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Research Article

Iterative Approximation of a Common Zero of a Countably Infinite Family of m-Accretive Operators in Banach Spaces

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Let E be a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm and let C be a closed convex nonempty subset of E. Strong convergence theorems for approximation of a common zero of a countably infinite family of m-accretive mappings from C to E are proved. Consequently, we obtained strong convergence theorems for a countably infinite family of pseudocontractive mappings.

1. Introduction

Let E be a real Banach space with dual E^*. The *normalized duality mapping* is the mapping $J : E \to 2^{E^*}$ defined for all $x \in E$ by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \ \|f^*\| = \|x\|\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between members of E and E^*. It is well known that if E^* is strictly convex, then J is single valued. In what follows, the single-valued normalized duality mapping will be denoted by j.

Let $(E, \|\cdot\|)$ be a normed linear space. The norm $\|\cdot\|$ is said to be *uniformly Gâteaux differentiable* if for each $y \in S = \{x \in E : \|x\| = 1\}$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists uniformly for $x \in S$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see, e.g., [1]). Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E.

Let C be a nonempty subset of a normed linear space E. A mapping $T : C \to E$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.3}$$

Most published results on nonexpansive mappings centered on existence theorems for fixed points of these mappings, and iterative approximation of such fixed points.

DeMarr [2] in 1963 studied the problem of existence of common fixed point for a family of nonlinear nonexpansive mappings. He proved the following theorem.

Theorem 1.1 (DM). *Let E be a Banach space and C be a nonempty compact convex subset of E. If Ω is a nonempty commuting family of nonexpansive mappings of C into itself, then the family Ω has a common fixed point in C.*

In 1965, Browder [3] proved the result of DeMarr in a uniformly convex Banach space, requiring that C be only bounded, closed, convex, and nonempty. For other fixed-point theorems for families of nonexpansive mappings, the reader may consult Belluce and Kirk [4], Lim [5], and Bruck Jr. [6].

In 1973, Bruck Jr. [7] considered the study of structure of the fixed-point set $F(T) = \{x \in C : Tx = x\}$ of nonexpansive mapping T and established several results.

Kirk [8] introduced an iterative process given by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \cdots + \alpha_r T^r x_n, \tag{1.4}$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$ and $\sum_{i=0}^{r} \alpha_i = 1$, for approximating fixed points of nonexpansive mappings on convex subset of uniformly convex Banach spaces. Maiti and Saha [9] worked on and improved the results of Kirk [8].

Considerable research efforts have been devoted to develop iterative methods for approximating common fixed points (when such fixed points exist) of families of several classes of nonlinear mappings (see, e.g., [10–18]).

Let C be a nonempty closed and bounded subset of a real Banach space E. Let $T_i : C \to C$, $i = 1, 2, \ldots, r$ be a finite family of nonexpansive mappings and let

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_r T_r, \tag{1.5}$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{r} \alpha_i = 1$. Then the family $\{T_i\}_{i=1}^{r}$ such that the common fixed-point set $F := \bigcap_{i=1}^{r} F(T_i) \neq \varnothing$ is said to satisfy condition A (see, e.g., [9, 19, 20]) if there exists a nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$, $\phi(\varepsilon) > 0$ for all $\varepsilon \in (0, +\infty)$, such that $\|x - Sx\| \geq \phi(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - z\| : z \in F\}$.

Liu et al. [19] introduced the following iteration process:

$$x_0 \in C, \quad x_{n+1} = Sx_n, \quad n \geq 0 \tag{1.6}$$

and showed that $\{x_n\}_{n \geq 0}$ defined by (1.6) converges to a common fixed point of $\{T_i\}_{i=1}^{r}$ in Banach spaces, provided that $\{T_i\}_{i=1}^{r}$ satisfy condition A. The result of Liu et al. [19] improves the corresponding results of Kirk [8], Maiti and Saha [9], Senter and Dotson [20] and those of a host of other authors. However, the assumption that the family $\{T_i\}_{i=1}^r$ satisfies condition A is strong.

Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $T_i : E \to E$, $i = 1,2,\dots,r$ be nonexpansive mappings and $\{x_n\}_{n\geq 0}$ a sequence in E defined iteratively by (1.6) and suppose that $J^{-1} : E^* \to E$ is weakly sequentially continuous at 0. If $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$, then Jung [21] in 2002 proved that, under this situation, $\{x_n\}_{n\geq 0}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$. In [22], Gossez and Lami Dozo proved that for any normed linear space E, the existence of a weakly sequentially continuous duality mapping implies that the space E satisfies Opial's condition (that is, for all sequences $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality $\liminf_{n\to\infty}\|x_n - y\| > \liminf_{n\to\infty}\|x_n - x\|$ holds for all $y \neq x$, see e.g., [23]). It is well known that L_p spaces, $1 < p < +\infty$, $p \neq 2$, do not satisfy Opial's condition. Consequently, the results of Jung [21] are not applicable in L_p spaces $1 < p < +\infty$, $p \neq 2$.

Another class of nonlinear mappings now studied is the class of accretive operators. Let E be a real normed linear space. A mapping $A : D(A) \subset E \to E$ is said to be *accretive* if the following inequality holds:

$$\|x - y\| \leq \big\|x - y + s(Ax - Ay)\big\| \quad \forall s > 0, \forall x, y \in D(A), \tag{1.7}$$

where $D(A)$ denotes the domain of the operator A. It is not difficult to deduce from (1.7) that the mapping A is accretive if and only if $(I + sA)^{-1}$ is nonexpansive on the range of $(I + sA)$, where I denotes the identity operator defined on E. We note that the range, $R(I+sA)$, of $(I+sA)$ needs not be all of E. When A is accretive and, in addition, the range of $(I + sA)$ is all of E, then A is called *m-accretive.*

Our presentation in this paper is primarily motivated by the study of equations of the form

$$u'(t) + Au(t) = f, \quad u(0) = u_0, \quad f \in E. \tag{1.8}$$

It is well known that many physically significant problems can be modeled by equations of the form (1.8) (where A is accretive), which is generally called *Evolution Equation.* Typical examples where such evolution equations occur can be found in the heat, wave, and Schrödinger equations (see, e.g., [24]). One of the fundamental results in the theory of accretive operators, due to Browder [25], states that if A is locally Lipschitzian and accretive, then A is m-accretive and this implies that (1.8) has a solution $u^* \in D(A)$ for any $f \in E$ (in particular for $f = 0$). This result was subsequently generalized by Martin [26] to continuous accretive operators. If in (1.8), $f = 0$ and $u(t)$ is independent of t, then (1.8) reduces to

$$Au = 0 \tag{1.9}$$

whose solutions correspond to the equilibrium points of (1.8). There is no known method to obtain a closed form solution of (1.9). The general approach for approximating a solution of (1.9) is to transform it into a fixed-point problem. Defining $T := I - A$, we observe that x^* is a solution of (1.9) if and only if x^* is a fixed point of T (i.e., $x^* \in Tx^*$). Browder [25] called such an operator T *pseudocontractive*.

Consequently, the study of methods of approximating fixed points of pseudocontractive maps, which correspond to equilibrium points of the system (1.8), became a flourishing area of research for numerous mathematicians (see, e.g., [27–31]).

Remark 1.2. We observe that a mapping $A := I - T$ is accretive if and only if the mapping T is pseudocontractive. It is, therefore, not difficult to see (using (1.7)) that every nonexpansive mapping is pseudocontractive. The converse, however, does not hold. The following illustrates this fact.

Example 1.3. Let $T : [0,1] \to (\mathbb{R}, |\cdot|)$ be defined by

$$Tx = \begin{cases} x - \dfrac{1}{2} & \text{if } x \in \left[0, \dfrac{1}{2}\right), \\ x - 1 & \text{if } x \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{1.10}$$

Clearly, T is not continuous and thus cannot be nonexpansive. Now, let $s > 0$, then for $x, y \in [0, 1/2) \cup (1/2, 1]$ we obtain that $|x - y + s((I-T)x - (I-T)y)| \geq |x - y|$. So, T is pseudocontrative but not nonexpansive. Thus, the class of pseudocontractive mappings properly contains the class of nonexpansive mappings. Moreover, we see in particular that the operator A is accretive, if and only if the mapping $J_A := (I + A)^{-1}$ is a single-valued nonexpansive mapping from $R(I + A)$ to $D(A)$ and that $F(J_A) = N(A)$, where $N(A) = \{x \in D(A) : Ax = 0\}$ and $F(J_A) = \{x \in E : J_A x = x\}$. (see, e.g., [1]).

