Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 346519, 16 pages
doi:10.1155/2009/346519

Research Article

Minimal Nielsen Root Classes and Roots of Liftings

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Received 24 April 2009; Accepted 26 May 2009

Recommended by Robert Brown

Given a continuous map $f : K \to M$ from a 2-dimensional CW complex into a closed surface, the Nielsen root number $N(f)$ and the minimal number of roots $\mu(f)$ of f satisfy $N(f) \leq \mu(f)$. But, there is a number $\mu_C(f)$ associated to each Nielsen root class of f, and an important problem is to know when $\mu(f) = \mu_C(f)N(f)$. In addition to investigate this problem, we determine a relationship between $\mu(f)$ and $\mu(\widetilde{f})$, when $\widetilde{f}$ is a lifting of f through a covering space, and we find a connection between this problems, with which we answer several questions related to them when the range of the maps is the projective plane.

1. Introduction

Let $f : X \to Y$ be a continuous map between Hausdorff, normal, connected, locally path connected, and semilocally simply connected spaces, and let $a \in Y$ be a given base point. A *root* of f at a is a point $x \in X$ such that $f(x) = a$. In root theory we are interested in finding a lower bound for the number of roots of f at a. We define the *minimal number of roots* of f at a to be the number

$$\mu(f, a) = \min\left\{\#\varphi^{-1}(a) \text{ such that } \varphi \text{ is homotopic to } f\right\}. \tag{1.1}$$

When the range Y of f is a manifold, it is easy to prove that this number is independent of the selected point $a \in Y$, and, from [1, Propositions 2.10 and 2.12], $\mu(f, a)$ is a finite number, providing that X is a finite CW complex. So, in this case, there is no ambiguity in defining the *minimal number of roots* of f:

$$\mu(f) := \mu(f, a) \quad \text{for some } a \in Y. \tag{1.2}$$

Definition 1.1. If $\varphi : X \to Y$ is a map homotopic to f and $a \in Y$ is a point such that $\mu(f) = \#\varphi^{-1}(a)$, we say that the pair (φ, a) provides $\mu(f)$ or that (φ, a) is a pair providing $\mu(f)$.

According to [2], two roots x_1, x_2 of f at a are said to be *Nielsen root f equivalent* if there is a path $\gamma : [0,1] \to X$ starting at x_1 and ending at x_2 such that the loop $f \circ \gamma$ in Y at a is fixed-end-point homotopic to the constant path at a. This relation is easily seen to be an equivalence relation; the equivalence classes are called *Nielsen root classes of f at a*. Also a homotopy H between two maps f and f' provides a correspondence between the Nielsen root classes of f at a and the Nielsen root classes of f' at a. We say that such two classes under this correspondence are *H-related*. Following Brooks [2] we have the following definition.

Definition 1.2. A Nielsen root class $\mathfrak{R}$ of a map f at a is essential if given any homotopy $H : f \simeq f'$ starting at f, and the class $\mathfrak{R}$ is H-related to a root class of f' at a. The number of essential root classes of f at a is the *Nielsen root number of f at a*; it is denoted by $N(f, a)$.

The number $N(f, a)$ is a homotopy invariant, and it is independent of the selected point $a \in Y$, provid that Y is a manifold. In this case, there is no danger of ambiguity in denot it by $N(f)$.

In a similar way as in the previous definition, Gonçalves and Aniz in [3] define the minimal cardinality of Nielsen root classes.

Definition 1.3. Let $\mathfrak{R}$ be a Nielsen root class of $f : X \to Y$. We define $\mu_C(f, \mathfrak{R})$ to be the minimal cardinality among all Nielsen root classes $\mathfrak{R}'$, of a map f', H-related to $\mathfrak{R}$, for H being a homotopy starting at f and ending at f':

Again in [3] was proved that if Y is a manifold, then the number $\mu_C(f, \mathfrak{R})$ is independent of the Nielsen root class of $f : X \to Y$. Then, in this case, there is no danger of ambiguity in defining the *minimal cardinality of Nielsen root classes of f*

$$\mu_C(f) := \mu_C(\mathfrak{R}) \text{ for some Nielsen rootclass } \mathfrak{R}. \tag{1.3}$$

An important problem is to know when it is possible to deform a map f to some map f' with the property that all its Nielsen root classes have minimal cardinality. When the range Y of f is a manifold, this question can be summarized in the following: when $\mu(f) = \mu_C(f)N(f)$?

Gonçalves and Aniz [3] answered this question for maps from CW complexes into closed manifolds, both of same dimension greater or equal to 3. Here, we study this problem for maps from 2-dimensional CW complexes into closed surfaces. In this context, we present several examples of maps having liftings through some covering space and not having all Nielsen root classes with minimal cardinality.

Another problem studied in this article is the following. Let $p_k : \overline{Y} \to Y$ be a k-fold covering. Suppose that $f : X \to Y$ is a map having a lifting $\widetilde{f} : X \to \overline{Y}$ through p_k. What is the relationship between the numbers $\mu(f)$ and $\mu(\widetilde{f})$? We answer completely this question for the cases in which X is a connected, locally path connected and semilocally simply connected space, and Y and $\overline{Y}$ are manifolds either compact or triangulable. We show that $\mu(f) \geq k\mu(\widetilde{f})$, and we find necessary and sufficient conditions to have the identity.

Related results for the Nielsen fixed point theory can be found in [4].

In Section 4, we find an interesting connection between the two problems presented. This whole section is devoted to the demonstration of this connection and other similar results.

In the last section of the paper, we answer several questions related to the two problems presented when the range of the considered maps is the projective plane.

Throughout the text, we simplify write f is a map instead of f is a continuous map.

2. The Minimizing of the Nielsen Root Classes

In this section, we study the following question: given a map $f : K \to M$ from a 2-dimensional CW complex into a closed surface, under what conditions we have $\mu(f) = \mu_C(f)N(f)$? In fact, we make a survey on the main results demonstrated by Aniz [5], where he studied this problem for dimensions greater or equal to 3. After this, we present several examples and a theorem to show that this problem has many pathologies in dimension two.

In [5] Aniz shows the following result.

Theorem 2.1. *Let $f : K \to M$ be a map from an n-dimensional CW complex into a closed n-manifold, with $n \geq 3$. If there is a map $f\prime : K \to M$ homotopic to f such that one of its Nielsen root classes $\mathfrak{R}'$ has exactly $\mu_C(f)$ roots, each one of them belonging to the interior of n-cells of K, then $\mu(f) = \mu_C(f)N(f)$.*

In this theorem, the assumption on the dimension of the complex and of the manifold is not superfluous; in fact, Xiaosong presents in [6, Section 4] a map $f : \mathbb{T}^2\#\mathbb{T}^2 \to \mathbb{T}^2$ from the bitorus into the torus with $\mu(f) = 4$ and $\mu_C(f)N(f) = 3$.

In [3, Theorem 4.2], we have the following result.

Theorem 2.2. *For each $n \geq 3$, there is an n-dimensional CW complex K_n and a map $f_n : K_n \to \mathbb{R}\mathrm{P}^n$ with $N(f_n) = 2$, $\mu_C(f_n) = 1$ and $\mu(f_n) \geq 3$.*

This theorem shows that, for each $n \geq 3$, there are maps $f : K^n \to M^n$ from n-dimensional CW complexes into closed n-manifolds with $\mu(f) \neq \mu_C(f)N(f)$. Here, we will show that maps with this property can be constructed also in dimension two. More precisely, we will construct three examples in this context for the cases in which the range-of the maps are, respectively, the closed surfaces $\mathbb{R}\mathrm{P}^2$ (the projective plane), $\mathbb{T}^2$ (the torus), and $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ (the Klein bottle). When the range is the sphere S^2, it is obvious that every map $f : K \to S^2$ satisfies $\mu(f) = \mu_C(f)N(f)$, since in this case there is a unique Nielsen root class.

Before constructing such examples, we present the main results that will be used.

