx = u_0 + \lambda(x - u_0) \text{ for some } \lambda > 1\} \tag{2.19}$$

is bounded, then the path $\{x_t\}$, $t \in [0,1)$ described by (2.7) converges strongly to the fixed point of T, which is the unique solution of the variational inequality

$$p \in F(T) \text{ such that } \langle p - f(p), j(p - x^*)\rangle \le 0, \quad x^* \in F(T). \tag{2.20}$$

Proof. It follows from Lemma 2.3 that for each contraction map $f : K \to K$ there exists a unique continuous path $t \to x_t \in K$, $t \in [0,1)$ satisfying (2.7). Let there exists $u_0 \in K$ such that the set B is bounded. Then by Proposition 2.4(i), the path $\{x_t\}$ described by (2.7) is bounded. It is easy to see that this implies that the set $\{f(x_t) : t \in [0,1)\}$ is bounded. The boundedness of the set $\{Tx_t : t \in [0,1)\}$ follows from Proposition 2.4(iv). Let $\sup_{t\in[0,1)} \|x_t\| \le M$. Then $\|x_t - x_s\| \le 2M$ for any $t,s \in [0,1)$. Set $x_n = x_{t_n}$ for $t_n \to 1^-$. Define $\psi : K \to \mathbb{R}$ by $\psi(x) = \operatorname{LIM}_n \|x_n - x\|^2$ $\forall x \in K$. Since E is reflexive, ψ is convex, continuous and $\psi(x) \to \infty$ as $\|x\| \to \infty$, we have that the set $C := \{y \in K : \psi(y) = \inf_{x\in K} \psi(x)\}$ is nonempty, closed and convex. We show that C is bounded. Let $y \in C$. Then $\psi(y) \le \operatorname{LIM}_n \|x_n - x_0\|^2 \le 4M^2$, where $x_0 \equiv x_{t_0}$. Applying the convexity of the functional $(1/2)\|\cdot\|^2 : K \to \mathbb{R}$, we deduce that

$$\|y\|^2 \le 2\operatorname{LIM}_n \|x_n - y\|^2 + 2\operatorname{LIM}_n \|x_n\|^2 \le 2\psi(y) + 2M^2 \le 10M^2, \tag{2.21}$$

that is, $\|y\| \le \sqrt{10}M$, $\forall y \in C$. Thus, C is bounded. The mapping $J_1 = (2I - T)^{-1}$ is a nonexpansive self-mapping of K (see [5, Theorem 6]). C is invariant under J_1. Indeed, let $y \in C$. Then

$$\begin{aligned}\psi(J_1(y)) &= \operatorname{LIM}_n \|x_n - J_1(y)\|^2 \le \operatorname{LIM}_n (\|x_n - J_1(x_n)\| + \|x_n - y\|)^2 \\ &\le \operatorname{LIM}_n (\|x_n - Tx_n\| + \|x_n - y\|)^2 = \operatorname{LIM}_n \|x_n - y\|^2 = \psi(y).\end{aligned} \tag{2.22}$$

By hypothesis, J_1 has a fixed point $p \in C$. Thus, $Tp = p$. Let $\tau \in (0,1)$. Then $\psi(p) \le \psi((1-\tau)p + \tau x)$, $x \in K$, and using Lemma 2.1, we have that $0 \le (\psi((1-\tau)p + \tau x) - \psi(p))/\tau \le -2\operatorname{LIM}_n \langle x - p, j(x_n - p - \tau(x - p))\rangle$. Thus

$$\operatorname{LIM}_n \langle x - p, j(x_n - p - \tau(x - p))\rangle \le 0. \tag{2.23}$$

Since, in this setting, J is norm-to-weak* uniformly continuous on bounded sets, letting $\tau \to 0$, we have that

$$\operatorname{LIM}_n \langle x - p, j(x_n - p)\rangle \le 0, \quad x \in K. \tag{2.24}$$