Let C be a nonempty closed convex subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $A_i : C \to E$, $i = 1, 2, \ldots, r$ be *a finite family* of m-accretive mappings with $N = \bigcap_{i=1}^{r} N(A_i) \neq \emptyset$. Suppose that every bounded closed convex nonempty subset of E has the fixed-point property for nonexpansive mappings; Zegeye and Shahzad [32] constructed an iterative sequence which converges strongly to a common solution of the equations $A_i x = 0$, $i = 1, 2, \ldots, r$.

It is our purpose in this paper to construct an iterative algorithm for the approximation of a common zero of a *countably infinite family of* m*-accretive operators* in Banach spaces. As a result, we obtain strong convergence theorems for approximation of a common fixed point of a *countably inftinite family* $\{T_k\}_{k \in \mathbb{N}}$ of *pseudocontractive mappings*, provided that $I - T_k$ is m-accretive for all $k \in \mathbb{N}$. Our theorems improve, generalize, and extend the correponding results of Zegeye and Shahzad [32] and several other results recently announced (see Remark 3.18 of this paper) from a *finite family* $\{A_i\}_{i=1}^{r}$ of m-accretive mappings to *a countably infinite family* $\{A_k\}_{k \in \mathbb{N}}$ of m-accretive mappings. Furthermore, our theorems are applicable, in particular in L_p spaces $1 < p < +\infty$, and our method of proof is of independent interest.

2. Preliminaries

In the sequel, the following Lemmas and Theorems will be used.

Lemma 2.1 (see, e.g., [18, 27, 33]). *Let $\{\lambda_n\}_{n \geq 1}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \sigma_n, \quad n \geq 0, \tag{2.1}$$

where $\{\alpha_n\}_{n \geq 0}$ and $\{\sigma_n\}_{n \geq 0}$ are sequences of real numbers such that $\{\alpha_n\}_{n \geq 1} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Suppose that $\sigma_n = o(\alpha_n)$, $n \geq 0$ (i.e., $\lim_{n \to \infty}(\sigma_n / \alpha_n) = 0$) or $\sum_{n=1}^{\infty} |\sigma_n| < +\infty$ or $\limsup_{n \to \infty}(\sigma_n / \alpha_n) \leq 0$, then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.2. *Let E be a real normed linear space. Then the following inequality holds: for all $x, y \in E$, for all $j(x+y) \in J(x+y)$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.2}$$

Lemma 2.3 (see [7, Lemma 3, page 257]). *Let C be a nonempty closed and convex subset of a real strictly convex Banach space E. Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of nonself nonexpansive mappings $T_k : C \to E$. Then there exists a nonexpansive mapping $T : C \to E$ such that $F(T) = \bigcap_{k=1}^{\infty} F(T_k)$.*

Proof. If the sequence $\{T_k\}_{k\in\mathbb{N}}$ does not have a common fixed point, we can assume T to be translation by any nonzero vector in which case $F(T) = \bigcap_{k=1}^{\infty} F(T_k) = \varnothing$. Otherwise, let x^* be a common fixed point of $\{T_k\}_{k\in\mathbb{N}}$. Let $\{\xi_k\}_{k\geq 1}$ be any sequence of positive real numbers such that $\sum_{k=1}^{\infty}\xi_k = 1$ and set $T := \sum_{k=1}^{\infty}\xi_k T_k$. Then the mapping T is well defined, since

$$\left\|T_k x\right\| \leq \left\|T_k x - T_k x^*\right\| + \left\|T_k x^*\right\| \leq \left\|x - x^*\right\| + \left\|x^*\right\|. \tag{2.3}$$

