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Research Article

Iterative Algorithms for Nonexpansive Mappings

Yeong-Cheng Liou[1] and Yonghong Yao[2]

[1] *Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan*
[2] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*

Correspondence should be addressed to Yeong-Cheng Liou, simplex_liou@hotmail.com

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We suggest and analyze two new iterative algorithms for a nonexpansive mapping T in Banach spaces. We prove that the proposed iterative algorithms converge strongly to some fixed point of T.

1. Introduction

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a nonexpansive mapping; namely,

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.1}$$

for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Throughout the paper, we assume that $F(T) \neq \emptyset$.

Construction of fixed points of nonlinear mappings is an important and active research area. In particular, iterative methods for finding fixed points of nonexpansive mappings have received a vast investigation; see [1–28].

It is well known that the Picard iteration $x_{n+1} = Tx_n = \cdots = T^{n+1}x$ of the mapping T at a point $x \in C$ may, in general, not behave well. This means that it may not converge even in the weak topology. One way to overcome this difficulty is to use Mann's iteration method that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. For example, Reich [9] proved that if E is a uniformly convex Banach space with a Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of T. However, this scheme has only a weak convergence even in a Hilbert space.

Some attempts to construct iteration method so that strong convergence is guaranteed have recently been made. For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in C$, let the sequence $\{x_n\}$ in C be iteratively defined by $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n. \tag{1.3}$$

The iterative method (1.3) is now referred to as the Halpern iterative method. The interest and importance of Halpern iterative method lie in the fact that strong convergence of the sequence $\{x_n\}$ is achieved under certain mild conditions on parameter $\{\alpha_n\}$ in a general Banach space.

Recently, Su and Li [21] introduced the following two new iterative algorithms for a nonexpansive mapping T: for fixed $u \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(\beta_n u + (1-\beta_n)x_n), \tag{1.4}$$

$$y_{n+1} = \alpha_n(\beta_n u + (1-\beta_n)Ty_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n}T^i y_n, \tag{1.5}$$

respectively. Su and Li proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to some fixed point of T under some assumptions. Subsequently, Liu and Li [22] extended and improved the corresponding results of Su and Li [21]. But we note that all of the above results have imposed some additional assumptions on parameters. Please see [21, 22] for more details.

Motivated and inspired by the above works, in this paper we construct two new iterative algorithms for approximating fixed points of a nonexpansive mapping T. We prove that the proposed iterative algorithms converge strongly to a fixed point of T under some mild conditions.

2. Preliminaries

Let E be a real Banach space with its dual E^*. Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The norm on E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$ and in this case E is said to be smooth. E is said to have a uniformly Frechet differentiable norm if the limit (2.1) is attained uniformly for $x, y \in S$ and in this case E is said to be uniformly smooth. It is well known that if E is uniformly smooth, then the duality map is norm-to-norm uniformly continuous on bounded subsets of E.

Let $C \subset E$ be closed convex and P a mapping of E onto C. Then, P is said to be sunny if $P(Px + t(x - Px)) = Px$, for all $x \in E$ and $t \geq 0$. A mapping P of E into E is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, where $R(P)$ is the range of P. A subset C of E is said to be a sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto C and it is said to be a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C. If $E = H$, the metric projection P_C is a sunny nonexpansive retraction from H to any closed convex subset of H.

We need the following lemmas for proving our main results.

Lemma 2.1 (see [16]). *Let E be a real Banach space and J the normalized duality map on E. Then, for any given $x, y \in E$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{2.2}$$

Lemma 2.2 (see [20]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies the condition $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1-\alpha_n) z_n$, $n \geq 0$, and $\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3 (see [17]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-b_n)a_n + c_n$, $n \geq 0$, where $\{b_n\}$ is a sequence in $(0,1)$ and $\{c_n\}$ is a sequence in R such that*

(i) $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) $\limsup_{n\to\infty} c_n/b_n \leq 0$ *or* $\sum_{n=0}^{\infty} |c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (see [17]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, there exists a continuous path $t \to z_t$, $0 < t < 1$, satisfying $z_t = tu + (1-t)Tz_t$, for arbitrary but fixed $u \in C$, which converges strongly to a fixed point of T. Further, if $Pu = \lim_{t\to 0} z_t$, for each $u \in C$, then P is a sunny nonexpansive retraction of C onto $F(T)$.*

Lemma 2.5 (see [17]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence. Suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then, $\langle u - Pu, j(x_n - Pu)\rangle \leq 0$.*

3. Main results

Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. First, for fixed two different anchors $u, v \in C$ we define $x_t = tu + (1-t)Tx_t$ and $y_t = tv + (1-t)Ty_t$. It follows from Lemma 2.4 that x_t and y_t converge strongly to Pu and Pv, respectively. Now we assume that anchors u and v satisfy condition (P): $Pu = Pv$, where P is a sunny nonexpansive retraction from C onto $F(T)$.