In particular,

$$\operatorname{LIM}_n \langle f(p) - p, j(x_n - p)\rangle \le 0. \tag{2.25}$$

Observe that

$$(1-\alpha)\|x_n - p\|^2 \le \langle x_n - f(x_n), j(x_n - p)\rangle + \langle f(p) - p, j(x_n - p)\rangle. \tag{2.26}$$

Using Proposition 2.4(iii) and (2.25), we have find that $\operatorname{LIM}_n \|x_n - p\| = 0$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. Assume that there is another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \to q \in F(T)$ as $l \to \infty$. With $x_{n_k} \to p$ and setting $x^* = q$, it follows from Proposition 2.4(iii) that

$$\langle p - f(p), j(p - q)\rangle \le 0. \tag{2.27}$$

Also, with $x_{n_l} \to q$ and setting $x^* = p$ in Proposition 2.4(iii), we have that

$$\langle q - f(q), j(q-p)\rangle \le 0. \tag{2.28}$$

Inequalities (2.27) and (2.28) yield that

$$\|p-q\|^2 \le \langle f(p) - f(q), j(p-q)\rangle \le \alpha\|p-q\|^2, \tag{2.29}$$

which implies that $p = q$, since $\alpha \in [0,1)$. Thus, $x_n \to p$ as $n \to \infty$ and $p \in F(T)$ is unique. Again, using Proposition 2.4(iii), we observe that

$$\langle p - f(p), j(p - x^*)\rangle \le 0, \quad \forall x^* \in F(T). \tag{2.30}$$

Hence, p is the unique solution of the variational inequality (2.20). This concludes the proof of Lemma 2.5. □

3. Main results

In the results that follow, if the map T is uniformly continuous and $\delta(\epsilon)$ denotes the modulus of continuity of T, we will let ϕ denote the pseudoinverse of δ and will assume that the set $\{\phi(t)/t : 0 < t < 1\}$ is bounded. Observe that if T is Lipschitz, then it is clear that the set $\{\phi(t)/t : 0 < t < 1\}$ is bounded.

Theorem 3.1. *Let K be a nonempty closed convex and bounded subset of a real Banach space E. Let $T : K \to K$ be a uniformly continuous pseudocontraction and let $f : K \to K$ be a contraction map with contraction constant* $\alpha \in [0,1)$. *Let $\{z_n\}$ be a sequence generated from an arbitrary $z_1 \in K$ by (1.8), where $\{\mu_n\}$, $\{\alpha_n\}$ are real sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\{\alpha_n\}$ *is decreasing and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \mu_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\mu_n) = \infty$;

(iii) (a) $\lim_{n\to\infty}(1-\mu_n)/\alpha_n = 0$,

(b) $\lim_{n\to\infty} \alpha_n^2/(1-\mu_n) = 0$,

(c) $\lim_{n\to\infty} |\mu_n - \mu_{n-1}|/(1-\mu_n)^2 = 0$,

(d) $\lim_{n\to\infty}(\alpha_{n-1} - \alpha_n)/\alpha_{n-1}(1-\mu_n) = 0$.

Then $\|z_n - Tz_n\| \to 0$ *as* $n \to \infty$.

Proof. We first prove that $\|z_n - x_n\| \to 0$ as $n \to \infty$, where $\{x_n\}$ is a sequence satisfying (2.7).