Let $f : X \to Y$ be a map between connected, locally path connected, and semilocally simply connected spaces. Then f induces a homomorphism $f_\# : \pi_1(X) \to \pi_1(Y)$ between fundamental groups. Since the image $f_\#\pi_1(X)$ of $\pi_1(X)$ by $f_\#$ is a subgroup of $\pi_1(Y)$, there is a covering space $p^+ : Y^+ \to Y$ such that $p^+_\#\pi_1(Y^+) = f_\#\pi_1(X)$. Thus, f has a lifting $f^+ : X \to Y^+$ through p^+. The map f^+ is called a *Hopf lift* of f, and $p^+ : Y^+ \to Y$ is called a *Hopf covering* for f.

The next result corresponds to [2, Theorem 3.4].

Proposition 2.3. *The sets $(f^+)^{-1}(a_i)$, for $a_i \in (p^+)^{-1}(a)$, that are nonempty, are exactly the Nielsen root class of f at a and a class $(f^+)^{-1}(a_i)$ is essential if and only if $(f_1^+)^{-1}(a_i)$ is nonempty for every map $f_1^+ : X \to Y^+$ homotopic to f^+.*

In [3], Gonçalves and Aniz exhibit an example which we adapt for dimension two and summarize now. Take the bouquet of m copies of the sphere S^2, and let $f : \vee_{i=1}^{m} S^2 \to \mathbb{R}P^2$ be the map which restricted to each S^2 is the natural double covering map. If m is at least 2, then $N(f) = 2$, $\mu_C(f) = 1$, and $\mu(f) = m + 1$.

Now, we present a little more complicated example of a map $f : K \to \mathbb{R}P^2$, for which we also have $\mu(f) \neq \mu_C(f)N(f)$. Its construction is based in [3, Theorem 4.2].

Example 2.4. Let $p_2 : S^2 \to \mathbb{R}P^2$ be the canonical double covering. We will construct a 2-dimensional CW complex K and a map $f : K \to \mathbb{R}P^2$ having a lifting $\widetilde{f} : K \to S^2$ through p_2 and satisfying:

(i) $N(f) = 2$,

(ii) $\mu_C(f) = 1$,

(iii) $\mu(f) \geq 3$,

(iv) $\mu(\widetilde{f}) = 1$.

We start by constructing the 2-complex K. Let S_1, S_2, and S_3 be three copies of the 2-sphere regarded as the boundary of the standard 3-simplex Δ^3:

$$S_1 = \partial\langle x_0, x_1, x_2, x_3\rangle, \qquad S_2 = \partial\langle y_0, y_1, y_2, y_3\rangle, \qquad S_3 = \partial\langle z_0, z_1, z_2, z_3\rangle. \tag{2.1}$$

Let K be the 2-dimensional (simplicial) complex obtained from the disjoint union $S_1 \sqcup S_2 \sqcup S_3$ by identifying $[x_0, x_1] = [y_0, y_1]$ and $[y_0, y_2] = [z_0, z_1]$. Thus, each S_i, $i = 1, 2, 3$, is imbedded into K so that

$$S_1 \cap S_2 = [x_0, x_1] = [y_0, y_1], \qquad S_2 \cap S_3 = [y_0, y_2] = [z_0, z_1]. \tag{2.2}$$

Then, $S_1 \cap S_2 \cap S_3$ is a single point $x_0 = y_0 = z_0$. The (simplicial) 2-dimensional complex K is illustrated in Figure 1.

Two simplicial complexes A and B are homeomorphic if there is a bijection ϕ between the set of the vertices of A and of B such that $\{v_1, \ldots, v_s\}$ is a simplex of A if and only if $\{\phi(v_1), \ldots, \phi(v_s)\}$ is a simplex of B (see [7, page 128]). Using this fact, we can construct homeomorphisms $h_{21} : S_2 \to S_1$ and $h_{32} : S_3 \to S_2$ such that $h_{21}|_{S_1 \cap S_2} =$ *identity map* and $h_{32}|_{S_2 \cap S_3} =$ *identity map*.

Let $\widetilde{f}_1 : S_1 \to S^2$ be any homeomorphism from S_1 onto S^2. Define $\widetilde{f}_2 = \widetilde{f}_1 \circ h_{21} : S_2 \to S^2$ and note that $\widetilde{f}_2(x) = \widetilde{f}_1(x)$ for $x \in S_1 \cap S_2$. Now, define $\widetilde{f}_3 = \widetilde{f}_2 \circ h_{32} : S_3 \to S^2$ and note that $\widetilde{f}_3(x) = \widetilde{f}_2(x)$ for $x \in S_2 \cap S_3$. In particular, $\widetilde{f}_1(x_0) = \widetilde{f}_2(x_0) = \widetilde{f}_3(x_0)$. Thus, $\widetilde{f}_1$, $\widetilde{f}_2$, and $\widetilde{f}_3$ can be used to define a map $\widetilde{f} : K \to S_2$ such that $\widetilde{f}|_{S_i} = \widetilde{f}_i$ for $i = 1, 2, 3$.

Let $f : K \to \mathbb{R}P^2$ be the composition $f = p_2 \circ \widetilde{f}$, where $p_2 : S^2 \to \mathbb{R}P^2$ is the canonical double covering. Note that $f_{\#}\pi_1(K) = (p_2)_{\#}\pi_1(S^2)$. Thus, we can use Proposition 2.3 to study the Nielsen root classes of f through the lifting $\widetilde{f}$.

Let $a = f(x_0) \in \mathbb{R}P^2$, and let $p_2^{-1}(a) = \{\widetilde{a}, -\widetilde{a}\}$ be the fiber of p_2 over a.

Clearly, the homomorphism $\widetilde{f}_* : H_2(K) \to H_2(S^2)$ is surjective, with $H_2(K) \approx \mathbb{Z}^3$ and $H_2(S^2) \approx \mathbb{Z}$. Hence, every map from K into S^2 homotopic to f is surjective. It follows that, for every map $\widetilde{g} : K \to S^2$ homotopic to $\widetilde{f}$, we have $\widetilde{g}^{-1}(\widetilde{a}) \neq \emptyset$ and $\widetilde{g}^{-1}(-\widetilde{a}) \neq \emptyset$. By

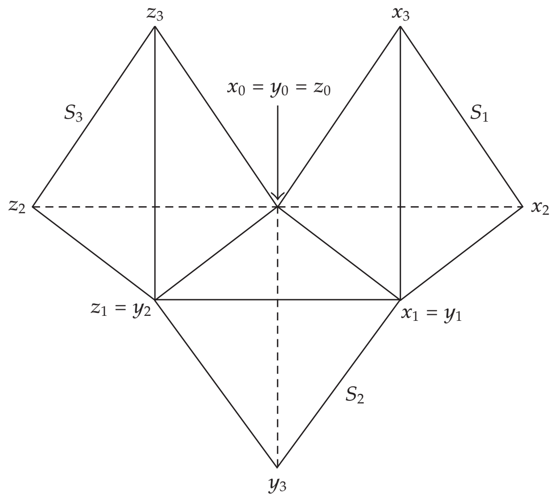

Figure 1: A simplicial 2-complex.

Proposition 2.3, $\widetilde{f}^{-1}(\widetilde{a})$ and $\widetilde{f}^{-1}(-\widetilde{a})$ are the Nielsen root classes of f, and both are essential classes. Therefore, $N(f) = 2$.

Now, since $a = f(x_0)$, either $x_0 \in \widetilde{f}^{-1}(\widetilde{a})$ or $x_0 \in \widetilde{f}^{-1}(-\widetilde{a})$. Without loss of generality, suppose that $x_0 \in \widetilde{f}^{-1}(\widetilde{a})$. Then, by the definition of $\widetilde{f}$, we have $\widetilde{f}^{-1}(\widetilde{a}) = \{x_0\}$. Hence, one of the Nielsen root classes is unitary. Furthermore, since such class is essential, it follows that its minimal cardinality is equal to one. This proves that $\mu_C(f) = 1$.