Thus, $\sum_{k=1}^{\infty}\xi_k T_k x$ converges absolutely for each $x \in C$. It is easy to see that T is nonexpansive and maps C into E. Next, we claim that $F(T) = \bigcap_{k=1}^{\infty} F(T_k)$. The inclusion $\bigcap_{k=1}^{\infty} F(T_k) \subset F(T)$ is obvious. We prove the reverse inclusion only. Suppose that $Tx_0 = x_0$. Then

$$\begin{aligned}
\left\|x_0 - x^*\right\| = \left\|Tx_0 - x^*\right\| &= \left\|\sum_{k=1}^{\infty}\xi_k T_k x_0 - x^*\right\| \\
&= \left\|\sum_{k=1}^{\infty}\xi_k \left(T_k x_0 - x^*\right)\right\| \\
&\leq \sum_{k=1}^{\infty}\xi_k \left\|T_k x_0 - x^*\right\|.
\end{aligned} \tag{2.4}$$

But $T_k x^* = x^*$ and T_k are nonexpansive for all $k \in \mathbb{N}$, so $\|T_k x_0 - x^*\| \leq \|x_0 - x^*\|$. Since $\sum_{k=1}^{\infty}\xi_k = 1$, (2.4) implies that

$$\begin{gathered}
\left\|\sum_{k=1}^{\infty}\xi_k T_k x_0 - x^*\right\| = \left\|x_0 - x^*\right\|, \\
\left\|T_k x_0 - x^*\right\| = \left\|x_0 - x^*\right\| \quad \forall\, k \in \mathbb{N}.
\end{gathered} \tag{2.5}$$

Since E is strictly convex and each $\xi_k > 0$ while $\sum_{k=1}^{\infty}\xi_k = 1$, (2.5) implies that $T_k x_0 - x^* = T_m x_0 - x^*$ for all $k, m \in \mathbb{N}$, that is, $T_k x_0 = T_m x_0$ for all $k, m \in \mathbb{N}$. Hence,

$$x_0 = Tx_0 = \sum_{k=1}^{\infty}\xi_k T_k x_0 = \sum_{k=1}^{\infty}\xi_k T_m x_0 = T_m x_0 \quad \forall\, m \in \mathbb{N}. \tag{2.6}$$

Thus, $x_0 \in \bigcap_{m=1}^{\infty} F(T_m)$. This completes the proof. $\square$

Remark 2.4. The proof of Lemma 2.3 is as given by Bruck Jr. [7]. We included it here for completeness of our presentation in this paper.

Theorem 2.5 (I). *(see e.g., [1]). Let A be a continuous accretive operator defined on a real Banach space E with $D(A) = E$. Then A is m-accretive.*

Theorem 2.6 (MJ). *(see [34]). Let C be a closed convex nonempty subset of a real reflexive Banach space E which has uniformly Gâteaux differentiable norm and $T : C \to E$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every bounded closed convex nonempty subset of C has the fixed-point property for nonexpansive mappings, then there exists a continuous path $t \to z_t$, $0 < t < 1$ satisfying $z_t = tu + (1-t)Tz_t$, for arbitrary but fixed $u \in C$, which converges strongly to a fixed point of T.*

3. Main results

For the rest of this paper, $\{\alpha_n\}_{n\geq 1}$ is a real sequence such that $\{\alpha_n\}_{n\geq 1} \subset [0,1]$ and satisfies (i) $\lim_{n\to\infty}\alpha_n = 0$; (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$ and either (iii) $\lim_{n\to\infty}|\alpha_n - \alpha_{n-1}|/\alpha_n = 0$ or (iii)$'$ $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$. The sequence $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{k=1}^{\infty}\xi_k = 1$.

We now state and prove our main theorems.

3.1. *Strong convergence theorems for a countably infinite family of m-accretive mappings*

Theorem 3.1. *Let C be a closed convex nonempty subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $A_k : C \to E$, $k \in \mathbb{N}$ be a countably infinite family of m-accretive mappings such that $N' = \bigcap_{k=1}^{\infty} N(A_k) \neq \emptyset$. Suppose that every bounded closed convex nonempty subset of C has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n, \quad n \geq 1, \tag{3.1}$$

where $S = \sum_{k=1}^{\infty}\xi_k J_{A_k}$; $J_{A_k} = (I + A_k)^{-1}$, $k \in \mathbb{N}$. Then, $\{x_n\}_{n\geq 1}$ converges strongly to a common zero of $\{A_k\}_{k\in\mathbb{N}}$.