Set $t_n = \alpha_n/(1-\mu_n+\alpha_n)$, $\forall n \in \mathbb{N}$. Then $t_n \in (0,1)$ for each $n \in \mathbb{N}$. By the given condition (iii)(a), $t_n \to 1$ as $n \to \infty$. It follows from Lemma 2.3 that there exists a unique sequence $\{x_n\} \subset K$ satisfying the following conditions:

$$x_n = t_n T x_n + (1-t_n) f(x_n), \quad n \in \mathbb{N}. \tag{3.1}$$

Equation (3.1) can be rewritten as follows:

$$x_n = \mu_n(\alpha_n T x_n + (1-\alpha_n)x_n) + (1-\mu_n) f(x_n) + (1-\mu_n)\alpha_n(Tx_n - x_n). \tag{3.2}$$

Using the pseudocontractivity of T, we make the following estimates:

$$\begin{aligned}
\|z_{n+1}-x_n\|^2 &= \mu_n\alpha_n\langle Tz_n - Tx_n, j(z_{n+1}-x_n)\rangle + \mu_n(1-\alpha_n)\langle z_n - x_n, j(z_{n+1}-x_n)\rangle \\
&\quad + (1-\mu_n)\langle f(z_n) - f(x_n), j(z_{n+1}-x_n)\rangle \\
&\quad + (1-\mu_n)\alpha_n\langle x_n - Tx_n, j(z_{n+1}-x_n)\rangle \\
&= \mu_n\alpha_n\langle Tz_{n+1} - Tx_n, j(z_{n+1}-x_n)\rangle + \mu_n\alpha_n\langle Tz_n - Tz_{n+1}, j(z_{n+1}-x_n)\rangle \\
&\quad + \mu_n(1-\alpha_n)\langle z_n - x_n, j(z_{n+1}-x_n)\rangle \\
&\quad + (1-\mu_n)\langle f(z_n) - f(x_n), j(z_{n+1}-x_n)\rangle \\
&\quad + (1-\mu_n)\alpha_n\langle x_n - Tx_n, j(z_{n+1}-x_n)\rangle \\
&\leq \mu_n\alpha_n\|z_{n+1}-x_n\|^2 + \mu_n\alpha_n\|Tz_n - Tz_{n+1}\|\|z_{n+1}-x_n\| \\
&\quad + \mu_n(1-\alpha_n)\|z_n - x_n\|\|z_{n+1}-x_n\| + (1-\mu_n)\|f(z_n) - f(x_n)\|\|z_{n+1}-x_n\| \\
&\quad + (1-\mu_n)\alpha_n\|x_n - Tx_n\|\|z_{n+1}-x_n\|.
\end{aligned} \tag{3.3}$$

Thus, we have that

$$\begin{aligned}
\|z_{n+1}-x_n\| &\leq \mu_n\alpha_n\|z_{n+1}-x_n\| + \mu_n\alpha_n\|Tz_n - Tz_{n+1}\| \\
&\quad + [\mu_n(1-\alpha_n) + (1-\mu_n)\alpha]\|z_n - x_n\| + (1-\mu_n)\alpha_n\|x_n - Tx_n\| \\
&\leq \mu_n\alpha_n\|z_{n+1}-x_n\| + \mu_n\alpha_n\phi(\|z_n - z_{n+1}\|) \\
&\quad + [\mu_n(1-\alpha_n) + (1-\mu_n)\alpha]\|z_n - x_n\| + (1-\mu_n)\alpha_n\|x_n - Tx_n\|,
\end{aligned} \tag{3.4}$$

so that

$$\begin{aligned}
\|z_{n+1}-x_n\| &\leq \left[1 - \frac{(1-\alpha)(1-\mu_n)}{1-\alpha_n\mu_n}\right]\|z_n - x_{n-1}\| + \|x_{n-1} - x_n\| \\
&\quad + \frac{\alpha_n}{1-\alpha_n\mu_n}\phi(\|z_n - z_{n+1}\|) + \frac{(1-\mu_n)\alpha_n}{1-\alpha_n\mu_n}\|x_n - Tx_n\|.
\end{aligned} \tag{3.5}$$

Since the mapping $\widetilde{J}_n := [I + (\alpha_n/(1-\mu_n))(I-T)]^{-1}$ is nonexpansive and $x_n = \widetilde{J}_n(f(x_n))$,

