Research Article

Best Proximity Pairs for Upper Semicontinuous Set-Valued Maps in Hyperconvex Metric Spaces

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A best proximity pair for a set-valued map $F : A \multimap B$ with respect to a map $g : A \to A$ is defined, and new existence theorems of best proximity pairs for upper semicontinuous set-valued maps with respect to a homeomorphism are proved in hyperconvex metric spaces.

1. Introduction and preliminaries

Let (M, d) be a metric space and let A and B be nonempty subsets of M. Let $g : A \to A$ and let $F : A \multimap B$ be a set-valued map. Now, $(g(a), F(a))$ is called a *best proximity pair* for F with respect to g if $d(g(a), F(a)) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Best proximity pair theorems establish conditions under which the problem of minimizing the real-valued function $x \to d(g(x), F(x))$ has a solution. In the setting of normed linear spaces, the best proximity pair problem has been studied by many authors for $g = I$, see [1–5]. Very recently, Al-Thagafi and Shahzad [1] proved some existence theorems for a finite family of Kakutani set-valued maps in a normed space setting. In the present paper, our aim is to prove new results in hyperconvex metric spaces. In the rest of this section, we recall some definitions and theorems which are used in Section 2.

Let X and Y be topological spaces with $A \subseteq X$ and $B \subseteq Y$. Let $F : X \multimap Y$ be a set-valued map with nonempty values. The image of A under F is the set $F(A) = \bigcup_{x \in A} F(x)$ and the inverse image of B under F is $F^-(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$. Now, F is said to be upper semicontinuous, if for each closed set $B \subseteq Y$, $F^-(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$ is closed in X.

A topological space X is said to be *contractible* if the identity map I_X of X is homotopic to a constant map and *acyclic* if all of its reduced Čech homology groups over the rationals vanish. Note that a contractible space is acyclic. For topological spaces X and Y, we define

$$F \in \mathbb{V}(X, Y) \Longleftrightarrow F : X \multimap Y \text{ is an acyclic map; that is,}$$
$$F \text{ is upper semicontinuous with compact acyclic values.} \tag{1.1}$$

We denote by $\mathbb{V}_c(X, Y)$ the set of all finite composites of maps in $\mathbb{V}(X, Y)$. Let (M, d) be a metric space and let $B(x, r) = \{y \in M : d(x, y) \leq r\}$ denote the closed ball with center x and radius r. Let

$$\mathrm{co}(A) = \bigcap \{B \subseteq M : B \text{ be a closed ball in } M \text{ such that } A \subseteq B\}. \tag{1.2}$$

If $A = \mathrm{co}(A)$, we say that A is admissible subset of M. Note that $\mathrm{co}(A)$ is admissible and the intersection of any family of admissible subsets of M is admissible. The following definition of a hyperconvex metric space is due to Aronszajn and Pantichpakdi [6].

Definition 1.1. A metric space (M, d) is said to be a hyperconvex metric space if for any collection of points x_α of M and any collection r_α of nonnegative real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, one has

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \varnothing. \tag{1.3}$$

The simplest examples of hyperconvex spaces are finite dimensional real Banach spaces endowed with the maximum norm. For other examples of hyperconvex metric spaces which are not linear spaces, see [7]. Note that an admissible subset of a hyperconvex metric space is hyperconvex and contractible [8]. Let A be a subset of M. The r-parallel of A is defined as

$$A + r = \bigcup_{a \in A} B(a, r). \tag{1.4}$$

The following result is due to Sine [9].

Lemma 1.2. *The r-parallel sets of an admissible subset of a hyperconvex metric space are also admissible sets.*

For $A \subseteq M$, the set $P_A(x) = \{a \in A : d(a, x) = d(x, A)\}$ is called the set of best approximations in A to $x \in M$. The map $P_A : M \multimap A$ is called the metric projection on A. The following lemma is well known. We give its proof for completeness.

Lemma 1.3. *Let A be a nonempty, admissible, and compact subset of a hyperconvex metric space (M, d). Then $P_A \in \mathbb{V}(M, A)$.*

Proof. Since A is compact, then P_A is nonempty. We now show that P_A is contractible and so is acyclic. To see this, notice that

$$P_A(x) = A \cap \{a \in M : d(a,x) \leq d(x,A)\} = A \cap B(x, d(x,A)). \tag{1.5}$$

Then $P_A(x)$ is admissible (note that A is admissible) and therefore is contractible. Now, we show that P_A is upper semicontinuous. Let C be a closed subset of A, $x_n \in \{x : P_A(x) \cap C \neq \varnothing\}$, and $x_n \to x_0$. Then there exists a sequence $a_n \in P_A(x_n) \cap C$ such that $d(a_n, x_n) = d(x_n, A)$. Since A is compact and $a_n \in A$, without loss of generality, we may assume that $a_n \to a$. Thus,

$$d(a, x_0) = \lim_{n \to \infty} d(a_n, x_n) = \lim_{n \to \infty} d(x_n, A) = d(x_0, A). \tag{1.6}$$

Therefore, $x_0 \in \{x : P_A(x) \cap C \neq \varnothing\}$ and the set $\{x : P_A(x) \cap C \neq \varnothing\}$ is closed. $\square$

To prove our main result, we need the following fixed point theorem, which is particular form of Theorem 4 in [10].

