Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2008, Article ID 528476, 11 pages
doi:10.1155/2008/528476

Research Article

Strong Convergence Theorem by a New Hybrid Method for Equilibrium Problems and Relatively Nonexpansive Mappings

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Received 6 September 2007; Accepted 8 January 2008

Recommended by Simeon Reich

We prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space by using a new hybrid method. Using this theorem, we obtain two new results for finding a solution of an equilibrium problem and a fixed point of a relatively nonexpnasive mapping in a Banach space.

1. Introduction

Let E be a Banach space and let C be a closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find

$$\hat{x} \in C \text{ such that } f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such solutions $\hat{x}$ is denoted by $\mathrm{EP}(f)$.

A mapping S of C into E is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

We denote by $F(S)$ the set of fixed points of S.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [1], Combettes and Hirstoaga [2], and Moudafi [3]. On the other hand, there are some methods for approximation of fixed points of a nonexpansive mapping; see, for instance, [4–10]. Recently, Tada and Takahashi [11, 12] and S. Takahashi and Takahashi [13] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [12] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced in Nakajo and Takahashi [14]. The authors also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space. Very recently, Takahashi et al. [15] proved the following theorem by a hybrid method which is different from Nakajo and Takahashi's hybrid method. We call such a method the shrinking projection method.

Theorem 1.1 (Takahashi et al. [15]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{aligned} y_n &= \alpha_n u_n + (1-\alpha_n) T u_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

In this paper, motivated by Takahashi et al. [15], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method. Using this theorem, we obtain two new strong convergence results for finding a solution of an equilibrium problem and a fixed point of a relatively nonexpnasive mapping in a Banach space.

2. Preliminaries

Throughout this paper, all the Banach spaces are real. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the topological dual of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. Then, the duality mapping J on E is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \tag{2.1}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [16] for more details. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x+y\|/2 \leq 1 - \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. A uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. The space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. We know that if E is smooth, strictly convex, and reflexive, then the duality mapping J is single valued, one to one, and onto; see [17] for more details.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \quad \forall y, x \in E. \tag{2.3}$$

Following Alber [18], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C}\phi(y, x), \quad \forall x \in E. \tag{2.4}$$

The generalized projection Π_C from E onto C is well defined, single valued and satisfies

$$(\|x\| - \|y\|)^2 \le \phi(y, x) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{2.5}$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C is the metric projection of H onto C. We know the following lemmas for generalized projections.

Lemma 2.1 (Alber [18] and Kamimura and Takahashi [19]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y), \quad \forall x \in C,\ y \in E.$$

Lemma 2.2 (Alber [18] and Kamimura and Takahashi [19]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz\rangle \le 0, \quad \forall y \in C. \tag{2.6}$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let T be a mapping from C into itself. We denoted by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T [20, 21] if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. Following Matsushita and Takahashi [22], a mapping T is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \le \phi(u, x)$, for all $u \in F(T)$, $x \in C$;

(3) $\widehat{F}(T) = F(T)$.

The following lemma is due to Matsushita and Takahashi [22].

Lemma 2.3 (Matsushita and Takahashi [22]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

We also know the following lemmas.

Lemma 2.4 (Kamimura and Takahashi [19]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If* $\lim_n \phi(x_n, y_n) = 0$, *then* $\lim_n \|x_n - y_n\| = 0$.

Lemma 2.5 (Xu [23] and Zălinescu [24, 25]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\left\|tx + (1-t)y\right\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g\left(\|x - y\|\right) \tag{2.7}$$

for all $x, y \in B_r$ and $t \in [0,1]$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.6 (Kamimura and Takahashi [19]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g\left(\|x - y\|\right) \le \phi(x, y) \tag{2.8}$$

for all $x, y \in B_r$.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all$x \in C$;

(A2) f is monotone, that is,$f(x, y) + f(y, x) \le 0$ for all$x, y \in C$;

(A3) for all$x, y, z \in C$,

$$\limsup_{t\downarrow 0} f\left(tz + (1-t)x, y\right) \le f(x, y); \tag{2.9}$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x\rangle, \quad \forall x, y \in C. \tag{2.10}$$

Then, f satisfies (A1)–(A4). The following result is in Blum and Oettli [1].

