

Research Article

An Extragradient Method for Mixed Equilibrium Problems and Fixed Point Problems

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The purpose of this paper is to investigate the problem of approximating a common element of the set of fixed points of a demicontractive mapping and the set of solutions of a mixed equilibrium problem. First, we propose an extragradient method for solving the mixed equilibrium problems and the fixed point problems. Subsequently, we prove the strong convergence of the proposed algorithm under some mild assumptions.

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . Let $\varphi : C \rightarrow \mathbf{R}$ be a real-valued function and $\Theta : C \times C \rightarrow \mathbf{R}$ be an equilibrium bifunction, that is, $\Theta(u, u) = 0$ for each $u \in C$. We consider the following mixed equilibrium problem (MEP) which is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \quad (\text{MEP})$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (EP), which is to find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \quad (\text{EP})$$

Denote the set of solutions of (MEP) by Ω and the set of solutions of (EP) by Γ . The mixed equilibrium problems include fixed point problems, optimization problems, variational

inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [1–5]. Some methods have been proposed to solve the equilibrium problems, see, for example, [5–21].

In 2005, Combettes and Hirstoaga [6] introduced an iterative algorithm of finding the best approximation to the initial data when $\Gamma \neq \emptyset$ and proved a strong convergence theorem. Recently by using the viscosity approximation method S. Takahashi and W. Takahashi [8] introduced another iterative algorithm for finding a common element of the set of solutions of (EP) and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $S : C \rightarrow H$ be a nonexpansive mapping and $f : C \rightarrow C$ be a contraction. Starting with arbitrary initial $x_1 \in H$, define the sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{aligned} \Theta(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \quad \forall n \geq 0. \end{aligned} \tag{TT}$$

S. Takahashi and W. Takahashi proved that the sequences $\{x_n\}$ and $\{u_n\}$ defined by (TT) converge strongly to $z \in \text{Fix}(S) \cap \Gamma$ with the following restrictions on algorithm parameters $\{\alpha_n\}$ and $\{r_n\}$:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $\liminf_{n \rightarrow \infty} r_n > 0$;
- (iii) (A1): $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; and (R1): $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Subsequently, some iterative algorithms for equilibrium problems and fixed point problems have further developed by some authors. In particular, Zeng and Yao [16] introduced a new hybrid iterative algorithm for mixed equilibrium problems and fixed point problems and Mainge and Moudafi [22] introduced an iterative algorithm for equilibrium problems and fixed point problems.

On the other hand, for solving the equilibrium problem (EP), Moudafi [23] presented a new iterative algorithm and proved a weak convergence theorem. Ceng et al. [24] introduced another iterative algorithm for finding an element of $\text{Fix}(S) \cap \Gamma$. Let $S : C \rightarrow C$ be a k -strict pseudocontraction for some $0 \leq k < 1$ such that $\text{Fix}(S) \cap \Gamma \neq \emptyset$. For given $x_1 \in H$, let the sequences $\{x_n\}$ and $\{u_n\}$ be generated iteratively by

$$\begin{aligned} \Theta(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \alpha_n u_n + (1 - \alpha_n) S u_n, \quad \forall n \geq 1, \end{aligned} \tag{CAY}$$

where the parameters $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$;
- (ii) $\{r_n\} \subset (0, \infty)$ and $\liminf_{n \rightarrow \infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (CAY) converge weakly to an element of $\text{Fix}(S) \cap \Gamma$.

At this point, we should point out that all of the above results are interesting and valuable. At the same time, these results also bring us the following conjectures.

Questions

- (1) Could we weaken or remove the control condition (iii) on algorithm parameters in S. Takahashi and W. Takahashi [8]?
- (2) Could we construct an iterative algorithm for k -strict pseudocontractions such that the strong convergence of the presented algorithm is guaranteed?
- (3) Could we give some proof methods which are different from those in [8, 12, 16, 24]?

It is our purpose in this paper that we introduce a general iterative algorithm for approximating a common element of the set of fixed points of a demicontractive mapping and the set of solutions of a mixed equilibrium problem. Subsequently, we prove the strong convergence of the proposed algorithm under some mild assumptions. Our results give positive answers to the above questions.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H .

Let $T : C \rightarrow C$ be a mapping. We use $\text{Fix}(T)$ to denote the set of the fixed points of T . Recall what follows.

- (i) T is called demicontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2 \quad (2.1)$$

for all $x \in C$ and $x^* \in \text{Fix}(T)$, which is equivalent to

$$\langle x - Tx, x - x^* \rangle \geq \frac{1-k}{2} \|x - Tx\|^2. \quad (2.2)$$

For such case, we also say that T is a k -demicontractive mapping.

