Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 50596, 9 pages
doi:10.1155/2007/50596

Research Article

Approximating Fixed Points of Nonexpansive Mappings in Hyperspaces

Zeqing Liu, Chi Feng, Shin Min Kang, and Jeong Sheok Ume

Received 29 March 2007; Revised 27 August 2007; Accepted 13 September 2007

Recommended by Wataru Takahashi

Two convergence theorems for the Ishikawa and Mann iteration sequences involving nonexpansive mappings in hyperspaces are established.

1. Introduction and preliminaries

Let X be a nonempty compact subset of a Banach space $(E,\|\cdot\|)$, and let $C(X)$ and $CC(X)$ denote the families of all nonempty compact and all nonempty compact convex subsets of X, respectively. It is well known that $(C(X),H)$ is compact, where H is the Hausdorff metric induced by $\|\cdot\|$. For $A,B \in CC(X)$ and $t \in \mathbb{R} = (-\infty,+\infty)$, let $A+B = \{a+b : a \in A,\ b \in B\}$, and let $tA = \{ta : a \in A\}$. In the sequel, we assume that X is a nonempty compact convex subset of E. Hu and Huang [1] proved that $(CC(X),H)$ is a compact subset of $(C(X),H)$. It is clear that $tA+(1-t)B \in CC(X)$ for all $A,B \in CC(X)$ and $t \in [0,1]$. That is, $CC(X)$ has convexity structure. Let $\mathfrak{I}$ be a nonempty subset of $CC(X)$. A mapping $T : (\mathfrak{I},H)\to(\mathfrak{I},H)$ is said to be *nonexpansive* if $H(TA,TB) \le H(A,B)$ for all $A,B \in \mathfrak{I}$.

Within the past 20 years or so, a few researchers have applied the Mann iteration method and the Ishikawa iteration method to approximate fixed points of nonexpansive mappings in several classes of subsets of Banach spaces. For details we refer to [2–11]. Recently, Hu and Huang [1] established the following result.

Theorem 1.1. *Let X be a nonempty compact convex subset of a Banach space $(E,\|\cdot\|)$, and let $\mathfrak{I}$ be a nonempty compact convex subset of $CC(X)$. Suppose that $T : (\mathfrak{I},H)\to(\mathfrak{I},H)$ is nonexpansive. Then for any $A_0 \in \mathfrak{I}$, the sequence defined by*

$$A_n = 2^{-1}(A_{n-1} + TA_{n-1}), \quad n \geq 1, \tag{1.1}$$

converges to a fixed point of T.

Inspired and motivated by the results in [1–11], in this paper we introduce the concepts of the Mann and Ishikawa iteration sequences in hyperspaces, and establish the convergence theorems for the Mann and Ishikawa iteration sequences dealing with nonexpansive mappings in hyperspaces. The results in this paper extend substantially Theorem 1.1.

In order to prove our results, we need the following concepts and results.

Definition 1.2. Let $\mathfrak{I}$ be a nonempty compact convex subset of $CC(X)$, and let $T:(\mathfrak{I},H)\to(\mathfrak{I},H)$ be a mapping.

(1) For any $A_0 \in \mathfrak{I}$, the sequence $\{A_n\}_{n\geq 0} \subseteq \mathfrak{I}$ defined by

$$\begin{aligned} B_n &= (1-s_n)A_n + s_n TA_n, \quad n \geq 0, \\ A_{n+1} &= (1-t_n)A_n + t_n TB_n, \quad n \geq 0, \end{aligned} \tag{1.2}$$

is called the *Ishikawa iteration sequence*, where $\{t_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ are real sequences in $[0,1]$ satisfying appropriate conditions.

(2) If $s_n = 0$ for all $n \geq 0$ in (1.2), the sequence $\{A_n\}_{n\geq 0} \subseteq \mathfrak{I}$ defined by

$$A_{n+1} = (1-t_n)A_n + t_n TA_n, \quad n \geq 0, \tag{1.3}$$

is called the *Mann iteration sequence*.