In order to show that $\mu(f) \geq 3$, note that since each restriction $\widetilde{f}|_{S_i}$ is a homeomorphism and $p_2 : S^2 \to \mathbb{R}P^2$ is a double covering, for each map g homotopic to f, the equation $g(x) = a$ must have at least two roots in each S_i, $i = 1, 2, 3$. By the decomposition of K this implies that $\mu(f) \geq 3$.

Moreover, it is very easy to see that $\mu(\widetilde{f}) = 1$, with the pair $(\widetilde{f}, \widetilde{f}(a_0))$ providing $\mu(\widetilde{f})$.

Now, we present a similar example where the range of the map f is the torus $\mathbb{T}^2$. Here, the complex K of the domain of f is a little bit more complicated.

Example 2.5. Let $p_2 : \mathbb{T}^2 \to \mathbb{T}^2$ be a double covering. We will construct a 2-dimensional CW complex K and a map $f : K \to \mathbb{T}^2$ having a lifting $\widetilde{f} : K \to \mathbb{T}^2$ through p_2 and satisfying the following:

(i) $N(f) = 2$,

(ii) $\mu_C(f) = 1$,

(iii) $\mu(f) = 3$,

(iv) $\mu(\widetilde{f}) = 1$.

We start constructing the 2-complex K. Consider three copies $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ of the torus with minimal celular decomposition. Let α_i (resp., β_i) be the longitudinal (resp., meridional) closed 1-cell of the torus $\mathbb{T}_i$, $i = 1, 2, 3$. Let K be the 2-dimensional CW complex obtained from the disjoint union $\mathbb{T}_1 \sqcup \mathbb{T}_2 \sqcup \mathbb{T}_3$ by identifying

$$\alpha_1 = \alpha_2, \qquad \alpha_3 = \beta_2. \tag{2.3}$$

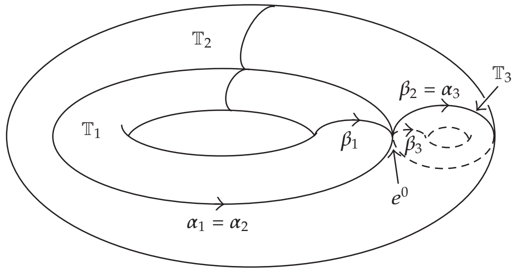

Figure 2: A 2-complex obtained by attaching three tori.

That is, K is obtained by attaching the tori $\mathbb{T}_1$ and $\mathbb{T}_2$ through the longitudinal closed 1-cell and, next, by attaching the longitudinal closed 1-cell of the torus $\mathbb{T}_3$ into the meridional closed 1-cell of the torus $\mathbb{T}_2$.

Each torus $\mathbb{T}_i$ is imbedded into K so that

$$\mathbb{T}_1 \cap \mathbb{T}_2 = \alpha_1 = \alpha_2, \qquad \mathbb{T}_2 \cap \mathbb{T}_3 = \alpha_3 = \beta_2, \qquad \mathbb{T}_1 \cap \mathbb{T}_3 = \mathbb{T}_1 \cap \mathbb{T}_2 \cap \mathbb{T}_3 = \left\{e^0\right\}, \tag{2.4}$$

where e^0 is the (unique) 0-cell of K, corresponding to 0-cells of $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ through the identifications. The 2-dimensional CW complex K is illustrate, in Figure 2.

Henceforth, we write $\mathbb{T}_i$ to denote the image of the original torus $\mathbb{T}_i$ into the 2-complexo K through the identifications above.

Certainly, there are homeomorphisms $h_{21} : \mathbb{T}_2 \to \mathbb{T}_1$ and $h_{32} : \mathbb{T}_3 \to \mathbb{T}_2$ with $h_{21}|_{\mathbb{T}_1 \cap \mathbb{T}_2} =$ *identity map* and $h_{32}|_{\mathbb{T}_2 \cap \mathbb{T}_3} =$ *identity map* such that h_{21} carries β_2 onto β_1, and h_{32} carries β_3 onto α_2. Thus, given a point $x_3 \in \beta_3$ we have $h_{32}(x_3) \in \alpha_1 = \mathbb{T}_1 \cap \mathbb{T}_2$. We should use this fact later.

Let $\widetilde{f}_1 : \mathbb{T}_1 \to \mathbb{T}^2$ be an arbitrary homeomorphism carrying longitude into longitude and meridian into meridian. Define $\widetilde{f}_2 = \widetilde{f}_1 \circ h_{21} : \mathbb{T}_2 \to \mathbb{T}^2$ and note that $\widetilde{f}_2(x) = \widetilde{f}_1(x)$ for $x \in \mathbb{T}_1 \cap \mathbb{T}_2$. Now, define $\widetilde{f}_3 = \widetilde{f}_2 \circ h_{32} : \mathbb{T}_3 \to \mathbb{T}^2$ and note that $\widetilde{f}_3(x) = \widetilde{f}_2(x)$ for $x \in \mathbb{T}_2 \cap \mathbb{T}_3$. In particular, $\widetilde{f}_1(e^0) = \widetilde{f}_2(e^0) = \widetilde{f}_3(e^0)$. Thus, $\widetilde{f}_1$, $\widetilde{f}_2$, and $\widetilde{f}_3$ can be used to define a map $\widetilde{f} : K \to \mathbb{T}^2$ such that $\widetilde{f}|_{\mathbb{T}_i} = \widetilde{f}_i$ for $i = 1, 2, 3$.

Let $p_2 : \mathbb{T}^2 \to \mathbb{T}^2$ be an arbitrary double covering. (We can consider, e.g., the longitudinal double covering $p_2(z) = (z_1^2, z_2)$ for each $z = (z_1, z_2) \in S^1 \times S^1 \cong \mathbb{T}^2$.)

We define the map $f : K \to \mathbb{T}^2$ to be the composition $f = p_2 \circ \widetilde{f}$.

In order to use Proposition 2.3 to study the Nielsen root classes of f using the information about $\widetilde{f}$, we need to prove that $f_\# \pi_1(K) = (p_2)_\# \pi_1(\mathbb{T}^2)$. Now, since $f_\# = (p_2)_\# \circ \widetilde{f}_\#$, it is sufficient to prove that $\widetilde{f}_\#$ is an epimorphism. This is what we will do. Consider the composition $\widetilde{f} \circ l : \mathbb{T}_1 \to \mathbb{T}^2$, where $l : \mathbb{T}_1 \hookrightarrow K$ is the obvious inclusion. This composition is exactly the homeomorphism $\widetilde{f}_1$, and therefore the induced homomorphism $\widetilde{f}_\# \circ l_\# = (\widetilde{f}_1)_\#$ is an isomorphism. It follows that $\widetilde{f}_\#$ is an epimorphism. Therefore, we can use Proposition 2.3.

Let $a = f(e^0) \in \mathbb{T}^2$, and let $p_2^{-1}(a) = \{\widetilde{a}, \widetilde{a}'\}$ be the fiber of p_2 over a. (If p_2 is the longitudinal double covering, as above, then if $\widetilde{a} = (\widetilde{a}_1, \widetilde{a}_2)$, we have $\widetilde{a}' = (-\widetilde{a}_1, \widetilde{a}_2)$.)

Clearly, the homomorphism $\widetilde{f}_* : H_2(K) \to H_2(\mathbb{T}^2)$ is surjective, with $H_2(K) \approx \mathbb{Z}^3$ and $H_2(\mathbb{T}^2) \approx \mathbb{Z}$. Hence, every map from K into $\mathbb{T}^2$ homotopic to f is surjective. It follows that, for every map $\widetilde{g} : K \to \mathbb{T}^2$ homotopic to $\widetilde{f}$, we have $\widetilde{g}^{-1}(\widetilde{a}) \neq \emptyset$ and $\widetilde{g}^{-1}(\widetilde{a}') \neq \emptyset$. By Proposition 2.3,

$\tilde{f}^{-1}(\tilde{a})$ and $\tilde{f}^{-1}(\tilde{a}')$ are Nielsen root classes of f, and both are essential classes. Therefore, $N(f) = 2$.