Proof. Since $J_{A_k} = (I + A_k)^{-1}$ is nonexpansive for each $k \in \mathbb{N}$, we obtain, by Lemma 2.3, that $S = \sum_{k=1}^{\infty}\xi_k J_{A_k}$ is well defined, nonexpansive, and $F(S) = \bigcap_{k=1}^{\infty} F(J_{A_k}) = N'$. Now, let $q \in F(S)$, then we obtain by induction (using (3.1)) that

$$\|x_n - q\| \leq \max\{\|x_1 - q\|, \|u - q\|\} \tag{3.2}$$

for all $n \in \mathbb{N}$; hence $\{x_n\}_{n\geq 1}$ and $\{Sx_n\}_{n\geq 1}$ are bounded. This implies that for some $M_0 > 0$,

$$\|x_{n+1} - Sx_n\| = \alpha_n\|u - Sx_n\| \leq \alpha_n M_0 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.3}$$

Moreover, from (3.1) we obtain that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n u + (1-\alpha_n)Sx_n - \alpha_{n-1}u - (1-\alpha_{n-1})Sx_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})(u - Sx_{n-1}) + (1-\alpha_n)(Sx_n - Sx_{n-1})\| \\ &\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_0. \end{aligned} \tag{3.4}$$

This results in the following two cases.

Case 1. Condition (iii) is satisfied. In this case, $\|x_{n+1} - x_n\| \leq (1 - \alpha_n)\|x_n - x_{n-1}\| + \sigma_n$, where $\sigma_n = \alpha_n \beta_n$; $\beta_n = (|\alpha_n - \alpha_{n-1}|M_0)/\alpha_n$, so that $\sigma_n = o(\alpha_n)$ (since $\lim_{n\to\infty}|\alpha_n - \alpha_{n-1}|/\alpha_n = 0$).

Case 2. Condition (iii)$'$ is satisfied. In this case, $\|x_{n+1} - x_n\| \leq (1 - \alpha_n)\|x_n - x_{n-1}\| + \sigma_n$, where $\sigma_n = |\alpha_n - \alpha_{n-1}|M_0$, so that $\sum_{n=0}^{\infty}\sigma_n < \infty$.

In either case, we obtain (by Lemma 2.1) that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. This implies that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ (since $\|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sx_n\| \to 0$ as $n \to \infty$). For all $t \in (0,1)$, define the mapping $G_t : E \to E$ by

$$G_t x := tu + (1-t)Sx, \quad x \in E. \tag{3.5}$$

It is easy to see that G_t is a contraction for each $t \in (0,1)$, and so has for each $t \in (0,1)$ a unique fixed point $z_t \in C$; using Theorem 2.6, we have that $z_t \to z^* \in F(S)$ as $t \to 0$. Now,

$$z_t - x_n = t(u - x_n) + (1-t)(Sz_t - x_n). \tag{3.6}$$

So, by Lemma 2.2 we have that

$$\begin{aligned}
\|z_t - x_n\|^2 &\leq (1-t)^2\|Sz_t - x_n\|^2 + 2t\langle u - x_n, j(z_t - x_n)\rangle \\
&\leq (1-t)^2(\|Sz_t - Sx_n\| + \|Sx_n - x_n\|)^2 + 2(\|z_t - x_n\|^2 + \langle u - z_t, j(z_t - x_n)\rangle) \\
&\leq (1+t^2)\|z_t - x_n\|^2 + 2t\langle u - z_t, j(z_t - x_n)\rangle + \|Sx_n - x_n\|(2\|z_t - x_n\| + \|Sx_n - x_n\|).
\end{aligned} \tag{3.7}$$

This implies that

$$\langle u - z_t, j(x_n - z_t)\rangle \leq \left(\frac{t}{2} + \frac{\|Sx_n - x_n\|}{2t}\right)M, \tag{3.8}$$

for some $M > 0$. Thus,

$$\limsup_{n\to\infty}\langle u - z_t, j(x_n - z_t)\rangle \leq \frac{t}{2}M. \tag{3.9}$$

Moreover, we have that

$$\langle u - z_t, j(x_n - z_t)\rangle = \langle u - z, j(x_n - z)\rangle + \langle u - z, j(x_n - z_t) - j(x_n - z)\rangle + \langle z - z_t, j(x_n - z_t)\rangle \tag{3.10}$$