- (ii) T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad (2.3)$$

for all $x, y \in C$.

- (iii) T is called quasi-nonexpansive if

$$\|Tx - x^*\| \leq \|x - x^*\| \quad (2.4)$$

for all $x \in C$ and $x^* \in \text{Fix}(T)$.

- (iv) T is called strictly pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 \quad (2.5)$$

for all $x, y \in C$.

It is worth noting that the class of demicontractive mappings includes the class of the nonexpansive mappings, the quasi-nonexpansive mappings and the strictly pseudo-contractive mappings as special cases.

Let us also recall that T is called demiclosed if for any sequence $\{x_n\} \subset H$ and $x \in H$, we have

$$x_n \longrightarrow x \text{ weakly, } (I - T)x_n \longrightarrow 0 \text{ strongly} \implies x \in \text{Fix}(T). \quad (2.6)$$

It is well-known that the nonexpansive mappings, strictly pseudo-contractive mappings are all demiclosed. See, for example, [25–27].

An operator $A : C \rightarrow H$ is said to be δ -strongly monotone if there exists a positive constant δ such that

$$\langle Ax - Ay, x - y \rangle \geq \delta \|x - y\|^2 \quad (2.7)$$

for all $x, y \in C$.

Now we concern the following problem: find $x^* \in \text{Fix}(T) \cap \Omega$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Omega. \quad (2.8)$$

In this paper, for solving problem (2.8) with an equilibrium bifunction $\Theta : C \times C \rightarrow \mathbf{R}$, we assume that Θ satisfies the following conditions:

- (H1) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;
- (H2) for each fixed $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous;
- (H3) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

A mapping $\eta : C \times C \rightarrow H$ is called Lipschitz continuous, if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda \|x - y\|, \quad \forall x, y \in C. \quad (2.9)$$

A differentiable function $K : C \rightarrow \mathbf{R}$ on a convex set C is called

(i) η -convex if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x) \rangle, \quad \forall x, y \in C, \quad (2.10)$$

where K' is the Frechet derivative of K at x ;

(ii) η -strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x) \rangle \geq \left(\frac{\sigma}{2}\right) \|x - y\|^2, \quad \forall x, y \in C. \quad (2.11)$$

Let C be a nonempty closed convex subset of a real Hilbert space H , $\varphi : C \rightarrow \mathbf{R}$ be real-valued function and $\Theta : C \times C \rightarrow \mathbf{R}$ be an equilibrium bifunction. Let r be a positive number. For a given point $x \in C$, the auxiliary problem for (MEP) consists of finding $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z, y) \rangle \geq 0, \quad \forall z \in C. \quad (2.12)$$

Let $S_r : C \rightarrow C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of the auxiliary problem, that is, $\forall x \in C$,

$$S_r(x) = \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z, y) \rangle \geq 0, \forall z \in C \right\}. \quad (2.13)$$

We need the following important and interesting result for proving our main results.

Lemma 2.1 ([16, 28]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \rightarrow \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \rightarrow \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Assume what follows.*

- (i) $\eta : C \times C \rightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that
 - (a) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in C$,
 - (b) $\eta(\cdot, \cdot)$ is affine in the first variable,
 - (c) for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.
- (ii) $K : C \rightarrow \mathbf{R}$ is η -strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology.
- (iii) For each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z_x, y) \rangle < 0. \quad (2.14)$$

Then there hold the following:

- (i) S_r is single-valued;
- (ii) S_r is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$ and

$$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2) \rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2) \rangle, \quad \forall (x_1, x_2) \in C \times C, \quad (2.15)$$

where $u_i = S_r(x_i)$ for $i = 1, 2$;

- (iii) $\text{Fix}(S_r) = \Omega$;
- (iv) Ω is closed and convex.

3. Main Results

Let H be a real Hilbert space, $\varphi : H \rightarrow \mathbf{R}$ be a lower semicontinuous and convex real-valued function, $\Theta : H \times H \rightarrow \mathbf{R}$ be an equilibrium bifunction. Let $A : H \rightarrow H$ be a mapping and $T : H \rightarrow H$ be a mapping. In this section, we first introduce the following new iterative algorithm.

Algorithm 3.1. Let r be a positive parameter. Let $\{\lambda_n\}$ be a sequence in $[0, \infty)$ and $\{\alpha_n\}$ be a sequence in $[0, 1)$. Define the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the following manner:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ \Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r} \langle K'(z_n) - K'(x_n), \eta(x, z_n) \rangle &\geq 0, \quad \forall x \in C, \\ y_n &= z_n - \lambda_n A z_n, \\ x_{n+1} &= (1 - \alpha_n) y_n + \alpha_n T y_n. \end{aligned} \tag{3.1}$$

Now we give a strong convergence result concerning Algorithm 3.1 as follows.