(3) If $s_n = 0$ and $t_n = 1$ for all $n \geq 0$ in (1.2), the sequence $\{A_n\}_{n\geq 0} \subseteq \mathfrak{I}$ defined by

$$A_{n+1} = TA_n, \quad n \geq 0, \tag{1.4}$$

is called the *Picard iteration sequence.*

Lemma 1.3. *Let A, B, U, and V be in $CC(X)$, and let t be in $[0,1]$. Then*

$$H\big(tA + (1-t)B, tU + (1-t)V\big) \leq tH(A,U) + (1-t)H(B,V). \tag{1.5}$$

Proof. Put $r = tH(A,U) + (1-t)H(B,V)$. For any $a \in A$ and $b \in B$, by Nadler's result we know that there exist $u \in U$, $v \in V$ such that $\|a-u\| \leq H(A,U)$ and $\|b-v\| \leq H(B,V)$ which yield that

$$\big|\big|ta + (1-t)b - tu - (1-t)v\big|\big| \leq t\|a-u\| + (1-t)\|b-v\| \leq r. \tag{1.6}$$

It follows that

$$\sup_{a\in A,b\in B}\left\{\inf_{u\in U,v\in V}\|ta+(1-t)b-tu-(1-t)v\|\right\}\leq r. \tag{1.7}$$

Similarly, we have

$$\sup_{u\in U,v\in V}\left\{\inf_{a\in A,b\in B}\|ta+(1-t)b-tu-(1-t)v\|\right\}\leq r. \tag{1.8}$$

Consequently, we infer that

$$\begin{aligned}
&H(tA+(1-t)B,tU+(1-t)V)\\
&\quad=\max\left\{\sup_{a\in A,b\in B}\inf_{u\in U,v\in V}\|ta+(1-t)b-tu-(1-t)v\|,\right.\\
&\qquad\qquad\left.\sup_{u\in U,v\in V}\inf_{a\in A,b\in B}\|ta+(1-t)b-tu-(1-t)v\|\right\}\leq r.
\end{aligned} \tag{1.9}$$

This completes the proof. □

Lemma 1.4 [9]. *Suppose that $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two sequences of nonnegative numbers such that $a_{n+1}\leq a_n+b_n$ for all $n\geq 0$. If $\sum_{n=0}^{\infty}b_n$ converges, then $\lim_{n\to\infty}a_n$ exists.*

2. Main results

Now we prove the following results.

Theorem 2.1. *Let X be a nonempty compact convex subset of a Banach space $(E,\|\cdot\|)$, and let $\mathfrak{I}$ be a nonempty compact convex subset of $CC(X)$. Suppose that $T:(\mathfrak{I},H)\to(\mathfrak{I},H)$ is nonexpansive and there exist constants a and b satisfying that*

$$0<a\leq t_n\leq b<1,\quad 0\leq s_n\leq 1,\quad n\geq 0, \tag{2.1}$$

$$\sum_{n=0}^{\infty}s_n<\infty. \tag{2.2}$$

Then for any $A_0\in\mathfrak{I}$, the Ishikawa iteration sequence $\{A_n\}_{n\geq 0}$ converges to a fixed point of T.

Proof. Let n and k be arbitrary nonnegative integers. Note that $tA+(1-t)A=A$ for any $A\in CC(X)$ and $t\in[0,1]$. Using (1.2), Lemma 1.3 and the nonexpansiveness of T, we infer that

$$\begin{aligned}
H(TB_n,A_n)&\leq H(TB_n,TA_n)+H(TA_n,A_n)\\
&\leq H(B_n,A_n)+H(TA_n,A_n)\leq(1+s_n)H(A_n,TA_n),
\end{aligned} \tag{2.3}$$

and that

$$H(A_{n+1},A_n) \le t_n H(TB_n,A_n) \le t_n(1+s_n)H(A_n,TA_n). \tag{2.4}$$

By virtue of (1.2), (2.3), (2.4), Lemma 1.3, and the nonexpansiveness of T, we get that