Thus, since $\{x_n\}_{n\geq 1}$ is bounded, we have that $\langle z^* - z_t, j(x_n - z_t)\rangle \to 0$ as $t \to 0$. Also, $\langle u - z^*, j(x_n - z_t) - j(x_n - z^*)\rangle \to 0$ as $t \to 0$ since the normalized duality mapping j is norm-to-weak* unformly continuous on bounded subsets of E. Thus as $t \to 0$, we obtian from (3.9) and (3.10) that

$$\limsup_{n\to\infty}\langle u - z^*, j(x_n - z^*)\rangle \leq 0. \tag{3.11}$$

Now, put

$$\mu_n := \max\{0, \langle u - z^*, j(x_n - z^*)\rangle\}. \tag{3.12}$$

Then, $0 \le \mu_n$ for all $n \ge 0$. It is easy to see that $\mu_n \to 0$ as $n \to \infty$ since by (3.11), if $\varepsilon > 0$ is given, there exists $n_\varepsilon \in \mathbb{N}$ such that $\langle u - z^*, j(x_n - z^*)\rangle < \varepsilon$ for all $n \ge n_\varepsilon$. Thus, $0 \le \mu_n < \varepsilon$ for all $n \ge n_\varepsilon$. So, $\lim_{n\to\infty}\mu_n = 0$.

Next, we obtain from the recursion formula (3.1) that

$$x_{n+1} - z^* = \alpha_n(u - z^*) + (1 - \alpha_n)(Sx_n - z^*). \tag{3.13}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &\le (1 - \alpha_n)^2\|Sx_n - z^*\|^2 + 2\alpha_n\langle u - z^*, j(x_{n+1} - z^*)\rangle \\
&\le (1 - \alpha_n)\|x_n - z^*\|^2 + 2\alpha_n\mu_{n+1} \\
&= (1 - \alpha_n)\|x_n - z^*\| + \gamma_n,
\end{aligned} \tag{3.14}$$

where $\gamma_n = 2\alpha_n\mu_{n+1}$. Therefore, $\gamma_n = o(\alpha_n)$ and by Lemma 2.1, we obtain that $\{x_n\}_{n\ge1}$ converges strongly to $z^* \in F(S)$. But $F(S) = \bigcap_{k=1}^{\infty}F(J_{A_k}) = \bigcap_{k=1}^{\infty}N(A_k) = N'$. Hence, $\{x_n\}_{n\ge1}$ converges strongly to the common zero of the family $\{A_k\}_{k\in\mathbb{N}}$ of m-accretive operators. This completes the proof. □

Corollary 3.2. *Let C be a closed convex nonempty subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $A_k : C \to E$, $k = 1, 2, \ldots, r$ be a finite family of m-accretive mappings such that $N' = \bigcap_{k=1}^{r}N(A_k) \ne \emptyset$. Suppose that every bounded closed convex nonempty subset of C has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\ge1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n, \quad n \ge 1, \tag{3.15}$$

where $S = \sum_{k=1}^{r}\alpha_k J_{A_k}$; $J_{A_k} = (I + A_k)^{-1}$; $\{\alpha_k\}_{k=1}^{r}$ is a finite collection of positive real numbers such that $\sum_{k=1}^{r}\alpha_k = 1$. Then, $\{x_n\}_{n\ge1}$ converges strongly to a common zero of $\{A_k\}_{k=1}^{r}$.

Proof. The mapping $S = \sum_{k=1}^{r}\alpha_k J_{A_k}$ is clearly nonexpansive. Following the arguement of the proof of Lemma 2.3 we get that $F(S) = \bigcap_{k=1}^{r}F(J_{A_k})$. The rest follows from Theorem 3.1. This completes the proof. □

Remark 3.3. If, in particular, we consider a single m-accretive operator A, the requirement that E be strictly convex will be dispensed, in this case, with $r = 1$ and S in Corollary 3.2 coincides with $J_A = (I + A)^{-1}$.

Remark 3.4. We note that if E is smooth, then E is reflexive and has a uniformly Gâteaux differentiable norm and with property that every bounded closed convex nonempty subset of E has the fixed point property for nonexpansive mappings (see e.g., [1]).

Thus, we have the following corollary.

