Research Article

On Coincidence and Fixed-Point Theorems in Symmetric Spaces

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We give an axiom (C.C) in symmetric spaces and investigate the relationships between (C.C) and axioms (W3), (W4), and (H.E). We give some results on coinsidence and fixed-point theorems in symmetric spaces, and also, we give some examples for the results of Imdad et al. (2006).

1. Introduction

In [1], the author introduced the notion of compatible mappings in metric spaces and proved some fixed-point theorems. This concept of compatible mappings was frequently used to show the existence of common fixed points. However, the study of the existence of common fixed points for noncompatible mappings is, also, very interesting. In [2], the author initially proved some common fixed-point theorems for noncompatible mappings. In [3], the authors gave a notion (E-A) which generalizes the concept of noncompatible mappings in metric spaces, and they proved some common fixed-point theorems for noncompatible mappings under strict contractive conditions. In [4], the authors proved some common fixed-point theorems for strict contractive noncompatible mappings in metric spaces. Recently, in [5] the authors extended the results of [3, 4] to symmetric(semimetric) spaces under tight conditions. In [6], the author gave a common fixed-point theorem for noncompatible self-mappings in a symmetric spaces under a contractive condition of integral type.

In this paper, we give some common fixed-point theorems in symmetric(semimetric) spaces and give counterexamples for the results of Imdad et al. [5].

In order to obtain common fixed-point theorems in symmetric spaces, some axioms are needed. In [5], the authors assumed axiom (W3), and in [6] the author assumed axioms (W3), (W4), and (H.E); see Section 2 for definitions.

We give another axiom for symmetric spaces and study their relationships in Section 2. We give common fixed-point theorems of four mappings in symmetric spaces and give some examples which justifies the necessity of axioms in Section 3.

2. Axioms on symmetric spaces

A *symmetric* on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions:

   (i) $d(x, y) = 0$, if and only if $x = y$ for $x, y \in X$,

   (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$.

Let d be a symmetric on a set X. For $x \in X$ and $\epsilon > 0$, let $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X defined as follows: $U \in \tau(d)$ if and only if for each $x \in U$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symmetric d is a *semimetric* if for each $x \in X$ and each $\epsilon > 0$, $B(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$.

A *symmetric* (resp., *semimetric*) *space* (X, d) is a topological space whose topology $\tau(d)$ on X is induced by symmetric(resp., semi-metric) d.

The difference of a symmetric and a metric comes from the triangle inequality. Actually a symmetric space need not be Hausdorff. In order to obtain fixed-point theorems on a symmetric space, we need some additional axioms. The following axioms can be found in [7].

   (W3) for a sequence $\{x_n\}$ in $X, x, y \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y) = 0$ imply $x = y$.

   (W4) for sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(y_n, x_n) = 0$ imply $\lim_{n \to \infty} d(y_n, x) = 0$.

Also the following axiom can be found in [6].

   (H.E) for sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(y_n, x) = 0$ imply $\lim_{n \to \infty} d(x_n, y_n) = 0$.

Now, we add a new axiom which is related to the continuity of the symmetric d.

   (C.C) for sequences $\{x_n\}$ in X and $x, y \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ implies $\lim_{n \to \infty} d(x_n, y) = d(x, y)$.

Note that if d is a metric, then (W3), (W4), (H.E), and (C.C) are automatically satisfied. And if $\tau(d)$ is Hausdorff, then (W3) is satisfied.

Proposition 2.1. *For axioms in symmetric space* (X, d), *one has*

   (1) *(W4)* $\Rightarrow$ *(W3)*,

   (2) *(C.C)* $\Rightarrow$ *(W3)*.

Proof. Let $\{x_n\}$ be a sequence in X and $x, y \in X$ with $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y) = 0$.

   (1) By putting $y_n = y$ for each $n \in N$, we have $\lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x_n, y) = 0$. By (W4), we have $0 = \lim_{n \to \infty} d(y_n, x) = d(y, x)$.

   (2) By (C.C), $\lim_{n \to \infty} d(x_n, x) = 0$ implies $d(x, y) = \lim_{n \to \infty} d(x_n, y) = 0$.

The following examples show that other relationships in Proposition 2.1 do not hold. $\square$

Example 2.2. (W4) $\not\Rightarrow$ (H.E) and (W4) $\not\Rightarrow$ (C.C) and so (W3) $\not\Rightarrow$ (H.E) and (W3) $\not\Rightarrow$ (C.C) by Proposition 2.1 (1).