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|\widetilde{J}_n(f(x_n)) - x_{n-1}\| = \|\widetilde{J}_n(f(x_n)) - \widetilde{J}_n(f(x_{n-1})) + \widetilde{J}_n(f(x_{n-1})) - x_{n-1}\| \\
&\leq \|f(x_n) - f(x_{n-1})\| + \|\widetilde{J}_n(f(x_{n-1})) - x_{n-1}\| \\
&\leq \alpha\|x_n - x_{n-1}\| + \|\widetilde{J}_n(f(x_{n-1})) - x_{n-1}\|,
\end{aligned} \tag{3.6}$$

so that

$$
\begin{aligned}
\|x_n - x_{n-1}\| &\le \frac{1}{1-\alpha}\|\widetilde{J}_n(f(x_{n-1})) - x_{n-1}\| \\
&\le \frac{1}{1-\alpha}\left\| f(x_{n-1}) - \left[x_{n-1} + \frac{\alpha_n}{1-\mu_n}(x_{n-1} - Tx_{n-1})\right]\right\| \\
&= \frac{1}{1-\alpha}\left|\frac{\alpha_{n-1}}{1-\mu_{n-1}} - \frac{\alpha_n}{1-\mu_n}\right| \|x_{n-1} - Tx_{n-1}\| \\
&= \frac{1}{1-\alpha}\left|1 - \frac{\alpha_n}{1-\mu_n}\frac{1-\mu_{n-1}}{\alpha_{n-1}}\right| \|f(x_{n-1}) - x_{n-1}\| \\
&= \frac{1}{1-\alpha}\left|\frac{(\alpha_{n-1}-\alpha_n)(1-\mu_n) + \alpha_n(\mu_{n-1}-\mu_n)}{\alpha_{n-1}(1-\mu_n)}\right| \|f(x_{n-1}) - x_{n-1}\| \\
&\le \frac{1}{1-\alpha}\left[\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}} + \frac{|\mu_{n-1}-\mu_n|}{1-\mu_n}\right] \|f(x_{n-1}) - x_{n-1}\|.
\end{aligned}
\tag{3.7}
$$

We estimate $\|z_n - z_{n+1}\|$. Let $c := \sup_{n\ge 1}\{(1-\mu_n)/\alpha_n\}$. Since the sequences $\{z_n\}$, $\{x_n\}$ and the set $\{\phi(t)/t : 0 < t < 1\}$ are bounded, let $\|z_n - Tz_n\| \le M$, $\|x_n - Tx_n\| \le M$, $\|f(z_n) - z_n\| \le M$, $\|f(x_n) - x_n\| \le M$ $\forall n \in \mathbb{N}$ and $\sup\{\phi(t)/t : 0 < t < 1\} \le M$ for some constant $M > 0$. Then

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \|\mu_n\alpha_n(Tz_n - z_n) + (1-\mu_n)(f(z_n) - z_n)\| \\
&\le \alpha_n\|Tz_n - z_n\| + (1-\mu_n)\|f(z_n) - z_n\| \\
&\le [\alpha_n + (1-\mu_n)]M \le \alpha_n(1+c)M,
\end{aligned}
\tag{3.8}
$$

for all $n \in \mathbb{N}$. It follows from (3.5) that

$$
\begin{aligned}
\|z_{n+1} - x_n\| \le{}& \left[1 - \frac{(1-\alpha)(1-\mu_n)}{1-\alpha_n\mu_n}\right]\|z_n - x_{n-1}\| + \frac{1}{1-\alpha}\left[\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}} - \frac{|\mu_n - \mu_{n-1}|}{1-\mu_n}\right]M \\
&+ \frac{\alpha_n}{1-\alpha_n\mu_n}\phi(\alpha_n(1+c)M) + \frac{(1-\mu_n)\alpha_n}{1-\alpha_n\mu_n}M.
\end{aligned}
\tag{3.9}
$$