Theorem 3.2. *Let H be a real Hilbert space. Let $\varphi : H \rightarrow \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \rightarrow \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : H \rightarrow H$ be an L -Lipschitz continuous and δ -strongly monotone mapping and $T : H \rightarrow H$ be a demiclosed and k -demicontractive mapping such that $\text{Fix}(T) \cap \Omega \neq \emptyset$. Assume what follows.*

- (i) $\eta : H \times H \rightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that
 - (a) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in H,$
 - (b) $\eta(\cdot, \cdot)$ is affine in the first variable,
 - (c) for each fixed $y \in H, x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.
- (ii) $K : H \rightarrow \mathbf{R}$ is η -strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$.
- (iii) For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x,$

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z_x, y) \rangle < 0. \tag{3.2}$$

- (iv) $\alpha_n \in [\gamma, (1 - k)/2]$ for some $\gamma > 0, \lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (3.1) converge strongly to x^* which solves the problem (2.8) provided S_r is firmly nonexpansive.

Proof. First, we prove that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded. Without loss of generality, we may assume that $0 < \delta < L$. Given $\mu \in (0, 2\delta/L^2)$ and $x, y \in H$, we have

$$\begin{aligned} \|(\mu A - I)x - (\mu A - I)y\|^2 &= \mu^2 \|Ax - Ay\|^2 + \|x - y\|^2 - 2\mu \langle Ax - Ay, x - y \rangle \\ &\leq \mu^2 L^2 \|x - y\|^2 + \|x - y\|^2 - 2\mu\delta \|x - y\|^2 \\ &= (1 - 2\mu\delta + \mu^2 L^2) \|x - y\|^2, \end{aligned} \quad (3.3)$$

that is,

$$\|(\mu A - I)x - (\mu A - I)y\| \leq \sqrt{1 - 2\mu\delta + \mu^2 L^2} \|x - y\|. \quad (3.4)$$

Take $x^* \in \text{Fix}(T) \cap \Omega$. From (3.1), we have

$$\begin{aligned} \|y_{n+1} - (x^* - \lambda_{n+1}Ax^*)\| &= \|(z_{n+1} - \lambda_{n+1}Az_{n+1}) - (x^* - \lambda_{n+1}Ax^*)\| \\ &= \left\| \left(1 - \frac{\lambda_{n+1}}{\mu}\right)(z_{n+1} - x^*) - \frac{\lambda_{n+1}}{\mu}((\mu A - I)z_{n+1} - (\mu A - I)x^*) \right\| \\ &\leq \left(1 - \frac{\lambda_{n+1}}{\mu}\right) \|z_{n+1} - x^*\| + \frac{\lambda_{n+1}}{\mu} \|(\mu A - I)z_{n+1} - (\mu A - I)x^*\|. \end{aligned} \quad (3.5)$$

Therefore,

$$\|y_{n+1} - (x^* - \lambda_{n+1}Ax^*)\| \leq \left(1 - \frac{\lambda_{n+1}\omega}{\mu}\right) \|z_{n+1} - x^*\|, \quad (3.6)$$

where $\omega = 1 - \sqrt{1 - 2\mu\delta + \mu^2 L^2} \in (0, 1)$.

Note that $z_{n+1} = S_r x_{n+1}$ and S_r are firmly nonexpansive. Hence, we have

$$\begin{aligned} \|z_{n+1} - x^*\|^2 &= \|S_r x_{n+1} - S_r x^*\|^2 \\ &\leq \langle S_r x_{n+1} - S_r x^*, x_{n+1} - x^* \rangle \\ &= \langle z_{n+1} - x^*, x_{n+1} - x^* \rangle \\ &= \frac{1}{2} \left(\|z_{n+1} - x^*\|^2 + \|x_{n+1} - x^*\|^2 - \|x_{n+1} - z_{n+1}\|^2 \right), \end{aligned} \quad (3.7)$$

which implies that

$$\|z_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 - \|x_{n+1} - z_{n+1}\|^2. \quad (3.8)$$

From (2.2) and (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)y_n + \alpha_n T y_n - x^*\|^2 \\
&= \|(y_n - x^*) - \alpha_n(y_n - T y_n)\|^2 \\
&= \|y_n - x^*\|^2 - 2\alpha_n \langle y_n - T y_n, y_n - x^* \rangle + \alpha_n^2 \|y_n - T y_n\|^2 \\
&\leq \|y_n - x^*\|^2 - 2\alpha_n \frac{1-k}{2} \|y_n - T y_n\|^2 + \alpha_n^2 \|y_n - T y_n\|^2 \\
&= \|y_n - x^*\|^2 - \alpha_n(1-k-\alpha_n) \|y_n - T y_n\|^2 \\
&\leq \|y_n - x^*\|^2.
\end{aligned} \tag{3.9}$$