Let $X = [0, \infty)$ and let

$$d(x, y) = \begin{cases} |x - y| & (x \neq 0, \ y \neq 0), \\ \dfrac{1}{x} & (x \neq 0). \end{cases} \tag{2.1}$$

Then, (X, d) is a symmetric space which satisfies (W4) but does not satisfy (H.E) for $x_n = n, y_n = n + 1$. Also (X, d) does not satisfy (C.C).

Example 2.3. (H.E) $\not\Rightarrow$ (W3), and so (H.E) $\not\Rightarrow$ (W4) and (H.E) $\not\Rightarrow$ (C.C).

Let $X = [0, 1] \cup \{2\}$ and let

$$d(x, y) = \begin{cases} |x - y| & (0 \leq x \leq 1, 0 \leq y \leq 1), \\ |x| & (0 < x \leq 1, y = 2) \end{cases} \tag{2.2}$$

and $d(0, 2) = 1$.

Then, (X, d) is a symmetric space which satisfies (H.E). Let $x_n = 1/n$. Then, $\lim_{n \to \infty} d(x_n, 0) = \lim_{n \to \infty} d(x_n, 2) = 0$. But $d(0, 2) \neq 0$ and hence the symmetric space (X, d) does not satisfy (W3).

Example 2.4. (C.C) $\not\Rightarrow$ (W4) and so (W3) $\not\Rightarrow$ (W4) by Proposition 2.1(2).

Let $X = \{1/n : n = 1, 2, \ldots\} \cup \{0\}$, and let $d(0, 1/n) = (1/n)$ (n is odd), $d(0, 1/n) = 1$ (n is even) and

$$d\left(\frac{1}{m}, \frac{1}{n}\right) = \begin{cases} \left|\dfrac{1}{m} - \dfrac{1}{n}\right| & (m + n \text{ is even}), \\ \left|\dfrac{1}{m} - \dfrac{1}{n}\right| & (m + n \text{ is odd and } |m - n| = 1), \\ 1 & (m + n \text{ is odd and } |m - n| > 2). \end{cases} \tag{2.3}$$

Then, the symmetric space (X, d) satisfies (C.C) but does not satisfy (W4) for $x_n = 1/(2n + 1)$ and $y_n = 1/2n$.

Example 2.5. (C.C) $\not\Rightarrow$ (H.E).

Let $X = \{1/n : n = 1, 2, \ldots\} \cup \{0\}$, and let

$$d\left(\frac{1}{m}, \frac{1}{n}\right) = \begin{cases} \left|\dfrac{1}{m} - \dfrac{1}{n}\right| & (|m - n| \geq 2), \\ 1 & (|m - n| = 1) \end{cases} \tag{2.4}$$

and $d(1/n, 0) = 1/n$. Then, (X, d) is a symmetric space which satisfies (C.C). Let $x_n = 1/n$, $y_n = 1/(n + 1)$. Then, $\lim_{n \to \infty} d(x_n, 0) = \lim_{n \to \infty} d(y_n, 0) = 0$. But $\lim_{n \to \infty} d(x_n, y_n) \neq 0$. Hence, the symmetric space (X, d) does not satisfy (H.E).

3. Common fixed points of four mappings

Let (X, d) be a symmetric(or semimetric) space and let f, g be self-mappings of X. Then, we say that the pair (f, g) satisfies property (E-A) [3] if there exists a sequence $\{x_n\}$ in X and a point $t \in X$ such that $\lim_{n \to \infty} d(fx_n, t) = \lim_{n \to \infty} d(gx_n, t) = 0$.

A subset S of a symmetric space (X, d) is said to be d-closed if for a sequence $\{x_n\}$ in S and a point $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ implies $x \in S$. For a symmetric space (X, d), d-closedness implies $\tau(d)$-closedness, and if d is a semimetric, the converse is also true.

At first, we prove coincidence point theorems of four mappings satisfying the property (E-A) under some contractive conditions.