There exists $N \in \mathbb{N}$ such that $\alpha_n(1+c)M < 1$ $\forall n \ge N$. Thus,

$$
\begin{aligned}
\|z_{n+1} - x_n\| \le{}& \left[1 - \frac{(1-\alpha)(1-\mu_n)}{1-\alpha_n\mu_n}\right]\|z_n - x_{n-1}\| \\
&+ \left[\frac{1}{1-\alpha}\left(\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}} - \frac{|\mu_n - \mu_{n-1}|}{1-\mu_n}\right)\right. \\
&\left. + \frac{\alpha_n^2(1+c)M}{1-\alpha_n\mu_n} + \frac{(1-\mu_n)\alpha_n}{1-\alpha_n\mu_n}\right]M, \quad \forall n \ge N.
\end{aligned}
\tag{3.10}
$$

Set $\beta_n := (1-\alpha)(1-\mu_n)/(1-\alpha_n\mu_n)$ and $\gamma_n := (1/(1-\alpha))((\alpha_{n-1}-\alpha_n)/\alpha_{n-1} - |\mu_n - \mu_{n-1}|/(1-\mu_n)) + \alpha_n^2(1+c)M/(1-\alpha_n\mu_n) + (1-\mu_n)\alpha_n/(1-\alpha_n\mu_n)$. Then the inequality

$$||z_{n+1}-x_n|| \le (1-\beta_n)||z_n-x_{n-1}|| + \gamma_n M \tag{3.11}$$

follows. By the assumptions on the sequences of numbers $\{\alpha_n\}$ and $\{\mu_n\}$ we find that $\gamma_n = o(\beta_n)$. Thus, by Lemma 2.2, $\|z_{n+1}-x_n\| \to 0$ as $n\to\infty$, so that

$$||z_n-x_n|| \le ||z_n-z_{n+1}|| + ||z_{n+1}-x_n|| \longrightarrow 0 \tag{3.12}$$

as $n\to\infty$.

Finally, we show that $\|z_n - Tz_n\| \to 0$ as $n\to\infty$.

Since $\|x_n - Tx_n\| = (1-\mu_n)/\alpha_n\|f(x_n)-x_n\| \le (1-\mu_n)/\alpha_n M \to 0$ as $n\to\infty$, and since T is uniformly continuous, we have that

$$||z_n-Tz_n|| \le ||z_n-x_n|| + ||x_n-Tx_n|| + ||Tx_n-Tz_n|| \longrightarrow 0 \tag{3.13}$$

as $n\to\infty$. Hence the proof of Theorem 3.1. □

Theorem 3.2. *Let K be a nonempty closed convex and bounded subset of a real reflexive Banach space E with a uniformly Gâteaux differentiable norm. Let $T: K\to K$ be a uniformly continuous pseudocontraction and let $f: K\to K$ be a contraction map. Suppose that every nonempty closed convex subset of K has the f.p.p. for nonexpansive self-mappings. Let $\{z_n\}$ be a sequence generated from an arbitrary $z_1\in K$ by (1.8), where $\{\mu_n\}$, $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying the same conditions in Theorem 3.1. Then $\{z_n\}$ converges strongly to the fixed point of T, which is the unique solution of the variational inequality (2.20).*

Proof. By Lemmas 2.3 and 2.5, a sequence $\{x_n\}$ given by $x_n = t_nTx_n + (1-t_n)f(x_n)$, with $t_n = \alpha_n/(1-\mu_n+\alpha_n)$, $n\in\mathbb{N}$ exists and converges strongly to the fixed point of T, which is the unique solution of the variational inequality (2.20). From the proof of Theorem 3.1, $\|z_n-x_n\|\to 0$ as $n\to\infty$. Hence, $\{z_n\}$ converges strongly to the same fixed point of T. □