$$\begin{aligned}
&H(B_n,A_{n+k+1})\\
&\quad\le H(B_n,A_{n+1})+\sum_{i=1}^{k}H(A_{n+i},A_{n+i+1})\\
&\quad\le (1-s_n)H(A_n,A_{n+1})+s_nH(TA_n,A_{n+1})+\sum_{i=1}^{k}t_{n+i}(1+s_{n+i})H(A_{n+i},TA_{n+i})\\
&\quad\le (1-s_n^2)t_nH(A_n,TA_n)+s_n\left[(1-t_n)H(A_n,TA_n)+t_nH(TB_n,TA_n)\right]\\
&\qquad+\sum_{i=1}^{k}(t_{n+i}+s_{n+i})H(A_{n+i},TA_{n+i})\\
&\quad\le (t_n+s_n(1-t_n))H(A_n,TA_n)+\sum_{i=1}^{k}(t_{n+i}+s_{n+i})H(A_{n+i},TA_{n+i})\\
&\quad\le \sum_{i=0}^{k}(t_{n+i}+s_{n+i})H(A_{n+i},TA_{n+i}),
\end{aligned} \tag{2.5}$$

and that

$$\begin{aligned}
H(TA_{n+1},A_{n+1}) &\le (1-t_n)H(A_n,TA_{n+1})+t_nH(TB_n,TA_{n+1})\\
&\le (1-t_n)(H(A_{n+1},TA_{n+1})+H(A_{n+1},A_n))+t_nH(B_n,A_{n+1})\\
&\le (1-t_n)H(A_{n+1},TA_{n+1})+(1-t_n)t_n(1+s_n)H(A_n,TA_n)\\
&\quad+t_n((1-t_n)H(A_n,B_n)+t_nH(TB_n,B_n)),
\end{aligned} \tag{2.6}$$

which together with (2.1) implies that

$$\begin{aligned}
H(A_{n+1},TA_{n+1}) &\le (1-t_n)(1+s_n)H(A_n,TA_n)\\
&\quad+(1-t_n)H(A_n,B_n)+t_nH(TB_n,B_n)\\
&\le (1-t_n)(1+2s_n)H(A_n,TA_n)\\
&\quad+t_n((1-s_n)H(A_n,TB_n)+s_nH(TA_n,TB_n))\\
&\le (1+2s_n(1-t_n))H(A_n,TA_n)\\
&\le (1+2(1-a)s_n)H(A_n,TA_n).
\end{aligned} \tag{2.7}$$

Notice that the compactness of $\Im$ implies that $\{H(A_n, TA_k) : n \geq 0, k \geq 0\}$ is bounded. It follows from Lemma 1.4, (2.2), and (2.7) that

$$\lim_{n\to\infty} H(A_n, TA_n) = r \geq 0, \tag{2.8}$$

which implies that for any $\varepsilon > 0$ there exists a positive integer N such that

$$r - \varepsilon \leq H(A_n, TA_n) \leq r + \varepsilon, \quad n \geq N. \tag{2.9}$$

It follows that

$$\begin{aligned} H(A_{n+1}, TC) &\leq (1-t_n)H(A_n, TC) + t_n H(TB_n, TC) \\ &\leq (1-t_n)H(A_n, TC) + t_n H(B_n, C), \quad C \in \Im,\ n \geq 0, \end{aligned} \tag{2.10}$$

which yields that

$$H(A_n, TC) \geq (1-t_n)^{-1}\big(H(A_{n+1}, TC) - t_n H(B_n, C)\big), \quad C \in \Im,\ n \geq 0. \tag{2.11}$$

Now we prove by induction that the following inequality holds for all $n \geq 1$:

$$\begin{aligned} H(A_p, TA_{p+n}) \geq{}& (r+\varepsilon)\left(1 + \sum_{i=0}^{n-1} t_{p+i}\right) - 2\varepsilon \prod_{i=0}^{n-1}(1-t_{p+i})^{-1} \\ &- (r+\varepsilon)\sum_{i=0}^{n-1}\left[t_{p+i}\left(\sum_{j=i}^{n-1} s_{p+j}\right)\prod_{k=0}^{i}(1-t_{p+k})^{-1}\right], \quad p \geq N. \end{aligned} \tag{2.12}$$