Theorem 3.1. *Let (X, d) be a symmetric(semimetric) space that satisfies (W3) and (H.E), and let $A, B, S,$ and T be self-mappings of X such that*

    (1) *$AX \subset TX$ and $BX \subset SX$,*

    (2) *the pair (B, T) satisfies property (E-A) (resp., (A, S) satisfies property (E-A)),*

    (3) *for any $x, y \in X$, $d(Ax, By) \le m(x, y)$, where*

$$m(x, y) = \max\{d(Sx, Ty), \min\{d(Ax, Sx), d(By, Ty)\}, \min\{d(Ax, Ty), d(By, Sx)\}\}, \quad (3.1)$$

    (4) *SX is a d-closed ($\tau(d)$-closed) subset of X (resp., TX is a d-closed($\tau(d)$-closed) subset of X).*

Then, there exist $u, w \in X$ such that $Au = Su = Bw = Tw$.

Proof. From (2), there exist a sequence $\{x_n\}$ in X, and a point $t \in X$ such that $\lim_{n \to \infty} d(Tx_n, t) = \lim_{n \to \infty} d(Bx_n, t) = 0$.

From (1), there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$ and hence $\lim_{n \to \infty} d(Sy_n, t) = 0$. By (H.E), $\lim_{n \to \infty} d(Bx_n, Tx_n) = \lim_{n \to \infty} d(Sy_n, Tx_n) = 0$.

From (4), there exists a point $u \in X$ such that $Su = t$.

From (3), we have

$$d(Au, Bx_n)$$
$$\le \max\{d(Su, Tx_n), \min\{d(Au, Su), d(Bx_n, Tx_n)\}, \min\{d(Au, Tx_n), d(Bx_n, Su)\}\}. \quad (3.2)$$

By taking $n \to \infty$, we have $\lim_{n \to \infty} d(Au, Bx_n) = 0$. By (W3), we get $Au = Su$.

Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$.

We show that $Tw = Bw$. From (3), we have

$$d(Au, Bw)$$
$$\le \max\{d(Su, Tw), \min\{d(Au, Su), d(Bw, Tw)\}, \min\{d(Au, Tw), d(Bw, Su)\}\}$$
$$= \max\{d(Tw, Tw), \min\{d(Au, Au), d(Bw, Tw)\}, \min\{d(Au, Au), d(Bw, Su)\}\} = 0. \quad (3.3)$$

Hence, $Au = Bw$ and hence $Au = Su = Bw = Tw$.

For the existence of a common fixed point of four self-mappings of a symmetric space, we need an additional condition, so-called weak compatibility.

Recall that for self-mappings f and g of a set, the pair (f, g) is said to be *weakly compatible* [8] if $fgx = gfx$, whenever $fx = gx$. Obviously, if f and g are commuting, the pair (f, g) is weakly compatible. $\qquad \square$

Theorem 3.2. *Let* (X, d) *be a symmetric(semimetric) space that satisfies (W3) and (H.E), and let* $A, B, S,$ *and* T *be self-mappings of* X *such that*

(1) $AX \subset TX$ *and* $BX \subset SX$,

(2) *the pair* (B, T) *satisfies property (E-A)(resp.,* (A, S) *satisfies property (E-A)),*

(3) *the pairs* (A, S) *and* (B, T) *are weakly compatible,*

(4) *for any* $x, y \in X(x \neq y), d(Ax, By) < m(x, y)$,

(5) SX *is a* d-*closed(*$\tau(d)$*-closed) subset of* X *(resp.,* TX *is a* d-*closed (*$\tau(d)$*-closed) subset of* X).

Then, $A, B, S,$ *and* T *have a unique common fixed point in* X.

Proof. From Theorem 3.1, there exist $u, w \in X$ such that $Au = Su = Tw = Bw$. From (3), $ASu = SAu$, $AAu = ASu = SAu = SSu$ and $BTw = TBw = TTw = BBw$.

If $Au \neq w$, then from (4) we have

$$
\begin{aligned}
d(Au, AAu) \\
&= d(AAu, Bw) \\
&< \max\left\{d(SAu, Tw), \min\left\{d(AAu, SAu), d(Bw, Tw)\right\}, \min\left\{d(AAu, Tw), d(Bw, SAu)\right\}\right\} \\
&= \max\left\{d(AAu, Au), 0, d(AAu, Au)\right\} \\
&= d(AAu, Au)
\end{aligned}
\tag{3.4}
$$

which is a contradiction.