Now we introduce the following iterative algorithm: for fixed $u, v \in C$ and given $x_0 \in C$ arbitrarily, find the approximate solution $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T(\delta_n v + (1-\delta_n) x_n), \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are real sequences in $[0,1]$.

Remark 3.1. Our iterative algorithm (3.1) can be reviewed as an extension of the iterative algorithm (1.4).

Now we state and prove the strong convergence of the iterative algorithm (3.1).

Theorem 3.2. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} \delta_n = 0$ *and* $\delta_n / \alpha_n \leq M$ *for some constant* $M \geq 0$.

For fixed anchors u and v satisfying condition (P) and arbitrary given $x_0 \in C$, the sequence $\{x_n\}$ defined by (3.1) *converges strongly to $Pu \in F(T)$.*

Proof. Setting $y_n = \delta_n v + (1-\delta_n)x_n$ and $\sigma_n = \alpha_n + \gamma_n \delta_n$. For $p \in F(T)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|u-p\| + \beta_n\|x_n - p\| + \gamma_n[\delta_n\|v-p\| + (1-\delta_n)\|x_n-p\|] \\ &= \alpha_n\|u-p\| + \gamma_n\delta_n\|v-p\| + [\beta_n + \gamma_n(1-\delta_n)]\|x_n-p\| \\ &\leq \sigma_n \max\{\|v-p\|, \|u-p\|\} + (1-\sigma_n)\|x_n-p\| \\ &\leq \max\{\|u-p\|, \|v-p\|, \|x_0-p\|\}, \end{aligned} \tag{3.2}$$

which implies that $\{x_n\}$ is bounded.

From (3.1), we write $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$. It is easily seen that

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}Ty_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n Ty_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(Ty_{n+1} - Ty_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)Ty_n. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|Ty_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|Ty_{n+1} - Ty_n\| \\ &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|Ty_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1} - y_n\|. \end{aligned} \tag{3.4}$$

At the same time, we note that

$$\|y_{n+1} - y_n\| \leq |\delta_{n+1} - \delta_n|(\|v\| + \|x_n\|) + (1-\delta_{n+1})\|x_{n+1} - x_n\|. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} \|z_{n+1} - z_n\| \leq{}& \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|Ty_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \\ &\times |\delta_{n+1} - \delta_n|(\|v\| + \|x_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\|. \end{aligned} \tag{3.6}$$

Then, we have

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \mid (\|u\| + \|Ty_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} |\delta_{n+1} - \delta_n| \\ &\quad \times (\|v\| + \|x_n\|) + \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|, \end{aligned} \tag{3.7}$$

which implies that

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.8}$$

From Lemma 2.2 and (3.8), we obtain $\lim_{n\to\infty} \|z_n - x_n\| = 0$. Consequently, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0$.

On the other hand, we observe that

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n \|u - Tx_n\| + \beta_n \|x_n - Tx_n\| + \gamma_n \|Ty_n - Tx_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n \|u - Tx_n\| + \beta_n \|x_n - Tx_n\| + \gamma_n \delta_n \|v - x_n\|, \end{aligned} \tag{3.9}$$

that is,

$$\|x_n - Tx_n\| \le \frac{1}{1-\beta_n} \{ \|x_{n+1} - x_n\| + \alpha_n \|u - Tx_n\| + \gamma_n \delta_n \|v - x_n\| \} \longrightarrow 0. \tag{3.10}$$

By Lemma 2.5 and (3.10), we can obtain

$$\limsup_{n\to\infty} \langle u - Pu, j(x_n - Pu) \rangle \le 0, \qquad \limsup_{n\to\infty} \langle v - Pv, j(x_n - Pv) \rangle \le 0. \tag{3.11}$$

Since $\|y_n - x_n\| = \delta_n \|v - x_n\| \to 0$ and $Pu = Pv$, we have

$$\limsup_{n\to\infty} \langle u - Pu, j(x_n - Pu) \rangle \le 0, \qquad \limsup_{n\to\infty} \langle v - Pu, j(y_n - Pu) \rangle \le 0. \tag{3.12}$$