Using (2.5), (2.9), and (2.11), we obtain that

$$\begin{aligned} H(A_p, TA_{p+1}) &\geq (1-t_p)^{-1}\big(H(A_{p+1}, TA_{p+1}) - t_p H(B_p, A_{p+1})\big) \\ &\geq (1-t_p)^{-1}\big(r - \varepsilon - (r+\varepsilon)t_p(t_p + s_p)\big) \\ &= (1-t_p)^{-1}\big[r - \varepsilon - (r+\varepsilon)\big(1 - 2(1-t_p) + (1-t_p)^2 + t_p s_p\big)\big] \\ &= (r+\varepsilon)(1+t_p) - 2\varepsilon(1-t_p)^{-1} - (r+\varepsilon)t_p s_p (1-t_p)^{-1}, \quad p \geq N. \end{aligned} \tag{2.13}$$

Hence (2.12) holds for $n = 1$. Suppose that (2.12) holds for $n = m \geq 1$. That is,

$$\begin{aligned} H(A_p, TA_{p+m}) \geq{}& (r+\varepsilon)\left(1 + \sum_{i=0}^{m-1} t_{p+i}\right) - 2\varepsilon \prod_{i=0}^{m-1}(1-t_{p+i})^{-1} \\ &- (r+\varepsilon)\sum_{i=0}^{m-1}\left[t_{p+i}\left(\sum_{j=i}^{m-1} s_{p+j}\right)\prod_{k=0}^{i}(1-t_{p+k})^{-1}\right], \quad p \geq N. \end{aligned} \tag{2.14}$$

According to (2.5), (2.9), (2.11), and (2.14), we infer that

$$
\begin{aligned}
&H(A_p, TA_{p+m+1}) \\
&\quad \geq (1-t_p)^{-1}\left(H(A_{p+1}, TA_{p+m+1}) - t_p H(B_p, A_{p+m+1})\right) \\
&\quad \geq (1-t_p)^{-1}\Bigg\{(r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{p+1+i}\right) - 2\varepsilon\prod_{i=0}^{m-1}(1-t_{p+1+i})^{-1} \\
&\qquad\qquad - (r+\varepsilon)\left[\sum_{i=0}^{m-1} t_{p+1+i}\left(\sum_{j=i}^{m-1} s_{p+1+j}\right)\prod_{k=0}^{i}(1-t_{p+1+k})^{-1}\right] \\
&\qquad\qquad - (r+\varepsilon)t_p\sum_{i=0}^{m}(t_{p+i}+s_{p+i})\Bigg\} \\
&\quad = (r+\varepsilon)(1-t_p)^{-1}\left[1+\sum_{i=0}^{m-1} t_{p+1+i} - \left(t_p^2 + t_p\sum_{i=1}^{m} t_{p+i} + t_p\sum_{i=0}^{m} s_{p+i}\right)\right] \\
&\qquad - 2\varepsilon\prod_{i=0}^{m}(1-t_{p+i})^{-1} - (r+\varepsilon)(1-t_p)^{-1}\sum_{i=0}^{m-1}\left[t_{p+1+i}\left(\sum_{j=i}^{m-1} s_{p+1+j}\right)\prod_{k=0}^{i}(1-t_{p+1+k})^{-1}\right] \\
&\quad = (r+\varepsilon)\left(1+\sum_{i=0}^{m} t_{p+i}\right) - (r+\varepsilon)(1-t_p)^{-1} t_p\sum_{i=0}^{m} s_{p+i} \\
&\qquad - 2\varepsilon\prod_{i=0}^{m}(1-t_{p+i})^{-1} - (r+\varepsilon)\sum_{i=1}^{m}\left[t_{p+i}\left(\sum_{j=i}^{m} s_{p+j}\right)\prod_{k=0}^{i}(1-t_{p+k})^{-1}\right] \\
&\quad = (r+\varepsilon)\left(1+\sum_{i=0}^{m} t_{p+i}\right) - 2\varepsilon\prod_{i=0}^{m}(1-t_{p+i})^{-1} \\
&\qquad - (r+\varepsilon)\sum_{i=0}^{m}\left[t_{p+i}\left(\sum_{j=i}^{m} s_{p+j}\right)\prod_{k=0}^{i}(1-t_{p+k})^{-1}\right], \quad p \geq N.
\end{aligned}
\tag{2.15}
$$