Similarly, if $u \neq Bw$, we have a contradiction. Thus, $Au = w = Su = Tw = Bw = u$, and w is a common fixed point of $A, B, S,$ and T.

For the uniqueness, let z be another common fixed point of $A, B, S,$ and T. If $w \neq z$, then from (4) we get

$$
\begin{aligned}
d(z, w) &= d(Az, Bw) \\
&< \max\left\{d(Sz, Tw), \min\left\{d(Az, Sz), d(Bw, Tw)\right\}, \min\left\{d(Az, Tw), d(Bw, Sz)\right\}\right\} \\
&= \max\left\{d(z, w), \min\left\{d(z, z), d(w, w)\right\}, \min\left\{d(z, w), d(w, z)\right\}\right\} \\
&= d(z, w)
\end{aligned}
\tag{3.5}
$$

which is a contradiction. Hence, $w = z$. □

Remark 3.3. In the case of $A = B = g$ and $S = T = f$ in Theorem 3.1 (resp., Theorem 3.2), we can show that f and g have a coincidence point (resp., f and g have a unique common fixed point) without making the assumption $gX \subset fX$.

Recently, R. P. Pant and V. Pant [4] obtained the existence of a common fixed point of the pair of (f, g) in a metric space (X, d) satisfying the condition

(P.P) for any $x, y \in X$,

$$
d(gx, gy) < \max\left\{d(fx, fy), \frac{k}{2}\{d(fx, gx) + d(fy, gy)\}, \frac{1}{2}\{d(fy, gx) + d(fx, gy)\}\right\}, \tag{3.6}
$$

where $1 \leq k < 2$.

Also in [5], the authors tried to extend the result of [4] to symmetric spaces which satisfy axiom (W3).

Now, we will extend R. P. Pant and V. Pant's result to symmetric spaces which satisfy additional conditions (H.E) and (C.C).

Theorem 3.4. *Let (X, d) be a symmetric(semimetric) space that satisfies (H.E) and (C.C) and let A, B, S, and T be self-mappings of X such that*

(1) $AX \subset TX$ and $BX \subset SX$,

(2) *the pair (B, T) satisfies property (E-A) (resp., (A, S) satisfies property (E-A)),*

(3) *for any $x, y \in X$, $d(Ax, By) \leq m_1(x, y)$, where $m_1(x, y) = \max\{d(Sx, Ty), (k/2)\{d(Ax, Sx) + d(By, Ty)\}, (k/2)\{d(Ax, Ty) + d(By, Sx)\}\}$, $0 < k < 2$,*

(4) *SX is a d-closed ($\tau(d)$-closed) subset of X (resp., TX is a d-closed ($\tau(d)$-closed) subset of X).*

Then, there exist $u, w \in X$ such that $Au = Su = Bw = Tw$.

Proof. As in the proof of Theorem 3.1, there exist sequences $\{x_n\}, \{y_n\}$ in X and a point $t \in X$ such that $\lim_{n \to \infty} d(Tx_n, t) = \lim_{n \to \infty} d(Bx_n, t) = 0$ and $Bx_n = Sy_n$. Hence, $\lim_{n \to \infty} d(Sy_n, t) = 0$.

From (4), there exists a point $u \in X$ such that $Su = t$.

We show $Au = Su$. From (3), we have

$$
d(Au, Bx_n)
$$
$$
\leq \max\left\{d(Su, Tx_n), \frac{k}{2}\{d(Au, Su) + d(Bx_n, Tx_n)\}, \frac{k}{2}\{d(Au, Tx_n) + d(Bx_n, Su)\}\right\}
$$

(3.7)

In the above inequality, we take $n \to \infty$, by (C.C) and (H.E), we have

$$
d(Au, Su) \leq \max\left\{0, \frac{k}{2}d(Au, Su), \frac{k}{2}\{d(Au, Su)\}\right\}
$$
$$
= \frac{k}{2}d(Au, Su).
$$

(3.8)

Since $0 < k/2 < 1$, we get $d(Au, Su) = 0$ and hence $Au = Su$.

Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$.