Now, since $a = f(e^0)$, either $e^0 \in \tilde{f}^{-1}(\tilde{a})$ or $e^0 \in \tilde{f}^{-1}(\tilde{a}')$. Without loss of generality, suppose that $e^0 \in \tilde{f}^{-1}(\tilde{a})$. Then, by the definition of $\tilde{f}$, we have $\tilde{f}^{-1}(\tilde{a}) = \{e^0\}$. Thus, one of the Nielsen root classes is unitary. Furthermore, since such class is essential, it follows that its minimal cardinality is equal to one. Therefore, $\mu_C(f) = 1$.

In order to prove that $\mu(f) = 3$, note that since each restriction $\tilde{f}|_{\mathbb{T}_i}$ is a homeomorphism and $p_2 : \mathbb{T}^2 \to \mathbb{T}^2$ is a double covering, for each map g homotopic to f, the equation $g(x) = a$ must have at least two roots in each $\mathbb{T}_i$, $i = 1,2,3$. By the decomposition of K, this implies that $\mu(f) \geq 3$. Now, let x_3 be a point in β_3, $x_3 \neq e^0$. As we have seen, $h_{32}(x_3) \in \alpha_1 \subset \mathbb{T}_1 \cap \mathbb{T}_2$. Write $x_{12} = h_{32}(x_3)$. By the definition of $\tilde{f}$, we have $\tilde{f}(x_{12}) = \tilde{f}(x_3) \neq \tilde{f}(e^0)$. Denote $y_0 = \tilde{f}(e^0)$ and $y_1 = \tilde{f}(x_{12})$.

Let $a \in \mathbb{T}^2$ be a point, and let $p_2^{-1}(a) = \{\tilde{a}, \tilde{a}\prime\}$ be the fiber of p_2 over a. Since $\mathbb{T}^2$ is a surface, there is a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity map such that $h(y_0) = \tilde{a}$ and $h(y_1) = \tilde{a}'$. Let $q_2 : \mathbb{T}^2 \to \mathbb{T}^2$ be the composition $q_2 = p_2 \circ h$, and let $\varphi : K \to \mathbb{T}^2$ be the composition $\varphi = q_2 \circ \tilde{f}$. Then, φ is homotopic to f and $\varphi^{-1}(a) = \{e^0, x_{12}, x_3\}$. Since $\mu(f) \geq 3$, this implies that $\mu(f) = 3$.

Moreover, it is very easy to see that $\mu(\tilde{f}\ ) = 1$, with the pair $(\tilde{f}, \tilde{f}(e^0))$ providing $\mu(\tilde{f}\ )$.

Note that in this example, for every pair (φ, a) providing $\mu(f)$ (which is equal to 3), we have necessarily $\varphi^{-1}(a) = \{e^0, x_1, x_2\}$ with either $x_1 \in \alpha_1$ and $x_2 \in \beta_3$ or $x_1 \in \beta_1$ and $x_2 \in \beta_2$.

For the same complex K of Example 2.5, we can construct a similar example with the range of f being the Klein bottle. The arguments here are similar to the previous example, and so we omit details.

Example 2.6. Let $\overline{p}_2 : \mathbb{T}^2 \to \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ be the orientable double covering. We will construct a 2-dimensional CW complex K and a map $f : K \to \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ having a lifting $\tilde{f} : K \to \mathbb{T}^2$ through $\overline{p}_2$ and satisfying the following:

(i) $N(f) = 2$,

(ii) $\mu_C(f) = 1$,

(iii) $\mu(f) = 3$,

(iv) $\mu(\tilde{f}) = 1$.

We repeat the previous example replacing the double covering $p_2 : \mathbb{T}^2 \to \mathbb{T}^2$ by the orientable double covering $\overline{p}_2 : \mathbb{T}^2 \to \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$. Also here, we have $\mu(\tilde{f}) = 1$, with the pair $(\tilde{f}, \tilde{f}(e^0))$ providing $\mu(\tilde{f})$.

Small adjustments in the construction of the latter two examples are sufficient to prove the following theorem.

Theorem 2.7. *Let K be the 2-dimensional CW complex of the previous two examples. For each positive integer n, there are cellular maps $f_n : K \to \mathbb{T}^2$ and $g_n : K \to \mathbb{R}\mathrm{P}^2\mathbb{R}\mathrm{P}^2$ satisfying the following:*

(1) $N(f_n) = n$, $\mu_C(f_n) = 1$ *and* $\mu(f_n) = 2n - 1$.

(2) $N(g_n) = 2n$, $\mu_C(g_n) = 1$ *and* $\mu(g_n) = 4n - 1$.

Proof. In order to prove item (1), let $\tilde{f} : K \to \mathbb{T}^2$ be as in Example 2.5. Let $p_n : \mathbb{T}^2 \to \mathbb{T}^2$ be an n-fold covering (which certainly exists; e.g., for each $z \in \mathbb{T}^2$ considered as a pair $z = (z_1, z_2) \in S^1 \times S^1$, we can define $p_n(z) = (z_1^n, z_2))$. Define $f_n = p_n \circ \tilde{f} : K \to \mathbb{T}^2$. Then, the same arguments of Example 2.5 can be repeated to prove the desired result.

In order to prove item (2), let $\tilde{f} : K \to \mathbb{T}^2$ be as in Example 2.6. Let $p_n : \mathbb{T}^2 \to \mathbb{T}^2$ be an n-fold covering (e.g., as in the first item), and let $\overline{p}_2 : \mathbb{T}^2 \to \mathbb{RP}^2\#\mathbb{RP}^2$ be the orientable double covering. Define $q_{2n} : \mathbb{T}^2 \to \mathbb{RP}^2\#\mathbb{RP}^2$ to be the composition $q_{2n} = \overline{p}_2 \circ p_n$. Then q_{2n} is a $2n$-fold covering. Define $f_n = q_{2n} \circ \tilde{f} : K \to \mathbb{RP}^2\#\mathbb{RP}^2$. Now proceed with the arguments of Example 2.6. □

Observation 2.8. It is obvious that if m and n are different positive integers, then the maps f_m, f_n and g_m, g_n satisfying the previous theorem are such that f_m is not homotopic to f_n and g_m is not homotopic to g_n.

3. Roots of Liftings through Coverings

In the previous section, we saw several examples of maps from 2-dimensional CW complexes into closed surfaces having lifting through some covering space and not having all Nielsen root classes with minimal cardinality. In this section, we study the relationship between the minimal number of roots of a map and the minimal number of roots of one of its liftings through a covering space, when such lifting exists.

Throughout this section, M and N are topological n-manifolds either compact or triangulable, and X denotes a compact, connected, locally path connected, and semilocally simply connected spaces All these assumptions are true, for example, if X is a finite and connected CW complex.

Lemma 3.1. *Let $p_k : \overline{Y} \to Y$ be a k-fold covering, and let $f : X \to Y$ be a map having a lifting $\tilde{f} : X \to \overline{Y}$ through p_k. Let $a \in Y$ be a point, and let $p_k^{-1}(a) = \{a_1, \ldots, a_k\}$ be the fiber of p_k over a. Then $\mu(f, a) \geq \sum_{i=1}^{k} \mu(\tilde{f}, a_i)$.*

Proof. Let $\varphi : X \to Y$ be a map homotopic to f such that $\#\varphi^{-1}(a) = \mu(f, a)$. Then, since p_k is a covering, we may lift φ through p_k to a map $\tilde{\varphi} : X \to \overline{Y}$ homotopic to $\tilde{f}$. It follows that $\varphi^{-1}(a) = \cup_{i=1}^{k} \tilde{\varphi}^{-1}(a_i)$, with this union being disjoint, and certainly $\#\tilde{\varphi}^{-1}(a_i) \geq \mu(\tilde{f}, a_i)$ for all $1 \leq i \leq k$. Therefore,

$$\mu(f, a) = \#\left(\bigcup_{i=1}^{k} \tilde{\varphi}^{-1}(a_i)\right) \geq \sum_{i=1}^{k} \mu\left(\tilde{f}, a_i\right). \tag{3.1}$$