Corollary 3.5. *Let C be a closed convex nonempty subset of a real uniformly smooth Banach space E. Let $A : C \to E$ be an m-accretive operator with $N(A) \neq \emptyset$. For arbitrary $u, x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_A x_n, \quad n \geq 1, \tag{3.16}$$

where $J_A := (I + A)^{-1}$. Then $\{x_n\}_{n\geq 1}$ converges strongly to some $x^* \in N(A)$.

Remark 3.6. If in Theorem 3.1 we consider $C = E$, then the condition that A_k is m-accretive for each $k \in \mathbb{N}$ could be replaced with the continuity of each A_k.

Thus, we have the following theorem.

Theorem 3.7. *Let E be a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm. Let $A_k : E \to E$, $k \in \mathbb{N}$ be a countably infinite family of continuous accretive operators such that $N' = \bigcap_{k=1}^{\infty} N(A_k) \neq \emptyset$. Suppose that every bounded closed convex nonempty subset of E has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in E$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \quad n \geq 1, \tag{3.17}$$

where $S = \sum_{k=1}^{\infty} \xi_k J_{A_k}$; $J_{A_k} = (I + A_k)^{-1}$. Then, $\{x_n\}_{n\geq 1}$ converges strongly to a common zero of $\{A_k\}_{k\in\mathbb{N}}$.

Proof. By Theorem 2.5, we have that A_k is m-accretive for each $k \in \mathbb{N}$. The rest follows from Theorem 3.1. □

Corollary 3.8. *Let E be a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm. Let $A_k : E \to E$, $k = 1, 2, \ldots, r$ be a finite family of continuous accretive operators such that $N' = \bigcap_{k=1}^{r} N(A_k) \neq \emptyset$. Suppose that every bounded closed convex nonempty subset of E has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in E$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \quad n \geq 1, \tag{3.18}$$

where $S = \sum_{k=1}^{r} \alpha_k J_{A_k}$; $J_{A_k} = (I + A_k)^{-1}$, where $\{\alpha_k\}_{k=1}^{r}$ is a finite collection of positive real numbers such that $\sum_{k=1}^{r} \alpha_k = 1$. Then, $\{x_n\}_{n\geq 1}$ converges strongly to a common zero of $\{A_k\}_{k=1}^{r}$.

3.2. Strong convergence theorem for countably infinite family of pseudocontractive mappings

Theorem 3.9. *Let C be a closed convex nonempty subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_k : C \to E$, $k \in \mathbb{N}$ be a countably infinite family of pseudocontractive mappings such that for each $k \in \mathbb{N}$, $(I - T_k)$ is m-accretive on C and $F' = \bigcap_{k=1}^{\infty} F(T_k) \neq \emptyset$. Let $J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1}$ for each $k \in \mathbb{N}$. Suppose that every* *bounded closed convex nonempty subset of C has the fixed-point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{3.19}$$

where $T = \sum_{k=1}^{\infty}\xi_k J_{T_k}$. Then, $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_k\}_{k\in\mathbb{N}}$.

Proof. Put $A_k := (I - T_k)$ for each $k \in \mathbb{N}$. It is then obvious that $N(A_k) = F(T_k)$ and hence $\bigcap_{k=1}^{\infty} N(A_k) = F' = \bigcap_{k=1}^{\infty} F(T_k)$. Besides, A_k is m-accretive for each $k \in \mathbb{N}$. Thus, the proof follows from Theorem 3.1. □

Corollary 3.10. *Let C be a closed convex nonempty subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_k : C \to E$, $k = 1, 2, \ldots, r$ be a finite family of pseudocontractive mappings such that for each $k = 1, 2, \ldots, r$, $(I - T_k)$ is m-accretive on C and $F = \bigcap_{k=1}^{r} F(T_k) \neq \emptyset$. Let $J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1}$ for each $k = 1, 2, \ldots, r$. Suppose that every nonempty bounded closed convex subset of C has the fixed-point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{3.20}$$

where $T = \sum_{k=1}^{r}\alpha_k J_{T_k}$ and $\{\alpha_k\}_{k=1}^{r}$ is a finite collection of positive numbers such that $\sum_{k=1}^{r}\alpha_k = 1$. Then, $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_k\}_{k=1}^{r}$.