That is, (2.12) holds for $n = m+1$. Hence (2.12) holds for any $n \geq 1$.

We next assert that $r = 0$. Otherwise $r > 0$. Let m be an arbitrary positive integer, and let $\varepsilon = 2^{-1}(1-b)^m \min\{r, 1\}$. It follows from (2.2) and (2.8) that there exists a positive integer $N = N(\varepsilon)$ satisfying (2.9) and that

$$\left|\sum_{i=0}^{q} s_{n+i}\right| \leq \varepsilon, \quad n \geq N,\ q \geq 0. \tag{2.16}$$

According to (2.1), (2.2), (2.9), (2.12), and (2.16), we easily conclude that

$$\begin{aligned}
&H(A_N, TA_{N+m}) \\
&\quad \geq (r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{N+i}\right) - 2\varepsilon\prod_{i=0}^{m-1}\left(1-t_{N+i}\right)^{-1} \\
&\qquad -(r+\varepsilon)\sum_{i=0}^{m-1}\left[t_{N+i}\left(\sum_{j=i}^{m-1} s_{N+j}\right)\prod_{k=0}^{i}\left(1-t_{N+k}\right)^{-1}\right] \\
&\quad \geq (r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{N+i}\right) - 2\varepsilon(1-b)^{-m} - (r+\varepsilon)\varepsilon\sum_{i=0}^{m-1} t_{N+i}(1-b)^{-i-1} \\
&\quad \geq r+\varepsilon - 2\varepsilon(1-b)^{-m} + (r+\varepsilon)\left(1-\varepsilon(1-b)^{-m}\right)\sum_{i=0}^{m-1} t_{N+i} \\
&\quad \geq r+\varepsilon - 2\cdot 2^{-1} r(1-b)^{m}(1-b)^{-m} \\
&\qquad +(r+\varepsilon)\left(1-2^{-1}(1-b)^{m}(1-b)^{-m}\right)\sum_{i=0}^{m-1} t_{N+i} \\
&\quad \geq 2^{-1} r\sum_{i=0}^{m-1} t_{N+i} \geq 2^{-1} rma \longrightarrow +\infty \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{2.17}$$

That is, $\{H(A_n, TA_k) : n \geq 0, k \geq 0\}$ is unbounded, which is a contradiction. Hence $r = 0$. The compactness of $\mathfrak{I}$ yields that there exists a subsequence $\{A_{n_k}\}_{k\geq 0}$ of $\{A_n\}_{n\geq 0}$ satisfying that

$$\lim_{k\to\infty} H(A_{n_k}, A) = 0 \quad \text{for some } A \in \mathfrak{I}. \tag{2.18}$$

In view of (2.8), (2.18) and the nonexpansiveness of T, we have

$$\begin{aligned}
H(A, TA) &\leq H(A, A_{n_k}) + H(A_{n_k}, TA_{n_k}) + H(TA_{n_k}, TA) \\
&\leq 2H(A, A_{n_k}) + H(A_{n_k}, TA_{n_k}) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned} \tag{2.19}$$

That is, $A = TA$. From (1.2) and Lemma 1.3, we know that

$$\begin{aligned}
H(A_{n+1}, A) &\leq (1-t_n)H(A_n, A) + t_n H(TB_n, A) \\
&\leq (1-t_n)H(A_n, A) + t_n H(B_n, A) \\
&\leq (1-t_n)H(A_n, A) + t_n\left((1-s_n)H(A_n, A) + s_n H(TA_n, A)\right) \\
&\leq H(A_n, A), \quad n \geq 0.
\end{aligned} \tag{2.20}$$

It follows from (2.18) and (2.20) that $\lim_{n\to\infty} H(A_n, A) = 0$. This completes the proof. □

From Theorem 2.1 we have the following.