□

Theorem 3.2. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to N$ be a map having a lifting $\tilde{f} : X \to M$ through p_k. Then $\mu(f) \geq k\mu(\tilde{f})$. Moreover, $\mu(f) = 0$ if and only if $\mu(\tilde{f}) = 0$.*

Proof. Let $a \in N$ be an arbitrary point, and let $p_k^{-1}(a) = \{a_1, \ldots, a_k\}$ be the fiber of p_k over a. Since M and N are manifolds, we have $\mu(f) = \mu(f, a)$ and $\mu(\tilde{f}) = \mu(\tilde{f}, a_i)$ for all $1 \leq i \leq k$. Hence, by the previous lemma, $\mu(f) \geq k\mu(\tilde{f})$. It follows that $\mu(\tilde{f}) = 0$ if $\mu(f) = 0$. On the other hand, suppose that $\mu(\tilde{f}) = 0$. Then $N(\tilde{f}) = 0$ and by [8, Theorem 2.3], there is a map

$\widetilde{g} : X \to M$ homotopic to $\widetilde{f}$ such that $\dim \widetilde{g}(X) \leq n - 1$, (where n is the dimension of M and N). Let $\varphi : X \to N$ be the composition $\varphi = p_k \circ \widetilde{\varphi}$. Then φ is homotopic to f and $\dim \varphi(X) \leq n - 1$. Therefore $\mu(f) = 0$. □

Note that if in the previous theorem we suppose that $k = 1$, then the covering $p_k : M \to N$ is a homeomorphism and $\mu(f) = \mu(\widetilde{f})$.

In Examples 2.4, 2.5, and 2.6 of the previous section, we presented maps $f : K \to N$ from 2-dimensional CW complexes into closed surfaces (here N is the projective plane, the torus, and the Klein bottle, resp.) for which we have

$$\mu(f) \geq 3 > 2 = 2\mu\left(\widetilde{f}\right). \tag{3.2}$$

This shows that there are maps $f : K \to N$ from 2-dimensional CW complexes into closed surfaces having liftings $\widetilde{f} : K \to M$ through a double covering $p_2 : M \to N$ and satisfying the strict inequality

$$\mu(f) > 2\mu\left(\widetilde{f}\right). \tag{3.3}$$

Moreover, Theorem 2.7 shows that there is a 2-dimensional CW complex K such that, for each integer $n > 1$, there is a map $f_n : K \to \mathbb{T}^2$ and a map $g_n : K \to \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ having liftings $\widetilde{f}_n : K \to \mathbb{T}^2$ through an n-fold covering $p_n : \mathbb{T}^2 \to \mathbb{T}^2$ and $\widetilde{g}_n : K \to \mathbb{T}^2$ through a $2n$-fold covering $q_{2n} : \mathbb{T}^2 \to \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$, respectively, satisfying the relations $\mu(f_n) = 2n - 1 > n = n\mu(\widetilde{f}_n)$ and $\mu(g_n) = 4n - 1 < 2n = 2n\mu(\widetilde{g}_n)$.

The proofs of the latter two theorems can be used to create a necessary and sufficient condition for the identity $\mu(f) = k\mu(\widetilde{f})$ to be true. We show this after the following lemma.

Lemma 3.3. *Let $p_k : M \to N$ be a k-fold covering, let $a_1, \ldots, a_k$ be different points of M, and let $a \in N$ be a point. Then, there is a k-fold covering $q_k : M \to N$ isomorphic and homotopic to p_k such that $q_k^{-1}(a) = \{a_1, \ldots, a_k\}$.*

Proof. Let $p_k^{-1}(a) = \{b_1, \ldots, b_k\}$ be the fiber of p_k over a. It can occur that some a_i is equal to some b_j. In this case, up to reordering, we can assume that $a_i = b_i$ for $1 \leq i \leq r$ and $a_i \neq b_i$ for $i > r$, for some $1 \leq r \leq k$. If $a_i \neq b_j$ for any i, j, then we put $r = 0$. If $r = k$, then there is nothing to prove. Then, we suppose that $r \neq k$. For each $i = r + 1, \ldots, k$, let U_i be an open subset of M homeomorphic to an open n-ball, containing a_i and b_i and not containing any other point a_j and b_j. Let $h_i : M \to M$ be a homeomorphism homotopic to the identity map, being the identity map outside U_i and such that $h_i(a_i) = b_i$. Let $h : M \to M$ be the homeomorphism $h = h_k \circ \cdots \circ h_{r+1}$. Then h is homotopic to the identity map and $h(a_i) = b_i$ for each $1 \leq i \leq k$. Let $q_k : M \to N$ be the composition $q_k = p_k \circ h$. Then q_k is a k-fold covering isomorphic and homotopic to p_k. Moreover, $q_k^{-1}(a) = \{a_1, \ldots, a_k\}$. □

Theorem 3.4. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to N$ be a map having a lifting $\widetilde{f} : X \to M$ through p_k. Then $\mu(f) = k\mu(\widetilde{f})$ if and only if, for each pair (φ, a) providing $\mu(f)$, each pair $(\widetilde{\varphi}, a_i)$ provides $\mu(\widetilde{f})$, where $\widetilde{\varphi}$ is a lifting of φ homotopic to $\widetilde{f}$ and $p_k^{-1}(a) = \{a_1, \ldots, a_k\}$.*

Proof. Let (φ, a) be a pair providing $\mu(f)$, let $p_k^{-1}(a) = \{a_1, \dots, a_k\}$ be the fiber of p_k over a, and let $\tilde{\varphi}$ be a lifting of φ homotopic to $\tilde{f}$. Then $\varphi^{-1}(a) = \cup_{i=1}^{k} \tilde{\varphi}^{-1}(a_i)$, with this union being disjoint. Hence $\mu(f) = \sum_{i=1}^{k} \#\tilde{\varphi}^{-1}(a_i)$. Now, $\#\tilde{\varphi}^{-1}(a_i) \geq \mu(\tilde{f})$ for each $1 \leq i \leq k$. Therefore, $\mu(f) = k\mu(\tilde{f})$ if and only if $\#\tilde{\varphi}^{-1}(a_i) = \mu(\tilde{f})$ for each $1 \leq i \leq k$, that is, each pair $(\tilde{\varphi}, a_i)$ provides $\mu(\tilde{f})$. □

Theorem 3.5. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to N$ be a map having a lifting $\tilde{f} : X \to M$ through p_k. Then $\mu(f) = k\mu(\tilde{f})$ if and only if, given k different points of M, say $a_1, \dots, a_k$, there is a map $\tilde{\varphi} : X \to M$ such that, for each $1 \leq i \leq k$: the pair $(\tilde{\varphi}, a_i)$ provides $\mu(\tilde{f})$.*

Proof. Let (φ, a) be a pair providing $\mu(f)$, and let $q_k : M \to N$ be a covering isomorphic and homotopic to p_k, such that $q_k^{-1}(a) = \{a_1, \dots, a_k\}$, as in Lemma 3.3.

Suppose that $\mu(f) = k\mu(\tilde{f})$. Let $\tilde{\varphi} : X \to M$ be a lifting of φ through q_k homotopic to $\tilde{f}$. Then, by the previous theorem, $(\tilde{\varphi}, a_i)$ provides $\mu(\tilde{f})$ for each $1 \leq i \leq k$.