Corollary 3.3. *Let K be a nonempty closed convex and bounded subset of a real Banach space E with a uniformly Gâteaux differentiable norm. Let $T: K\to K$ be a uniformly continuous pseudocontraction and let $f: K\to K$ be a contraction map. Suppose that K has normal structure. Let $\{z_n\}$ be a sequence generated from an arbitrary $z_1\in K$ by (1.8), where $\{\mu_n\}$, $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying the same conditions in Theorem 3.1. Then $\{z_n\}$ converges strongly to the fixed point of T, which is the unique solution of the variational inequality (2.20).*

Corollary 3.4. *Let K be a nonempty closed convex and bounded subset of a real Banach space E with a uniformly Gâteaux differentiable norm and let $T: K\to K$ be a unformly continuous pseudocontraction. Suppose that every nonempty closed convex subset of K has the f.p.p. for nonexpansive self-mappings. Fix any $w\in K$ and let $\{z_n\}$ be a sequence generated from an arbitrary $z_1\in K$ by (1.2), where $\{\mu_n\}$, $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying the same conditions in Theorem 3.1. Then $\{z_n\}$ converges strongly to the fixed point of T, which is the unique solution of the variational inequality (2.20).*

Remarks 3.5. (A) If the map T is assumed to be Lipschitz in the above results then the condition that the set K or the sequence $\{z_n\}_n$ be bounded can be dropped. It is proved in [10] that, in this case, the sequence $\{z_n\}_n$ is bounded.

(B) It is clear that the conditions on the iteration parameters $\{\alpha_n\}$, $\{\mu_n\}$ in Theorems 3.1, 3.2 and Corollaries 3.3, 3.4 are much simpler than those imposed on the parameters in Theorem 1.1. Examples of real sequences $\{\mu_n\}$ and $\{\alpha_n\}$ that satisfy the conditions (i), (ii), and (iii) of Theorem 3.1 are

$$\mu_n = 1-(n+1)^{-1/2} \text{ and } \alpha_n = (n+1)^{-1/3}, \tag{3.14}$$

respectively.

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References

[1] K. Deimling, *Zeros of accretive operators*, Manuscripta Mathematica **13** (1974), no. 4, 365–374.
[2] B. Halpern, *Fixed points of nonexpanding maps*, Bulletin of the American Mathematical Society **73** (1967), 957–961.
[3] T. Kato, *Nonlinear semigroups and evolution equations*, Journal of the Mathematical Society of Japan **19** (1967), 508–520.
[4] T.-C. Lim, *On characterizations of Meir-Keeler contractive maps*, Nonlinear Analysis. Theory, Methods & Applications. Series A: Theory and Methods **46** (2001), no. 1, 113–120.
[5] R. H. Martin Jr., *Differential equations on closed subsets of a Banach space*, Transactions of the American Mathematical Society **179** (1973), 399–414.
[6] C. H. Morales, *On the fixed-point theory for local k-pseudocontractions*, Proceedings of the American Mathematical Society **81** (1981), no. 1, 71–74.
[7] C. H. Morales and J. S. Jung, *Convergence of paths for pseudo-contractive mappings in Banach spaces*, Proceedings of the American Mathematical Society **128** (2000), no. 11, 3411–3419.
[8] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, Journal of Mathematical Analysis and Applications **241** (2000), no. 1, 46–55.
[9] J. Schu, *Approximating fixed points of Lipschitzian pseudocontractive mappings*, Houston Journal of Mathematics **19** (1993), no. 1, 107–115.
[10] A. Udomene, *Path convergence, approximation of fixed points and variational solutions of pseudo-contractions in Banach spaces*, submitted to Nonlinear Analysis, TMA.
[11] H.-K. Xu, *Iterative algorithms for nonlinear operators*, Journal of the London Mathematical Society. Second Series **66** (2002), no. 1, 240–256.
[12] ———, *Viscosity approximation methods for nonexpansive mappings*, Journal of Mathematical Analysis and Applications **298** (2004), no. 1, 279–291.

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