Theorem 2.2. *Let X be a nonempty compact convex subset of a Banach space $(E, \|\cdot\|)$, and let $\mathfrak{I}$ be a nonempty compact convex subset of $CC(X)$. Suppose that $T : (\mathfrak{I}, H) \to (\mathfrak{I}, H)$ is* *nonexpansive and there exist constants a and b satisfying that*

$$0 < a \le t_n \le b < 1, \quad n \ge 0. \tag{2.21}$$

Then for any $A_0 \in \mathfrak{I}$*, the Mann iteration sequence* $\{A_n\}_{n\ge 0}$ *converges to a fixed point of* T*.*

Remark 2.3. In case $t_n = 1/2$ for all $n \ge 0$, Theorem 2.2 reduces to [1, Theorem 3.2] by Hu and Huang. The following example reveals that Theorem 2.2 extends properly the result of Hu and Huang.

Example 2.4. Let $E = \mathbb{R}$ with the usual norm $|\cdot|$, $X = [0,1]$, and let $\mathfrak{I} = \{[0,x] : x \in X\}$. Define $T : (\mathfrak{I}, H) \to (\mathfrak{I}, H)$ by

$$T[0,x] = [0,1-x], \quad x \in X. \tag{2.22}$$

Then $\mathfrak{I}$ is a nonempty compact convex subset of $CC(X)$ and

$$H\big(T[0,x], T[0,y]\big) = |x-y| = H\big([0,x],[0,y]\big), \quad x,y \in X. \tag{2.23}$$

That is, T is nonexpansive. Set $t_n = (n+1)/(10n+3)$ for all $n \ge 0$ and $a = 1/10$, $b = 1/3$. Thus all conditions of Theorem 2.2 are fulfilled. Therefore, we may invoke our Theorem 2.2 to show that T has a fixed point in $\mathfrak{I}$; but we cannot invoke [1, Theorem 3.2] by Hu and Huang to show that T has fixed points in $\mathfrak{I}$ since $t_n \ne 1/2$ for all $n \ge 0$.

Remark 2.5. The example below shows that the Picard iteration sequences of nonexpansive mappings in hyperspaces need not converge and the condition "$t_n \le b < 1$, $n \ge 0$" in Theorem 2.2 is necessary.

Example 2.6. Let E, X, $\mathfrak{I}$, and T be as in Example 2.4. Take $t_n = 1$ for all $n \ge 0$. For any $A_0 = [0,x]$ with $x \in X \setminus \{1/2\}$, the Picard iteration sequence $\{A_n\}_{n\ge 0} \subset \mathfrak{I}$ does not converge since $A_{2n} = [0,x]$ for all $n \ge 0$ and $A_{2n-1} = [0,1-x]$ for all $n \ge 1$.

Acknowledgments

The authors thank the referees sincerely for their valuable and useful comments and suggestions. This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (20060467) and the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-312-C00026).

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Zeqing Liu: Department of Mathematics, Liaoning Normal University, P.O. Box 200, Dalian, Liaoning 116029, China
Email address: zeqingliu@sina.com.cn

Chi Feng: Department of Science, Dalian Fisheries College, Dalian, Liaoning 116023, China
Email address: chifeng@x.cn

Shin Min Kang: Department of Mathematics and the Research Institute of Natural Science, Gyeongsang National University, Jinju 660-701, South Korea
Email address: smkang@nongae.gsnu.ac.kr

Jeong Sheok Ume: Department of Applied Mathematics, Changwon National University, Changwon 641-733, South Korea
Email address: jsume@sarim.changwon.ac.kr