On the other hand, suppose that there is a map $\tilde{\varphi} : X \to M$ such that, for each $1 \leq i \leq k$, the pair $(\tilde{\varphi}, a_i)$ provides $\mu(\tilde{f})$. Let $\varphi : X \to N$ be the composition $\varphi = q_k \circ \tilde{\varphi}$. Then $\tilde{\varphi}$ is a lifting of φ through q_k homotopic to $\tilde{f}$ and $\mu(f) \leq \#\varphi^{-1}(a) = \sum_{i=1}^{k} \#\tilde{\varphi}^{-1}(a_i) = k\mu(\tilde{f})$. But, by Theorem 3.2, we have $\mu(f) \geq k\mu(\tilde{f})$. Therefore $\mu(f) = k\mu(\tilde{f})$. □

Theorem 3.6. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to N$ be a map having a lifting $\tilde{f} : X \to M$ through p_k. Then $\mu(f) > k\mu(\tilde{f})$ if and only if, for every map $\tilde{\varphi} : X \to M$ homotopic to $\tilde{f}$, there are at most $k - 1$ points in M whose preimage by $\tilde{\varphi}$ has exactly $\mu(\tilde{f})$ points.*

Proof. From Theorem 3.2, $\mu(f) \neq k\mu(\tilde{f})$ if and only if $\mu(f) > k\mu(\tilde{f})$. Thus, a trivial argument shows that this theorem is equivalent to Theorem 3.5. □

Example 3.7. Let $f : K \to N$, $p_2 : M \to N$ and $\tilde{f} : K \to M$ be the maps of Examples 2.4, 2.5, or 2.6. Then, we have proved that $\mu(f) \geq 3 > 2 = 2\mu(\tilde{f})$. (More precisely, in Examples 2.5 and 2.6 we have $\mu(f) = 3$.) Therefore, by Theorem 3.6, if $\tilde{\varphi} : K \to M$ is a map providing $\mu(\tilde{f})$ (which is equal to 1), then there is a unique point of M whose preimage by $\tilde{\varphi}$ is a single point.

Now, we present a proposition showing equivalences between the vanishing of the Nielsen numbers and the minimal number of roots of f and its liftings $\tilde{f}$ through a covering.

Proposition 3.8. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to M$ be a map having a lifting $\tilde{f} : X \to M$ through p_k. Then, the following statements are equivalent:*

(i) $N(f) = 0$,

(ii) $N(\tilde{f}) = 0$,

(iii) $\mu(f) = 0$,

(iv) $\mu(\tilde{f}) = 0$.

Proof. First, we should remember that, by Theorem 3.2, (iii)⇔(iv). Also, since $N(g) \leq \mu(g)$ for every map g, it follows that (iii)⇒(i) and (iv)⇒(ii). On the other hand, by [8, Theorem 2.1], we have that (i)⇒(iii) and (ii)⇒(iv). This completes the proof. □

Until now, we have studied only the cases in which a given map f has a lifting through a finite fold covering. When f has a lifting through an infinite fold covering, the problem is easily solved using the results of Gonçalves and Wong presented in [8].

Theorem 3.9. *Let $f : X \to N$ be a map having a lifting $\widetilde{f} : X \to M$ through an infinite fold covering $p_\infty : M \to N$. Then the numbers $N(f)$, $N(\widetilde{f})$, $\mu(f)$ and $\mu(\widetilde{f})$ are all zero.*

Proof. Certainly, the subgroup $f_\#\pi_1(X)$ has infinite index in the group $\pi_1(N)$. Thus, by [8, Corollary 2.2], $\mu(f) = 0$ and so $N(f) = 0$. Now, it is easy to check that also $\mu(\widetilde{f}) = 0$ and so $N(\widetilde{f}) = 0$. □

4. Minimal Classes versus Roots of Liftings

In this section we present some results relating the problems of Sections 2 and 3. We start remembering and proving general results which will be used in here.

Also in this section, X is always a compact, connected, locally path connected and semilocally simply connected space and M and N are topological n-manifolds either compact or triangulable.

Let $f : X \to Y$ be a map with Y having the same properties of X. We denote the *Riedemeister number* of f by $\mathcal{R}(f)$, which is defined to be the index of the subgroup $f_\#\pi_1(X)$ in the group $\pi_1(M)$. In symbols, $\mathcal{R}(f) = |\pi_1(M) : f_\#\pi_1(X)|$. When Y is a topological manifold (not necessarily compact), it follows from [2] that $N(f) > 0 \Rightarrow N(f) = \mathcal{R}(f) < \infty$. Thus, if $\mathcal{R}(f) = \infty$, then $N(f) = 0$.

Corollary 4.1. *Let $f : X \to N$ be a map with $\mathcal{R}(f) = k$, let $p_k : M \to N$ be a k-fold covering and let $\widetilde{f} : X \to M$ be a lifting of f through p_k. Then the following statements are equivalent:*

(i) $N(f) \neq 0$,

(ii) $N(f) = k$,

(iii) $\mu(f) \neq 0$,

(iv) $\mu(\widetilde{f}) \neq 0$.

Proof. The equivalences (i)⇔(iii)⇔(iv) are proved in Proposition 3.8. The implication (ii)⇒(i) is trivial. For a proof that (i) implies (ii) see [2]. □

Theorem 4.2. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to N$ be a map having a lifting $\widetilde{f} : X \to M$. If $\mathcal{R}(f) = k$, then $\mu(\widetilde{f}) = \mu_C(f)$.*

Proof. If $N(f) = 0$, then all $\mu(f)$, $\mu(\widetilde{f})$, and $\mu_C(f)$ also are zero. In this case, there is nothing to prove. Now, suppose that $N(f) \neq 0$. Then, by Corollary 4.1, $N(f) = k$ and $\mu(f)$ and $\mu(\widetilde{f})$ are both nonzero. Thus, also $\mu_C(f) \neq 0$. Let $\mathfrak{R}$ be a Nielsen root class of f, and let $H : f \simeq f_1$ be a homotopy starting at f and ending at f_1. Moreover, let $\mathfrak{R}_1$ be the Nielsen root class of f_1 that is H-related with $\mathfrak{R}$. Let $\widetilde{f}_1$ be a lifting of f_1 through p_k homotopic to $\widetilde{f}$. By Proposition 2.3, $\mathfrak{R}_1 = \widetilde{f}_1^{-1}(\widetilde{a})$ for some point $\widetilde{a} \in M$ over a specific point a of N. Thus, the cardinality $\#\mathfrak{R}_1$ is minimal if and only if the cardinality $\#\widetilde{f}_1^{-1}(\widetilde{a})$ is minimal; that is, $\#\mathfrak{R}_1 = \mu_C(f)$ if and only if $\#\widetilde{f}_1(\widetilde{a}) = \mu(\widetilde{f})$. □

Theorem 4.3. *Let $p_k : M \to N$ be a k-fold covering, and let $f : X \to N$ be a map having a lifting $\widetilde{f} : X \to M$ through p_k. If $\mathcal{R}(f) = k$, then the following statements are equivalent:*

(i) $\mu(f) = \mu_C(f)N(f)$,

(ii) $\mu(f) = k\mu(\widetilde{f})$,

(iii) $\mu(f) = \mu(\widetilde{f})N(f)$.

Proof. By the previous results, we have $N(f) = 0 \Leftrightarrow \mu(f) = 0 \Leftrightarrow \mu(\widetilde{f}) = 0$. Thus, if one of these numbers are zero, then the three statements are automatically equivalent. Now, if $N(f) \neq 0$, then $N(f) = \mathcal{R}(f) = k$ and, by Theorem 4.2, $\mu(\widetilde{f}) = \mu_C(f)$. This proves the desired equivalences. □

5. Maps into the Projective Plane

In this section, we use the capital letter K to denote finite and connected 2-dimensional CW complexes, and we use M to denote closed surfaces.

In the next two lemmas, we consider the 2-sphere in the domain of f with cellular decomposition $S^2 = e^0 \cup e^2$ and the 2-sphere in the range of f with cellular decomposition $S^2 = e^0_* \cup e^2_*$.

Lemma 5.1. *Let $f : S^2 \to S^2$ be a map with degree $d \neq 0$, and let $a \in S^2$ be a point, $a \neq e^0_*$. Then, there is a cellular map $\varphi : S^2 \to S^2$ such that $f \simeq \varphi\,\mathrm{rel}\{e^0\}$ and $\#\varphi^{-1}(a) = 1 = \#\varphi^{-1}(-a)$.*

Proof. Without loss of generality, suppose that a is the north pole and so $-a$ is the south pole.