Lemma 2.7 (Blum and Oettli [1]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $r > 0$ and let $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C. \tag{2.11}$$

We also know the following lemmas.

Lemma 2.8 (Takahashi and Zembayashi [26]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to 2^C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\} \tag{2.12}$$

for all $x \in E$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping [27], that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.13}$$

(3) *$F(T_r) = \widehat{F}(T_r) = \mathrm{EP}(f)$;*

(4) *$\mathrm{EP}(f)$ is closed and convex.*

Lemma 2.9 (Takahashi and Zembayashi [26]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Then for $r > 0$, $x \in E$, and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.14}$$

3. Strong convergence theorem

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap \mathrm{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x \end{aligned} \tag{3.1}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(f)} x$, where $\Pi_{F(S)\cap \mathrm{EP}(f)}$ is the generalized projection of E onto $F(S) \cap \mathrm{EP}(f)$.

Proof. Putting $u_n = T_{r_n} y_n$ for all $n \in \mathbb{N}$, we have from Lemma 2.9 that T_{r_n} are relatively nonexpansive.

We first show that C_n is closed and convex. It is obvious that C_n is closed. Since

$$\phi(z, u_n) \leq \phi(z, x_n) \Longleftrightarrow \|u_n\|^2 - \|x_n\|^2 - 2\langle z, Ju_n - Jx_n \rangle \geq 0, \tag{3.2}$$

C_n is convex. So, C_n is a closed convex subset of E for all $n \in \mathbb{N} \cup \{0\}$.

Next, we show by induction that $\mathrm{EP}(f) \cap F(S) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $C_0 = C$, we have

$$F(S) \cap \mathrm{EP}(f) \subset C_0. \tag{3.3}$$

Suppose that $F(S) \cap \mathrm{EP}(f) \subset C_k$ for some $k \in \mathbb{N} \cup \{0\}$. Let $u \in F(S) \cap \mathrm{EP}(f) \subset C_k$. Since T_{r_k} and S are relatively nonexpansive, we have

$$\begin{aligned}
\phi(u, u_k) &= \phi(u, t_{r_k} y_k) \leq \phi(u, y_k) \\
&= \phi\left(u, j^{-1}(\alpha_k j x_k + (1-\alpha_k) j s x_k)\right) \\
&= \|u\|^2 - 2\langle u, \alpha_k j x_k + (1-\alpha_k) j s x_k \rangle + \left\|\alpha_k j x_k + (1-\alpha_k) j s x_k\right\|^2 \\
&\leq \|u\|^2 - 2\alpha_k \langle u, j x_k \rangle - 2(1-\alpha_k)\langle u, j s x_k \rangle + \alpha_k \left\|x_k\right\|^2 + (1-\alpha_k)\left\|s x_k\right\|^2 \\
&= \alpha_k \phi(u, x_k) + (1-\alpha_k)\phi(u, s x_k) \leq \phi(u, x_k).
\end{aligned} \tag{3.4}$$

Hence, we have $u \in C_{k+1}$. This implies that

$$F(S) \cap \mathrm{EP}(f) \subset C_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

So, $\{x_n\}$ is well defined.

From the definition of x_n, we have

$$\phi(x_n, x) = \phi(\Pi_{C_n} x, x) \leq \phi(u, x) - \phi(u, \Pi_{C_n} x) \leq \phi(u, x) \tag{3.6}$$

for all $u \in F(S) \cap \mathrm{EP}(f) \subset C_n$. Then, $\phi(x_n, x)$ is bounded. Therefore, $\{x_n\}$ and $\{Sx_n\}$ are bounded.