Theorem 1.4. *Let X be a nonempty compact admissible subset of a hyperconvex metric space (M, d) and $F \in \mathbb{V}_c(X, X)$. Then F has a fixed point.*

Corollary 1.5. *Let X be a nonempty compact admissible subset of a hyperconvex metric space (M, d), $g : X \to X$ a homeomorphism, and $F \in \mathbb{V}_c(X, X)$. Then there exists an $x_0 \in X$ such that $g(x_0) \in F(x_0)$.*

Proof. Since g is a homeomorphism, then $g^{-1} \circ F \in \mathbb{V}_c(X, X)$. Hence, by Theorem 1.4, $g^{-1} \circ F$ has a fixed point, say x_0. Therefore, $g(x_0) \in F(x_0)$. $\square$

2. Best proximity theorems

Let A and B be nonempty subsets of M. Define

$$\begin{aligned}
A^0 &:= \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\}, \\
B^0 &:= \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.
\end{aligned} \tag{2.1}$$

Notice that A^0 is nonempty if and only if B^0 is nonempty.

Theorem 2.1. *Let A and B be nonempty subsets of M. Then the following statement holds.*

    (i) *If A^0 and B are admissible, then B^0 is admissible.*

    (ii) *If A^0 and B are compact, then B^0 is compact.*

Proof. To prove (a), notice that

$$
\begin{aligned}
B^0 &:= \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\} \\
&= \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A^0\} \\
&= B \cap \{b \in M : d(a,b) \le d(A,B) \text{ for some } a \in A^0\} \\
&= B \cap \left(\bigcup_{a \in A^0} B(a, d(A,B)) \right) = B \cap (A^0 + d(A,B)).
\end{aligned}
\tag{2.2}
$$

Since A^0 is admissible, then by Lemma 1.2, $A^0 + d(A,B)$ is also admissible. Thus, B^0 is admissible (note, that B is admissible).

(b) Let $\{b_n\}$ be a sequence in B^0 such that $b_n \to b \in B$. Then there exists sequence $\{a_n\}$ in A such that $d(a_n, b_n) = d(A,B)$. Since A^0 is compact, we may assume that $a_n \to a \in A^0$. Then

$$
d(a,b) = \lim_{n \to \infty} d(a_n, b_n) = d(A,B).
\tag{2.3}
$$

Thus, $b \in B^0$. Therefore, B^0 is closed and so compact. $\qquad\square$

Theorem 2.2. *Let (M,d) be a hyperconvex metric space, $A \subseteq M$ and $B \subseteq M$ are admissible. Let $g : A^0 \to A^0$ be a homeomorphism, and let $F : A \multimap B$ be an upper semicontinuous set-valued map with admissible values. Assume that A^0 is compact and admissible and $F(x) \cap B^0$ is nonempty, for each $x \in A^0$. Then there exists $a \in A^0$ such that $d(g(a), F(a)) = d(A,B)$.*

Proof. We use some ideas from [1, Theorem 3.2]. From [11, Proposition 2.8], A^0 is nonempty. Since A^0 and B are admissible, it follows from Theorem 2.1(a) that B^0 is admissible. From Lemma 1.3, $P_{A^0} \in \mathbb{V}(B^0, A^0)$ (note, that A^0 is nonempty, admissible, and compact and B^0 is hyperconvex since B^0 is an admissible subset of M). Define $G : A^0 \multimap B^0$ by $G(x) = F(x) \cap B^0$. Since F is upper semicontinuous with nonempty admissible values and B^0 is admissible, then G is upper semicontinuous with admissible (in particular acyclic) values. From Lemma 1.3 (see proof), $P_{A^0} : B^0 \multimap A^0$ is upper semicontinuous with admissible (in particular acyclic) values. Since $P_{A^0} \circ G \in \mathbb{V}_c(A^0, A^0)$, it follows from Corollary 1.5 that there exists $a \in A^0$ such that $g(a) \in P_{A^0} \circ G(a)$. Thus, there exists $b \in G(a)$ such that $g(a) \in P_{A^0}(b)$. Hence, $g(a) \in P_{A^0}(b) \subseteq A^0$ and $b \in G(a) = F(a) \cap B^0$. Since $b \in B^0$, there exists $a_0 \in A^0$ such that $d(a_0, b) = d(A,B)$ and hence

$$
d(A,B) \le d(g(a), F(a)) \le d(g(a), b) = d(b, A^0) \le d(a_0, b) = d(A,B).
\tag{2.4}
$$

Then

$$
d(g(a), F(a)) = d(A,B).
\tag{2.5}
$$

$\qquad\square$

If we take $g = I_{A^0}$, we get the following corollary.

Corollary 2.3. *Let (M, d) be a hyperconvex metric space, $A \subseteq M$ and $B \subseteq M$ are admissible. Let $F : A \multimap B$ be an upper semicontinuous set-valued map with admissible values. Assume that $F(x) \cap B^0$ is nonempty, for each $x \in A^0$ and A^0 is compact and admissible. Then there exists $a \in A^0$ such that $d(a, F(a)) = d(A, B)$.*

Corollary 2.4 (see [8, Corollary 5.6]). *Let (M, d) be a hyperconvex metric space and $A \subseteq M$ nonempty, compact, and admissible. Let $F : A \multimap M$ be an upper semicontinuous set-valued map with admissible values. Assume that $F(x) \cap A$ is nonempty, for each $x \in A$. Then F has a fixed point.*

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