There is a cellular map $g : S^2 \to S^2$ such that $g \simeq f$ and $\#g^{-1}(a) = 1 = \#g^{-1}(-a)$. In fact, consider the domain sphere S^2 fragmented in $|d|$ southern tracks by meridians $\mathfrak{m}_1, \ldots, \mathfrak{m}_{|d|}$ chosen so that e^0 is in $\mathfrak{m}_1$. Let $g : S^2 \to S^2$ be a map defined so that each meridian $\mathfrak{m}_i$, for $1 \leq i \leq |d|$, is carried homeomorphically onto a same distinguished meridian $\mathfrak{m}$ of the range 2-sphere containing e^0_*, and each of the $|d|$ tracks covers once the sphere S^2, always in the same direction, which is chosen according to the orientation of S^2, so that g is a map of degree d.

Since f and g have the same degree, they are homotopic. Moreover, $g^{-1}(a) = \{b\}$ and $g^{-1}(-a) = \{-b\}$, where b is the north pole of the domain 2-sphere, and so $-b$ is its south pole. Therefore, we have $\#g^{-1}(a) = 1 = \#g^{-1}(-a)$. What we cannot guarantee immediately is that the homotopy between f and g is a homotopy relative to $\{e^0\}$.

Now, if $H : S^2 \times I \to S^2$ is a homotopy starting at f and ending at g, then as in [9, Lemma 3.1], we can slightly modify H in a small closed neighborhood $V \times I$ of $e^0 \times I$, with V homeomorphic to a closed 2-disc and not containing a and $-a$, to obtain a new homotopy $\widehat{H} : S^2 \times I \to S^2$, which is relative to e^0. Let $\varphi : S^2 \to S^2$ be the end of this new homotopy, that is, $\varphi = \widehat{H}(\cdot, 1)$. Since H and $\widehat{H}$ differ only on $V \times I$ and a and $-a$ do not belong to V, we have $\varphi^{-1}(a) = \{b\}$ and $\varphi^{-1}(-a) = \{-b\}$.

This concludes the proof of this lemma. □

Lemma 5.2. *Let $f : S^2 \to S^2$ be a map with zero degree and let $\kappa^0 : S^2 \to S^2$ be the constant map at e^0_*. Then $f \simeq \kappa^0\,\mathrm{rel}\{e^0\}$. Moreover, if $a \in S^2$, $a \neq e^0_*$, then $(\kappa^0)^{-1}(a) = \emptyset = (\kappa^0)^{-1}(-a)$.*

Proof. This is [9, Lemma 3.2]. Also, it is an adaptation of the proof of the previous lemma. □

Now, we insert an important definition about the type of maps which provides the minimal number of roots of a given map.

Definition 5.3. Let $f : K \to M$ be a map. We say that f is of type ∇_2 if there is a pair (φ, a) providing $\mu(f)$ such that $\varphi^{-1}(a) \subset K \setminus K^1$. Moreover, we say that f is of type ∇_3 if in addition we can choose the map φ being a cellular map.

Proposition 5.4. *Every map $f : K \to M$ of type ∇_2 is also of the type ∇_3.*

Proof. Let $\varphi : K \to M$ be a map and let $a \in M$ be a point such that (φ, a) provides $\mu(f)$ and $\varphi^{-1}(a) \subset K \setminus K^1$. We can assume that a is in the interior of the unique 2-cell of M. (We consider M with a minimal cellular decomposition.) Let V be an open neighborhood of a in M homeomorphic to an open 2-disc and such that the closure $\overline{V}$ of V in M is contained in $M \setminus M^1$, where M^1 is the 1-skeleton of M. Let $\chi : D^2 \to M$ be the attaching map of the 2-cell of M, and let $h : \overline{V} \to D^2$ be a homeomorphism, where D^2 is the unitary closed 2-disc.

Certainly, there is a retraction $r : M \setminus V \to M^1$ such that for each $x \in \partial\overline{V}$ we have $r(x) = (\chi \circ h)(x)$. Then, the maps r and $\chi \circ h$ can be used to define a map $g : M \to M$ such that $g|_{M\setminus V} = r$ and $g|_V = \chi \circ h$. Now, it is easy to see that g is cellular and homotopic to the identity map $\mathrm{id} : M \to M$.

Let $\psi : K \to M$ be the composition $\psi = g \circ \varphi$ and call $a' = g(a)$. Then, ψ is a cellular map homotopic to f and $\psi^{-1}(a') = \varphi^{-1}(a) \subset K \setminus K^1$. This concludes the proof. □

Proposition 5.5. *Every map between closed surfaces is of type ∇_2 and so of type ∇_3.*

Proof. Let $f : M \to N$ be a map between closed surfaces. Suppose that $n = \mu(f)$, and let (φ, a) be a pair providing $\mu(f)$. Let $\varphi^{-1}(a) = \{x_1, \ldots, x_n\}$. If each x_j is in the interior of the 2-cell of M, then there is nothing to prove. Otherwise, let $y_1, \ldots, y_n$ be n different points of M belonging to its 2-cell. There is a homeomorphism $h : M \to M$ homotopic to the identity map $\mathrm{id} : M \to M$ such that $h(y_j) = x_j$ for each $1 \le j \le n$. Let $\psi : M \to N$ be the composition $\psi = \varphi \circ h$. Then ψ is homotopic to f and $\psi^{-1}(a) = \{y_1, \ldots, y_n\} \subset M \setminus M^1$. Now, we use the previous proposition to complete the proof. □

Theorem 5.6. *Let $f : K \to \mathbb{R}\mathrm{P}^2$ be a map having a lifting $\widetilde{f} : K \to S^2$ through the double covering $p_2 : S^2 \to \mathbb{R}\mathrm{P}^2$. If $\widetilde{f}$ is of type ∇_2, then $2\mu(\widetilde{f}) = \mu(f) = \mu_C(f)N(f)$.*

Proof. Since $\widetilde{f}$ is of type ∇_2, then $\widetilde{f}$ is also of type ∇_3, by Proposition 5.4. Let $\widetilde{\varphi} : K \to S^2$ be a cellular map, and let $a \in S^2 = e^0_* \cup e^2_*$ be a point different from e^0_* such that $\#\widetilde{\varphi}^{-1}(a) = \mu(\widetilde{f})$ and $\widetilde{\varphi}^{-1}(a) \subset K \setminus K^1$. Let $e^2_1, \ldots, e^2_m$ be the 2-cells of K. For each $1 \le i \le m$, we define the quotient map

$$\omega_i : K \twoheadrightarrow \frac{K}{(K \setminus e^2_i)}, \tag{5.1}$$

which collapses the complement of the interior of the 2-cell e^2_i to a point c^0_i. The image $\omega_i(K) = K/(K \setminus e^2_i)$ is naturally homeomorphic to a 2-sphere S^2_i which inherits from K a cellular decomposition $S^2_i = c^0_i \cup c^2_i$, where the interior of the 2-cell c^2_i corresponds homeomorphically to the image by ω_i of the interior of the 2-cell e^2_i of the 2-complex K.

Since $\widetilde{\varphi} : K \to S^2$ is a cellular map, the 1-skeleton K^1 of K is carried by $\widetilde{\varphi}$ into the 0-cell e_*^0 of S^2. Moreover, K^1 is carried by ω_i (which is also a cellular map) into the 0-cell c_i^0 of the sphere S_i^2, for all $1 \le i \le m$. Then we can define, for each $1 \le i \le m$, a unique cellular map $\widetilde{\varphi}_i : S_i^2 \to S^2$ such that $\widetilde{\varphi}|_{\overline{e}_i^2} = \widetilde{\varphi}_i \circ \omega_i|_{\overline{e}_i^2}$. In fact, for each $\overline{x} = \omega_i(x) \in S_i^2$, we define $\widetilde{\varphi}_i(\overline{x}) = \widetilde{\varphi}(x)$. Since $\widetilde{\varphi}$ is a cellular map, $\widetilde{\varphi}_i$ is well defined and is also a cellular map. Moreover, for each $x \in e_i^2$, we have $\widetilde{\varphi}(x) = (\widetilde{\varphi}_i \circ \omega_i)(x)$.