From $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n} x$, we have

$$\phi(x_n, x) \leq \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.7}$$

Thus, $\{\phi(x_n, x)\}$ is nondecreasing. So, the limit of $\{\phi(x_n, x)\}$ exists. Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x) \leq \phi(x_{n+1}, x) - \phi(\Pi_{C_n} x, x) = \phi(x_{n+1}, x) - \phi(x_n, x) \tag{3.8}$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_{n+1}} x \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.9}$$

Therefore, we also have

$$\lim_{n\to\infty}\phi(x_{n+1},u_n)=0. \tag{3.10}$$

Since $\lim_{n\to\infty}\phi(x_{n+1},x_n)=\lim_{n\to\infty}\phi(x_{n+1},u_n)=0$ and E is uniformly convex and smooth, we have from Lemma 2.4 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|x_{n+1}-u_n\|=0. \tag{3.11}$$

So, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty}\|x_n-u_n\|=0$, we have

$$\lim_{n\to\infty}\|Jx_n-Ju_n\|=0. \tag{3.13}$$

Let $r=\sup_{n\in\mathbb{N}}\{\|x_n\|,\|Sx_n\|\}$. Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 2.5, there exists a continuous, strictly increasing, and convex function g with $g(0)=0$ such that

$$\|\alpha x^*+(1-\alpha)y^*\|^2\le\alpha\|x^*\|^2+(1-\alpha)\|y^*\|^2-\alpha(1-\alpha)g(\|x^*-y^*\|) \tag{3.14}$$

for $x^*,y^*\in B_r^*$ and $\alpha\in[0,1]$. So, we have that for $u\in F(S)\cap\mathrm{EP}(f)$,

$$\begin{aligned}\phi(u,u_n)&=\phi(u,t_{r_n}y_n)\le\phi(u,y_n)=\phi(u,j^{-1}(\alpha_njx_n+(1-\alpha_n)jsx_n))\\&=\|u\|^2-2\langle u,\alpha_njx_n+(1-\alpha_n)jsx_n\rangle+\|\alpha_njx_n+(1-\alpha_n)jsx_n\|^2\\&\le\|u\|^2-2\alpha_n\langle u,jx_n\rangle-2(1-\alpha_n)\langle u,jsx_n\rangle+\alpha_n\|x_n\|^2+(1-\alpha_n)\|sx_n\|^2\\&\quad-\alpha_n(1-\alpha_n)g(\|jx_n-jsx_n\|)\\&=\alpha_n\phi(u,x_n)+(1-\alpha_n)\phi(u,sx_n)-\alpha_n(1-\alpha_n)g(\|jx_n-jsx_n\|)\\&\le\phi(u,x_n)-\alpha_n(1-\alpha_n)g(\|jx_n-jsx_n\|).\end{aligned} \tag{3.15}$$

Therefore, we have

$$\alpha_n(1-\alpha_n)g(\|Jx_n-JSx_n\|)\le\phi(u,x_n)-\phi(u,u_n),\quad\forall n\in\mathbb{N}\cup\{0\}. \tag{3.16}$$

Since

$$\begin{aligned}\phi(u,x_n)-\phi(u,u_n)&=\|x_n\|^2-\|u_n\|^2-2\langle u,jx_n-ju_n\rangle\\&\le\left|\|x_n\|^2-\|u_n\|^2\right|+2|\langle u,jx_n-ju_n\rangle|\\&\le\left|\|x_n\|-\|u_n\|\right|(\|x_n\|+\|u_n\|)+2\|u\|\|jx_n-ju_n\|\\&\le\|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|u\|\|jx_n-ju_n\|,\end{aligned} \tag{3.17}$$

we have

$$\lim_{n\to\infty}\left(\phi(u,x_n)-\phi(u,u_n)\right)=0. \tag{3.18}$$

From $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we have

$$\lim_{n\to\infty} g\left(\|jx_n-jsx_n\|\right)=0. \tag{3.19}$$

Therefore, from the property of g, we have

$$\lim_{n\to\infty}\|Jx_n-JSx_n\|=0. \tag{3.20}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.21}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup \hat{x}$. Since S is relatively nonexpansive, we have $\hat{x}\in\widehat{F}(S)=F(S)$. On the other hand, from $u_n=T_{r_n}y_n$, (3.4), and Lemma 2.9, we have that

$$\phi(u_n,y_n)=\phi(T_{r_n}y_n,y_n)\le\phi(u,y_n)-\phi(u,T_{r_n}y_n)\le\phi(u,x_n)-\phi(u,T_{r_n}y_n)=\phi(u,x_n)-\phi(u,u_n). \tag{3.22}$$

So, we have from (3.18) that

$$\lim_{n\to\infty}\phi(u_n,y_n)=0. \tag{3.23}$$

Since E is uniformly convex and smooth and $\{u_n\}$ is bounded, we have from Lemma 2.4 that

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.24}$$

From $x_{n_k}\rightharpoonup\hat{x}$, (3.24) and $\|x_n-u_n\|\to 0$, we have $y_{n_k}\rightharpoonup\hat{x}$.