We show that $Tw = Bw$. From (3), we have

$$
d(Tw, Bw) = d(Au, Bw)
$$
$$
\leq \max\left\{d(Su, Tw), \frac{k}{2}\{d(Au, Su) + d(Bw, Tw)\}, \frac{k}{2}\{d(Au, Tw) + d(Bw, Su)\}\right\}
$$
$$
= \max\left\{d(Tw, Tw), \frac{k}{2}\{d(Au, Au) + d(Bw, Tw)\}, \frac{k}{2}\{d(Au, Au) + d(Bw, Su)\}\right\}
$$
$$
= \max\left\{\frac{k}{2}d(Bw, Tw), \frac{k}{2}d(Bw, Su)\right\}
$$
$$
= \frac{k}{2}d(Bw, Tw).
$$

(3.9)

Since $0 < k/2 < 1$, we get $d(Tw, Bw) = 0$ and hence $Tw = Bw$. Therefore, we have $Au = Su = Bw = Tw$. $\square$

Theorem 3.5. *(X, d) be a symmetric(semimetric) space that satisfies (H.E) and (C.C) and let $A, B, S,$ and T be self-mappings of X such that*

(1) *$AX \subset TX$ and $BX \subset SX$,*

(2) *the pair (B, T) satisfies property (E-A) (resp., (A, S) satisfies property (E-A)),*

(3) *the pairs (A, S) and (B, T) are weakly compatible,*

(4) *for any $x, y \in X(x \neq y), d(Ax, By) < m_2(x, y),$ where $m_2(x, y) = \max\{d(Sx, Ty), (k/2)\{d(Ax, Sx) + d(By, Ty)\}, (1/2)\{d(Ax, Ty) + d(By, Sx)\}\},\ 0 < k < 2.$*

(5) *SX is a d-closed ($\tau(d)$-closed) subset of X (resp., TX is a d-closed($\tau(d)$-closed) subset of X).*

Then *$A, B, S,$ and T have a unique common fixed point in X.*

Proof. From Theorem 3.4, there exist points $u, w \in X$ such that $Au = Su = Tw = Bw, AAu = ASu = SAu = SSu,$ and $BTw = TBw = TTw = BBw.$

We show that $Au = w$. If $Au \neq w$, then from (4) we have

$$d(Au, AAu)$$

$$= d(AAu, Bw)$$

$$< \max\left\{d(SAu, Tw), \frac{k}{2}\{d(AAu, SAu) + d(Bw, Tw)\}, \frac{1}{2}\{d(AAu, Tw) + d(Bw, SAu)\}\right\}$$

$$= \max\left\{d(AAu, Au), 0, d(AAu, Au)\right\} = d(AAu, Au).$$

$$(3.10)$$

which is a contradiction.

Similarly, if $u \neq Bw$, we have a contradiction. Thus $Au = w = Su = Tw = Bw = u.$

For the uniqueness, let w be another common fixed point of $A, B, S,$ and T. If $w \neq z,$ then from (4) we get

$$d(z, w) = d(Az, Bw)$$

$$< \max\left\{d(Sz, Tw), \frac{k}{2}\{d(Az, Sz) + d(Bw, Tw)\}, \frac{1}{2}\{d(Az, Tw) + d(Bw, Sz)\}\right\}$$

$$= \max\left\{d(z, w), \frac{k}{2}\{d(z, z) + d(w, w)\}, \frac{1}{2}\{d(z, w) + d(w, z)\}\right\}$$

$$= \max\left\{d(z, w), 0, d(w, z)\right\} = d(z, w).$$

$$(3.11)$$

which is a contradiction. Hence $w = z.$ □

Example 3.6. Let $X = [0, 1]$ and $d(x, y) = (x - y)^2$. Define self-mappings $A, B, S,$ and T by $Ax = Bx = (1/2)x$ and $Sx = Tx = x$ for all $x \in X$. Then, we have the following:

(0) (X, d) is a symmetric space satisfying the properties (H.E) and (C.C),

(1) $AX \subset TX$ and $BX \subset SX$,

(2) the pair (B, T) satisfies property (E-A) for the sequence $x_n = 1/n,\ n = 1, 2, 3, \ldots,$

(3) the pairs (A, S) and (B, T) are weakly compatible,

(4) for any $x, y \in X(x \neq y), d(Ax, By) < d(Sx, Ty) \leq m_i(x, y),\ i = 1, 2,$

(5) SX is a d-closed($\tau(d)$-closed) subset of X,

(6) $A0 = B0 = S0 = T0 = 0.$

Remark 3.7. In the case of $A = B = g$ and $S = T = f$ in **Theorem 3.4** (resp., **Theorem 3.5**), we can show that f and g have a coincidence point (resp., f and g have a unique common fixed point) without the condition (1), that is, $gX \subset fX$.