Since $\widetilde{\varphi}^{-1}(a) \subset K \setminus K^1$, the set $\widetilde{\varphi}^{-1}(a)$ is in one-to-one correspondence with the set $\cup_{i=1}^m \widetilde{\varphi}_i^{-1}(a)$; in fact, we have $\widetilde{\varphi}^{-1}(a) = \cup_{i=1}^m (\widetilde{\varphi}_i \circ \omega_i)^{-1}(a)$. Now, by the proof of Theorem 4.1 of [9], for each $1 \le i \le m$, either $\#\widetilde{\varphi}_i^{-1}(a) = 1$ or $\widetilde{\varphi}$ is homotopic to a constant map. Then, by Lemmas 5.1 and 5.2, for each $1 \le i \le m$, there is a cellular map $\widetilde{\psi}_i : S_i^2 \to S^2$ such that $\widetilde{\varphi}_i \simeq \widetilde{\psi}_i \,\mathrm{rel}\{c_i^0\}$ and $\#\widetilde{\psi}_i^{-1}(a) = \#\widetilde{\varphi}_i^{-1}(a) = \#\widetilde{\psi}_i^{-1}(-a)$. Let $H_i : \widetilde{\varphi}_i \simeq \widetilde{\psi}_i \mathrm{rel}\{c_i^0\}$ be such homotopies, $1 \le i \le m$.

For each $x \in K$, choose once and for all an index $i(x) \in \{1, \ldots, m\}$ such that $x \in e_{i(x)}$. Then, define $\widetilde{\psi} : K \to S^2$ by $\widetilde{\psi}(x) = \widetilde{\psi}_{i(x)}(\omega_{i(x)}(x))$. This map is clearly well defined and cellular. Moreover, the homotopies H_i, $1 \le i \le m$, can be used to define a homotopy H starting at $\widetilde{\varphi}$ and ending at $\widetilde{\psi}$.

From this construction, we have $\#\widetilde{\psi}^{-1}(a) = \mu(\widetilde{f}) = \#\widetilde{\psi}^{-1}(-a)$. By Theorem 3.5, we have that $\mu(f) = 2\mu(\widetilde{f})$. Now, it is obvious that $\mathcal{R}(f) = 2$. So, by Theorem 4.3, $\mu(f) = \mu_C(f)N(f)$. □

Theorem 5.6 is not true, in general, when the map f is not of the type ∇_2. We present an example to illustrate this fact.

Example 5.7. Let $K = S_1^2 \vee S_2^2$ be the bouquet of two 2 spheres with minimal cellular decomposition with one 0-cell e^0 and two 2-cells e_1^2 and e_2^2. Let $\widetilde{f} : K \to S^2$ be a map which, restricted to each S_i^2, $i = 1, 2$, is homotopic to the identity map. Consider the sphere S^2 with its minimal cellular decomposition $S^2 = e_*^0 \cup e_*^2$. Then, there is a cellular map $\widetilde{\varphi} : K \to S^2$ homotopic to $\widetilde{f}$ such that $\widetilde{\varphi}^{-1}(e_*^0) = \{e^0\}$. Thus, the pair $(\widetilde{\varphi}, e_*^0)$ provides $\mu(\widetilde{f})$ (= 1, of course). Now, it is obvious that ,for every map g homotopic to $\widetilde{f}$, the restrictions $g|_{S_i^2}$, $i = 1, 2$, are surjective. Hence, for every such map g, the equation $g(x) = a$ has at least one root in each S_i^2, $i = 1, 2$, whatever the point $a \in S^2$. Therefore, if x_0 is a root of $g(x) = a$ belonging to the interior of one of the 2 cells of K, then the equation $g(x) = a$ must have a second root, which must belong to the closure of the other 2 cell of K. But in this case, $\#g^{-1}(a) \ge 2$, and so the pair (g, a) do not provide $\mu(\widetilde{f})$. This means that the map $\widetilde{f}$ is not of type ∇_2. Moreover, this shows that if $(\widetilde{\varphi}, a)$ is a pair providing $\mu(\widetilde{f}\,)$, then necessarily $\widetilde{\varphi}^{-1}(a) = \{e^0\}$. Thus, for every map $\widetilde{\varphi} : K \to S^2$ homotopic to $\widetilde{f}$, there is at most one point in S^2 whose preimage by $\widetilde{\varphi}$ is a set with $\mu(\widetilde{f})$ points. Now, let $p_2 : S^2 \to \mathbb{R}\mathrm{P}^2$ be a double covering, and let $f : K \to \mathbb{R}\mathrm{P}^2$ be the composition $f = p_2 \circ \widetilde{f}$. Then $\widetilde{f}$ is a lifting of f through p_2, and, by Theorem 3.6, we have $\mu(f) > 2\mu(\widetilde{f})$. More precisely, $\mu(f) = 3$. Moreover, $\mu_C(f) = 1$, $N(f) = 2$ and $\mu(f) \neq \mu_C(f)N(f)$.

In the next theorem, $A(f)$ denotes the absolute degree of the given map f (see [10] or [11]).

Theorem 5.8. *Let $f : M \to \mathbb{R}\mathrm{P}^2$ be a map inducing the trivial homomorphism on fundamental groups. Then, $\mu(f) = 0$ if $A(f) = 0$ and $\mu(f) = 2$ if $A(f) \neq 0$.*

Proof. Since $f_{\#}\pi_1(M)$ is trivial, f has a lifting $\widetilde{f} : M \to S^2$ through the (universal) double covering $p_2 : S^2 \to \mathbb{R}P^2$. By Proposition 5.5, $\widetilde{f}$ is of type ∇_2. Hence, by Theorem 5.6, we have $\mu(f) = 2\mu(\widetilde{f})$. Now, it is well known that $\mu(\widetilde{f}) = 0$ if $A(\widetilde{f}) = 0$ and $\mu(\widetilde{f}) = 1$ if $A(\widetilde{f}) \neq 0$. (see, e.g., [11] or [9] or [3]). But, by the definition of absolute degree (see [11, page 371]) it is easy to check that $A(f) = 2A(\widetilde{f})$. This concludes the proof. □

Theorem 5.8 is not true, in general, if the homomorphism $f_{\#} : \pi_1(M) \to \pi_1(\mathbb{R}P^2)$ is not the trivial homomorphism. To illustrate this, let $\mathrm{id} : \mathbb{R}P^2 \to \mathbb{R}P^2$ be the identity map. It is obvious that this map induces the identity isomorphism on fundamental groups and $\mu(\mathrm{id}) = 1$.

In the next theorem, X is a compact, connected, locally path connected, and semilocally simply connected space.

Theorem 5.9. *Let $f : X \to \mathbb{R}P^2$ be a map. Then $\mu(f) = \mu_C(f)N(f)$ if at least one of the following alternatives is true: < (i) $f_{\#}\pi_1(X) \neq 0$; (ii) X is a 2-dimensional CW complex, and f is of type ∇_2.*

Proof. Up to isomorphism, there are only two covering spaces for $\mathbb{R}P^2$, namely, the identity covering $p_1 : \mathbb{R}P^2 \to \mathbb{R}P^2$ and the double covering $p_2 : S^2 \to \mathbb{R}P^2$. Suppose that (i) is true. Then, $f_{\#}\pi_1(X) = \pi_1(\mathbb{R}P^2) \approx \mathbb{Z}_2$, and p_1 is a covering corresponding to $f_{\#}\pi_1(X)$. Thus, either $N(f) = 0$ or $N(f) = \mathcal{R}(f) = 1$. Now, if $N(f) = 0$, then also $\mu(f) = 0$ by Proposition 3.8. If $N(f) = 1$, then the result is obvious. Therefore, we have $\mu(f) = \mu_C(f)N(f)$. If, on the other hand, (ii) is true and (i) is false, then we use Theorem 5.6. □

Example 5.7 shows that the assumptions in Theorem 5.9 are not superfluous.

Acknowledgments

The authors would like to express their thanks to Daciberg Lima Gonçalves for his encouragement to the development of the project which led up to this article. This work is partially sponsored by FAPESP - Grant 2007/05843-5. They would like to thank the referee for his careful reading, comments, and suggestions which helped to improve the manuscript.

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