Corollary 3.11. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to E$ be pseudocontractive mappings such that $(I - T)$ is m-accretive on C and $F(T) \neq \emptyset$. Let $J_T = (I + (I - T)^{-1}) = (2I - T)^{-1}$. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_T x_n, \quad n \geq 1. \tag{3.21}$$

Then, $\{x_n\}_{n\geq 1}$ converges strongly to a fixed point of T.

Theorem 3.12. *Let E be a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_k : E \to E$, $k \in \mathbb{N}$ be a countably infinite family of continuous pseudocontractive mappings such that $F' = \bigcap_{k=1}^{\infty} F(T_k) \neq \emptyset$. Let $J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1}$ for each $k \in \mathbb{N}$. Suppose that every bounded closed convex nonempty subset of C has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{3.22}$$

where $T = \sum_{k=1}^{\infty}\xi_k J_{T_k}$. Then, $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_k\}_{k\in\mathbb{N}}$.

Proof. The proof follows from Theorem 3.9. □

Corollary 3.13. *Let E be a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_k : E \to E$, $k = 1, 2, \ldots, r$ be a finite family of continuous pseudocontractive mappings such that $F = \bigcap_{k=1}^{r} F(T_k) \neq \emptyset$. Let $J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1}$ for each $k = 1, 2, \ldots, r$. Suppose that every bounded closed convex nonempty subset of E has the fixed-point property for nonexpansive mappings. For arbitrary $u, x_1 \in E$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{3.23}$$

where $T = \sum_{k=1}^{r}\alpha_k J_{T_k}$*;* $J_{T_k} = (I + T_k)^{-1}$*, where* $\{\alpha_k\}_{k=1}^{r}$ *is a finite collection of positive numbers such that* $\sum_{k=1}^{r}\alpha_k = 1$*. Then,* $\{x_n\}_{n\geq 1}$ *converges strongly to a common fixed point of* $\{T_k\}_{k=1}^{r}$*.*

Corollary 3.14. *Let* E *be a real uniformly smooth Banach space. Let* $T : E \to E$ *be continuous pseudo-contractive mappings such that* $F(T) \neq \varnothing$*. Let* $J_T = (I+(I-T)^{-1}) = (2I-T)^{-1}$*. For arbitrary* $u, x_1 \in E$*, let* $\{x_n\}_{n\geq 1}$ *be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_T x_n, \quad n \geq 1. \tag{3.24}$$

Then, $\{x_n\}_{n\geq 1}$ *converges strongly to fixed point of* T*.*

Remark 3.15. A prototype for the sequence $\{\alpha_n\}_{n\geq 1}$ satisfying the conditions on our iteration parameter is the sequence $\{1/(n+1)\}_{n\geq 1}$. We note that conditions (iii) and (iii)$'$ are not comparable, since (e.g.) the sequence $\{\beta_n\}_{n\geq 1}$ given by

$$\beta_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\ \dfrac{1}{\sqrt{n}-1}, & \text{if } n \text{ is even} \end{cases} \tag{3.25}$$

satisfies (iii) but does not satisfy (iii)$'$ (see e.g., [33]).

Remark 3.16. The addition of bounded error terms to our recursion formulas leads to no further generalization.

Remark 3.17. If $f : K \to K$ is a contraction mapping and we replace u by $f(x_n)$ in the recursion formulas of our theorems, we obtain what some authors now call *viscosity* iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces u by $f(x_n)$, repeats the argument of this paper, using the fact that f is a contraction map.

Remark 3.18. Our theorems improve, extend, and generalize the corresponding results of Zegeye and Shahzad [32] and that of a host of other authors from approximation of a common zero (common fixed point) of *a finite family of accretive (pseudocontractive) operators* to approximation of a common zero (common fixed point) of *a countably infinite family of accretive (pseudocontractive) operators*. Furthermore, Theorem 3.12 extends the corresponding results of Liu et al. [19], Maiti and Saha [9], Senter and Dotson [20], Jung [17] from approximation of a common fixed point of a finite family of nonexpansive mappings to the approximation of common fixed points of a countably infinite family of continuous psedocontractive mappings, without assuming that our operators satisfy the so-called condition A. Our theorems are applicable, in particular, in L_p spaces, $1 < p < +\infty$.

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