Finally, we show that $x_n \to Pu$. As a matter of fact, we have

$$\begin{aligned} \|x_{n+1} - Pu\|^2 &\le \|\beta_n (x_n - Pu) + \gamma_n (Ty_n - Pu)\|^2 + 2\alpha_n \langle u - Pu, j(x_{n+1} - Pu) \rangle \\ &\le \beta_n \|x_n - Pu\|^2 + \gamma_n \|y_n - Pu\|^2 + 2\alpha_n \langle u - Pu, j(x_{n+1} - Pu) \rangle \\ &\le \beta_n \|x_n - Pu\|^2 + \gamma_n \{ (1-\delta_n) \|x_n - Pu\|^2 + 2\delta_n \langle v - Pu, j(y_n - Pu) \rangle \} \\ &\quad + 2\alpha_n \langle u - Pu, j(x_{n+1} - Pu) \rangle \\ &\le (1-\alpha_n) \|x_n - Pu\|^2 + 2\delta_n \langle v - Pu, j(y_n - Pu) \rangle + 2\alpha_n \langle u - Pu, j(x_{n+1} - Pu) \rangle \\ &= (1-\alpha_n) \|x_n - Pu\|^2 + \alpha_n \lambda_n, \end{aligned} \tag{3.13}$$

where $\lambda_n = (2\delta_n/\alpha_n)\langle v - Pu, j(y_n - Pu) \rangle + 2\langle u - Pu, j(x_{n+1} - Pu) \rangle$. It is easy to see from (3.12) that $\limsup_{n\to\infty} \lambda_n \le 0$. Lemma 2.3 and (3.13) ensure that $x_n \to Pu$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} \delta_n = 0$ *and* $\delta_n/\alpha_n \leq M$ *for some constant* $M \geq 0$.

For fixed anchor u and arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ as

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T(\delta_n u + (1-\delta_n)x_n). \tag{3.14}$$

Then, the sequence $\{x_n\}$ converges strongly to $Pu \in F(T)$.

Finally, we introduce another iterative algorithm: for fixed anchor $u \in C$ and given $x_0 \in C$ arbitrarily, find the approximate solution $\{x_n\}$ by the iterative algorithm:

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) A_n y_n, \\ y_n &= \alpha_n u + (1-\alpha_n) T x_n, \quad n \geq 0, \end{aligned} \tag{3.15}$$

where $A_n = (1/(n+1))\sum_{i=0}^{n} T^i$.

Now we prove the strong convergence of the iterative algorithm (3.15).

Theorem 3.4. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences in $(0,1)$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (3.15) *converges strongly to $Pu \in F(T)$, where P is a sunny nonexpansive retraction from C onto $F(T)$.*

Proof. First, we note that $A_n : C \to C$ is nonexpansive. In fact, for any $x, y \in C$, we have that

$$\|A_n x - A_n y\| = \left\| \frac{1}{n+1}\sum_{i=0}^{n} T^i x - \frac{1}{n+1}\sum_{i=0}^{n} T^i y \right\| \leq \frac{1}{n+1}\sum_{i=0}^{n} \|x-y\| = \|x-y\|. \tag{3.16}$$

At the same time, for any $x \in C$,

$$\|A_{n+1}x - A_n x\| = \left\| \frac{1}{n+2}\sum_{i=0}^{n+1} T^i x - \frac{1}{n+1}\sum_{i=0}^{n} T^i x \right\| \leq \frac{1}{(n+1)(n+2)}\sum_{i=0}^{n} \|T^i x\| + \frac{1}{n+2}\|T^{n+1}x\|. \tag{3.17}$$

Now we show that $\{x_n\}$ is bounded. Indeed, for $p \in F(T)$, we have

$$\|y_n - p\| \leq \alpha_n \|u-p\| + (1-\alpha_n)\|Tx_n - p\| \leq \alpha_n \|u-p\| + (1-\alpha_n)\|x_n - p\|. \tag{3.18}$$

Hence, we have

$$\|x_{n+1}-p\| \leq \beta_n\|x_n-p\| + (1-\beta_n)\|A_n y_n - p\| \leq \beta_n\|x_n-p\| + (1-\beta_n)\|y_n-p\|$$
$$\leq [1-\alpha_n(1-\beta_n)]\|x_n-p\| + (1-\beta_n)\alpha_n\|u-p\| \leq \max\{\|x_n-p\|, \|u-p\|\}. \tag{3.19}$$

Now, an induction yields

$$\|x_n-p\| \leq \max\{\|x_0-p\|, \|u-p\|\}. \tag{3.20}$$

Therefore, $\{x_n\}$ is bounded, so is $\{Tx_n\}$.

