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Research Article

Generalized Hyers-Ulam Stability of Quadratic Functional Equations: A Fixed Point Approach

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Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $f(2x+y) = 4f(x) + f(y) + f(x+y) - f(x-y)$ in Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space.Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [8] proved the generalized Hyers-Ulam stability of the quadratic functional equation and Park [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation in Banach modules over a C^*-algebra. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10–17]).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the following theorem by Diaz and Margolis.

Theorem 1.1 (see[18]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.2}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \ge n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \le (1/(1 - L)) d(y, Jy)$ *for all* $y \in Y$.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of the following quadratic functional equation:

$$f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y) \tag{1.3}$$

in Banach spaces.

Throughout this paper, assume that X is a normed vector space with norm $\|\cdot\|$ and that Y is a Banach space with norm $\|\cdot\|$.

In 1996, Isac and Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

2. Fixed points and generalized Hyers-Ulam stability of quadratic functional equations

For a given mapping $f : X \to Y$, we define

$$Cf(x, y) := f(2x + y) - 4f(x) - f(y) - f(x + y) + f(x - y) \tag{2.1}$$

for all $x, y \in X$.

Proposition 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y) \tag{2.2}$$

if and only if the mapping $f : X \to Y$ satisfies

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.3}$$

for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.2).

Letting $x = y = 0$ in (2.2), we get $f(0) = 0$.

Letting $y = 0$ in (2.2), we get $f(2x) = 4f(x)$ for all $x \in X$.

Letting $x = 0$ in (2.2), we get $f(-y) = f(y)$ for all $y \in X$.

Replacing y in (2.2) by $-y$, we get

$$f(2x - y) = 4f(x) + f(-y) + f(x - y) - f(x + y) \tag{2.4}$$

for all $x, y \in X$. It follows from (2.2) and (2.4) that

$$f(2x + y) + f(2x - y) = 8f(x) + f(y) + f(-y) = 2f(2x) + 2f(y) \tag{2.5}$$

for all $x, y \in X$. So the mapping $f : X \to Y$ satisfies

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.6}$$

for all $x, y \in X$.

Assume that $f : X \to Y$ satisfies $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Since

$$\begin{aligned} f(2x + y) &= f(x + y + x) \\ &= 2f(x + y) + 2f(x) - f(y) \\ &= f(x + y) + f(x + y) + 2f(x) - f(y) \\ &= f(x + y) + 2f(x) + 2f(y) - f(x - y) + 2f(x) - f(y) \\ &= 4f(x) + f(y) + f(x + y) - f(x - y) \end{aligned} \tag{2.7}$$

for all $x, y \in X$, the mapping $f : X \to Y$ satisfies (2.2). □

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $Cf(x, y) = 0$.

Theorem 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,0) \leq 4L\varphi(x/2,0)$ for all $x \in X$, and*

$$\sum_{j=0}^{\infty} 4^{-j}\varphi(2^j x, 2^j y) < \infty, \tag{2.8}$$

$$\|Cf(x,y)\| \leq \varphi(x,y) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying(2.2) *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{4-4L}\varphi(x,0) \tag{2.10}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \longrightarrow Y\} \tag{2.11}$$

and introduce the *generalized metric* on S as follows:

$$d(g,h) = \inf\left\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varphi(x,0),\ \forall x \in X\right\}. \tag{2.12}$$

It is easy to show that (S,d) is complete. (See the proof of Theorem 2.5 of [20].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{2.13}$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [21] that

$$d(Jg, Jh) \leq Ld(g,h) \tag{2.14}$$

for all $g, h \in S$.

Letting $y = 0$ in (2.9), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x,0) \tag{2.15}$$

for all $x \in X$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(x,0) \tag{2.16}$$

for all $x \in X$. Hence $d(f, Jf) \leq 1/4$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following.

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{2.17}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.18}$$

This implies that Q is a unique mapping satisfying (2.17) such that there exists $K \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \leq K\varphi(x, 0) \tag{2.19}$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x) \tag{2.20}$$

for all $x \in X$.

(3) $d(f, Q) \leq (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{4 - 4L}. \tag{2.21}$$

This implies that the inequality (2.10) holds.