Since J is uniformly norm-to-norm continuous on bounded sets, from (3.24), we have

$$\lim_{n\to\infty}\|Ju_n-Jy_n\|=0. \tag{3.25}$$

From $r_n\ge a$, we have

$$\lim_{n\to\infty}\frac{\|Ju_n-Jy_n\|}{r_n}=0. \tag{3.26}$$

By $u_n=T_{r_n}y_n$, we have

$$f(u_n,y)+\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\ge 0,\quad \forall y\in C. \tag{3.27}$$

Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y-u_{n_k}, Ju_{n_k}-Jy_{n_k}\rangle \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}), \quad \forall y \in C. \tag{3.28}$$

Since $f(x,\cdot)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. So, letting $k \to \infty$, we have from (3.26) and (A4) that

$$f(y, \hat{x}) \leq 0, \quad \forall y \in C. \tag{3.29}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and hence $f(y_t, \hat{x}) \leq 0$. So, from (A1) we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, \hat{x}) \leq tf(y_t, y). \tag{3.30}$$

Dividing by t, we have

$$f(y_t, y) \geq 0, \quad \forall y \in C. \tag{3.31}$$

Letting $t \downarrow 0$, from (A3), we have

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{3.32}$$

Therefore, $\hat{x} \in \mathrm{EP}(f)$.

Let $w = \Pi_{F(S)\cap \mathrm{EP}(f)}x$. From $x_n = \Pi_{C_n}x$ and $w \in F(S) \cap \mathrm{EP}(f) \subset C_n$, we have

$$\phi(x_n, x) \leq \phi(w, x). \tag{3.33}$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}
\phi(\hat{x}, x) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx\rangle + \|x\|^2 \\
&\leq \liminf_{k\to\infty}\left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx\rangle + \|x\|^2\right) \\
&= \liminf_{k\to\infty}\phi(x_{n_k}, x) \\
&\leq \limsup_{k\to\infty}\phi(x_{n_k}, x) \leq \phi(w, x).
\end{aligned} \tag{3.34}$$

From the definition of $\Pi_{F(S)\cap \mathrm{EP}(f)}$, we have $\hat{x} = w$. Hence, $\lim_{k\to\infty}\phi(x_{n_k}, x) = \phi(w, x)$. Therefore, we have

$$\begin{aligned}
0 &= \lim_{k\to\infty}\left(\phi(x_{n_k}, x) - \phi(w, x)\right) \\
&= \lim_{k\to\infty}\left(\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx\rangle\right) \\
&= \lim_{k\to\infty}\left(\|x_{n_k}\|^2 - \|w\|^2\right).
\end{aligned} \tag{3.35}$$

Since E has the Kadec-Klee property, we have that $x_{n_k} \to w = \Pi_{F(S)\cap \mathrm{EP}(f)}x$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(f)}x$. □

As direct consequences of Theorem 3.1, we can obtain two corollaries.

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{aligned} &u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ &C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ &x_{n+1} = \Pi_{C_{n+1}} x \end{aligned} \tag{3.36}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{\mathrm{EP}(f)} x$.

Proof. Putting $S = I$ in Theorem 3.1, we obtain Corollary 3.2. □

Corollary 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let S be a relatively nonexpansive mapping from C into itself. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{aligned} u_n &= \pi_c j^{-1}(\alpha_n j x_n + (1 - \alpha_n) j s x_n), \\ c_{n+1} &= \{z \in c_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \pi_{c_{n+1}} x \end{aligned} \tag{3.37}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)} x$.

Proof. Putting $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, we obtain Corollary 3.3. □

Theorem 1.1 is a simple consequence of this corollary.

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