Setting $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$, where $z_n = A_n y_n$ for all $n \geq 0$, then we have that

$$\begin{aligned}\|z_{n+1}-z_n\| &\leq \|A_{n+1}y_{n+1} - A_n y_{n+1}\| + \|A_n y_{n+1} - A_n y_n\| \\ &\leq \|A_{n+1}y_{n+1} - A_n y_{n+1}\| + \|y_{n+1}-y_n\|.\end{aligned} \tag{3.21}$$

Observe that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|(\alpha_{n+1}-\alpha_n)u + 1 - \alpha_{n+1}(Tx_{n+1}-Tx_n) + (\alpha_n - \alpha_{n+1})Tx_n\| \\ &\leq |\alpha_{n+1}-\alpha_n|(\|u\| + \|Tx_n\|) + (1-\alpha_{n+1})\|x_{n+1}-x_n\|.\end{aligned} \tag{3.22}$$

So, we get

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \leq \|A_{n+1}y_{n+1} - A_n y_{n+1}\| + |\alpha_{n+1}-\alpha_n|(\|u\| + \|Tx_n\|). \tag{3.23}$$

This together with (3.17) implies that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0. \tag{3.24}$$

Consequently, from Lemma 2.2, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|z_n - x_n\| &= 0, \\ \lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|x_n - z_n\| &= 0.\end{aligned} \tag{3.25}$$

Next, we show that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.26}$$

We note from [23] that

$$\|z_n - Tz_n\| \longrightarrow 0. \tag{3.27}$$

Therefore, from (3.25), and (3.27), we have

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_{n+1}-x_n\| + \|x_{n+1}-Tx_n\| \\ &\leq \|x_{n+1}-x_n\| + \|x_{n+1}-z_n\| + \|z_n - Tx_n\| \\ &\leq \|x_{n+1}-x_n\| + \|x_{n+1}-z_n\| + \|z_n - Tz_n\| + \|Tz_n - Tx_n\| \\ &\leq \|x_{n+1}-x_n\| + \|x_{n+1}-z_n\| + \|z_n - Tz_n\| + \|z_n - x_n\| \longrightarrow 0.\end{aligned} \tag{3.28}$$

From (3.28) and Lemma 2.5, we have

$$\limsup_{n\to\infty} \langle u - Pu, j(x_n - Pu)\rangle \leq 0. \tag{3.29}$$

Noting that

$$\|y_n - x_n\| \leq \alpha_n \|u - x_n\| + (1-\alpha_n)\|x_n - Tx_n\| \longrightarrow 0, \tag{3.30}$$

hence,

$$\limsup_{n\to\infty} \langle u - Pu, j(y_n - Pu)\rangle \leq 0. \tag{3.31}$$

Finally, applying Lemma 2.1 to (3.15), we obtain

$$\begin{aligned}
\|x_{n+1} - Pu\|^2 &\leq \beta_n \|x_n - Pu\|^2 + (1-\beta_n)\|z_n - Pu\|^2 \\
&\leq \beta_n \|x_n - Pu\|^2 + (1-\beta_n)\|y_n - Pu\|^2 \\
&= \beta_n \|x_n - Pu\|^2 + (1-\beta_n)\|\alpha_n(u - Pu) + (1-\alpha_n)(Tx_n - Pu)\|^2 \\
&\leq \beta_n \|x_n - Pu\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|Tx_n - Pu\|^2 + 2\alpha_n\langle u - Pu, j(y_n - Pu)\rangle\right] \\
&\leq \beta_n \|x_n - Pu\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, j(y_n - Pu)\rangle\right] \\
&= \left[1 - (1-\beta_n)\alpha_n\right]\|x_n - Pu\|^2 + 2(1-\beta_n)\alpha_n\langle u - Pu, j(y_n - Pu)\rangle.
\end{aligned} \tag{3.32}$$

By Lemma 2.3 and (3.32), we conclude that $x_n \to Pu$. This completes the proof. □

Remark 3.5. We drop the imposed assumptions in [21]: $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$ on parameters $\{\alpha_n\}$ and $\{\delta_n\}$, respectively.

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