The following example shows that the axioms (H.E) and (C.C) cannot be dropped in **Theorem 3.4**.

Example 3.8. Let (X, d) be the symmetric space as in **Example 2.2**. Then, the symmetric d does not satisfy both (H.E) and (C.C).

Let $S = T = f$ and $A = B = g$ be self-mappings of X defined as follows:

$$f x = x (x \geq 0), \qquad g x = \begin{cases} \dfrac{1}{3}x & (x > 0), \\ \dfrac{1}{3} & (x = 0). \end{cases} \tag{3.12}$$

Then, the condition (3) (resp., (4)) of **Theorem 3.4** (resp., **Theorem 3.5**) is satisfied for $k = 1$.

To show this, let $n_1(x, y) = \max\{d(fx, fy), (1/2)\{d(fx, gx) + d(fy, gy)\}, (1/2)\{d(fy, gx) + d(fx, gy)\}\}$. We consider two cases.

Case 1. $x = 0$, $y > 0$,

$$\begin{aligned} n_1(x, y) &= \max\left\{ d(0, y), \frac{1}{2}\left\{ d\left(0, \frac{1}{3}\right) + d\left(y, \frac{1}{3}y\right)\right\}, \frac{1}{2}\left\{ d\left(0, \frac{1}{3}y\right) + d\left(y, \frac{1}{3}\right)\right\}\right\} \\ &= \max\left\{ \frac{1}{y}, \frac{1}{2}\left\{ 3 + \frac{2}{3}y\right\}, \frac{1}{2}\left\{ \frac{3}{y} + \left|y - \frac{1}{3}\right|\right\}\right\} \tag{3.13} \\ &\geq \frac{1}{2}\left\{ 3 + \frac{2}{3}y\right\} = \frac{y}{3} + \frac{3}{2} > \frac{1}{3}|y - 1| = d\left(\frac{1}{3}, \frac{1}{3}y\right) = d(gx, gy). \end{aligned}$$

Case 2. $x > 0$, $y > 0 (x \neq y)$,

$$n_1(x, y) \geq d(fx, fy) = |x - y| > \frac{1}{3}|x - y| = d(gx, gy). \tag{3.14}$$

Thus, the condition (3) (resp., (4)) of **Theorem 3.4** (resp., **Theorem 3.5**) is satisfied. Note that fX is a d-closed($\tau(d)$-closed) subset of X. Also, the pair (f, g) satisfies property (E-A) for $x_n = n$, but the pair (f, g) has no coincidence points, and also the pair (f, g) has no common fixed points.

Remark 3.9. **Example 3.6** satisfies all conditions of [5, Theorems 2.1 and 2.2] and satisfies also all conditions of [5, Theorem 2.3].

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that

(ϕ1) ϕ is nondecreasing on $\mathbb{R}^+$,

(ϕ2) $0 < \phi(t) < t$ for all $t \in (0, \infty)$.

Note that from (ϕ1) and (ϕ2), we have $\phi(0) = 0$.

On the studying of fixed points, various conditions of ϕ have been studied by many different authors [3, 5, 6].

Remark 3.10. The functions $m_i(x, y)$ in Theorems 3.4 and 3.5 can be generalized to the compositions $\phi(m_i(x, y))$ for $i = 1, 2$.

Example 3.11. Let (X, d) be the symmetric space and $A, B, S,$ and T be the functions as in Example 3.8. Recall that (X, d) satisfies (W3) but does not satisfy both (H.E) and (C.C). Let $\phi(t) = (2/3)t,\ t \in \mathbb{R}^+$ and $k = 3/2$. Then, for any $x, y \in X, d(Ax, By) \leq \phi(m_i(x, y))$ for $i = 1, 2$. Note that the pairs (A, S) and (B, T) satisfy property (E-A), and $AX \subset TX, BX \subset SX$, and SX are d-closed($\tau(d)$-closed).

Therefore, $A, B, S,$ and T satisfy all conditions of [5, Theorem 2.4] and satisfy also all conditions of [5, Theorem 2.5]. But the pairs (A, S) and (B, T) have no points of coincidence, and also the pairs (A, S) and (B, T) have no common fixed points.

Acknowledgments

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