It follows from (2.8), (2.9), and (2.20) that

$$\begin{aligned} \|CQ(x, y)\| &= \lim_{n\to\infty} \frac{1}{4^n} \|Cf(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \\ &= 0 \end{aligned} \tag{2.22}$$

for all $x, y \in X$. So $CQ(x, y) = 0$ for all $x, y \in X$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic.

Therefore, there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.2) and (2.10), as desired. □

Corollary 2.3. *Let $p < 2$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Cf(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.23}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.2) and

$$\|f(x) - Q(x)\| \leq \frac{\theta}{4 - 2^p} \|x\|^p \tag{2.24}$$

for all $x \in X$.

Proof. The proof follows from Theorem2.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.25}$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

Remark 2.4. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.9) and $f(0) = 0$ such that

$$\sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.26}$$

for all $x, y \in X$. By a similar method to the proof of Theorem 2.2, one can show that if there exists an $L < 1$ such that $\varphi(x, 0) \leq (1/4)L\varphi(2x, 0)$ for all $x \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.2) and

$$\|f(x) - Q(x)\| \leq \frac{L}{4 - 4L}\varphi(x, 0) \tag{2.27}$$

for all $x \in X$.

For the case $p > 2$, one can obtain a similar result to Corollary 2.3

Theorem 2.5. *Let $f : X \to Y$ be an even mapping $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.8) *and* (2.9) *such that there exists an $L < 1$ such that $\varphi(x, -x) \leq 4L\varphi(x/2, -x/2)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.2) *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{4 - 4L}\varphi(x, -x) \tag{2.28}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \longrightarrow Y\} \tag{2.29}$$

and introduce the *generalized metric* on S as follows:

$$d(g, h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varphi(x, -x)\ \forall x \in X\}. \tag{2.30}$$

It is easy to show that (S, d) is complete. (See the proof of Theorem 2.5 of [20].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{2.31}$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [21] that

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.32}$$

for all $g, h \in S$.

Letting $y = -x$ in (2.9), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, -x) \tag{2.33}$$

for all $x \in X$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(x, -x) \tag{2.34}$$

for all $x \in X$. Hence $d(f, Jf) \leq 1/4$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following.

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{2.35}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{2.36}$$

This implies that Q is a unique mapping satisfying (2.35) such that there exists $K \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \le K\varphi(x,-x) \tag{2.37}$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x) \tag{2.38}$$

for all $x \in X$.

(3) $d(f,Q) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,Q) \le \frac{1}{4-4L}. \tag{2.39}$$

This implies that the inequality (2.38) holds.

It follows from (2.8), (2.9), and (2.38) that

$$\begin{aligned} \|CQ(x,y)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|Cf(2^n x, 2^n y)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y) \\ &= 0 \end{aligned} \tag{2.40}$$

for all $x, y \in X$. So $CQ(x,y) = 0$ for all $x, y \in X$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic.

Therefore, there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.2) and(2.28), as desired. □

Corollary 2.6. *Let $p < 1$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\|Cf(x,y)\| \le \theta\cdot\|x\|^p\cdot\|y\|^p \tag{2.41}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying(2.2) *and*

$$\|f(x) - Q(x)\| \le \frac{\theta}{4-4^p}\|x\|^{2p} \tag{2.42}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x, y) := \theta \cdot \|x\|^p \cdot \|y\|^p \tag{2.43}$$

for all $x, y \in X$. Then we can choose $L = 4^{p-1}$ and we get the desired result. □

Remark 2.7. Let $f : X \to Y$ be an even mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.9), (2.26), and $f(0) = 0$. By a similar method to the proof of Theorem 2.5, one can show that if there exists an $L < 1$ such that $\varphi(x, -x) \leq (1/4)L\varphi(2x, -2x)$ for all $x \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.2) and

$$\|f(x) - Q(x)\| \leq \frac{L}{4 - 4L}\varphi(x, -x) \tag{2.44}$$

for all $x \in X$.

For the case $p > 1$, one can obtain a similar result to Corollary 2.6.

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