From (3.6)–(3.9), we have

$$\begin{aligned}
\|y_{n+1} - x^*\| &\leq \|y_{n+1} - (x^* - \lambda_{n+1} A x^*)\| + \lambda_{n+1} \|A x^*\| \\
&\leq \left(1 - \frac{\lambda_{n+1} \omega}{\mu}\right) \|z_{n+1} - x^*\| + \lambda_{n+1} \|A x^*\| \\
&\leq \left(1 - \frac{\lambda_{n+1} \omega}{\mu}\right) \|x_{n+1} - x^*\| + \lambda_{n+1} \|A x^*\| \\
&\leq \left(1 - \frac{\lambda_{n+1} \omega}{\mu}\right) \|y_n - x^*\| + \lambda_{n+1} \|A x^*\| \\
&= \left(1 - \frac{\lambda_{n+1} \omega}{\mu}\right) \|y_n - x^*\| + \frac{\lambda_{n+1} \omega}{\mu} \left\{ \frac{\mu}{\omega} \|A x^*\| \right\} \\
&\leq \max \left\{ \|y_n - x^*\|, \frac{\mu \|A x^*\|}{\omega} \right\} \\
&\leq \dots \\
&\leq \max \left\{ \|y_0 - x^*\|, \frac{\mu \|A x^*\|}{\omega} \right\}.
\end{aligned} \tag{3.10}$$

This implies that $\{y_n\}$ is bounded, so are $\{x_n\}$ and $\{z_n\}$.

From (3.1), we can write $y_n - T y_n = (1/\alpha_n)(y_n - x_{n+1})$. Thus, from (3.9), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|y_n - x^*\|^2 - \alpha_n(1-k-\alpha_n) \|y_n - T y_n\|^2 \\
&\leq \|y_n - x^*\|^2 - \frac{1-k-\alpha_n}{\alpha_n} \|y_n - x_{n+1}\|^2.
\end{aligned} \tag{3.11}$$

Since $\alpha_n \in (0, (1-k)/2]$, $(1-k-\alpha_n)/\alpha_n \geq 1$. Therefore, from (3.8) and (3.11), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|y_n - x^*\|^2 - \|y_n - x_{n+1}\|^2 \\
&= \|z_n - x^* - \lambda_n Az_n\|^2 - \|z_n - x_{n+1} - \lambda_n Az_n\|^2 \\
&= \|z_n - x^*\|^2 - 2\lambda_n \langle Az_n, z_n - x^* \rangle + \lambda_n^2 \|Az_n\|^2 \\
&\quad - \|z_n - x_{n+1}\|^2 + 2\lambda_n \langle Az_n, z_n - x_{n+1} \rangle - \lambda_n^2 \|Az_n\|^2 \\
&= \|z_n - x^*\|^2 - 2\lambda_n \langle x_{n+1} - x^*, Az_n \rangle - \|x_{n+1} - z_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\lambda_n \langle x_{n+1} - x^*, Az_n \rangle - \|x_{n+1} - z_n\|^2.
\end{aligned} \tag{3.12}$$

We note that $\{x_n\}$ and $\{z_n\}$ are bounded. So there exists a constant $M \geq 0$ such that

$$|\langle x_{n+1} - x^*, Az_n \rangle| \leq M \quad \forall n \geq 0. \tag{3.13}$$

Consequently, we get

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \|x_{n+1} - z_n\|^2 + \|x_n - z_n\|^2 \leq 2M\lambda_n. \tag{3.14}$$

Now we divide two cases to prove that $\{x_n\}$ converges strongly to x^* .

Case 1. Assume that the sequence $\{\|x_n - x^*\|\}$ is a monotone sequence. Then $\{\|x_n - x^*\|\}$ is convergent. Setting $\lim_{n \rightarrow \infty} \|x_n - x^*\| = d$.

(i) If $d = 0$, then the desired conclusion is obtained.

(ii) Assume that $d > 0$. Clearly, we have

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \longrightarrow 0, \tag{3.15}$$

this together with $\lambda_n \rightarrow 0$ and (3.14) implies that

$$\|x_{n+1} - z_n\|^2 + \|x_n - z_n\|^2 \longrightarrow 0, \tag{3.16}$$

that is to say

$$\|x_{n+1} - z_n\| \longrightarrow 0, \quad \|x_n - z_n\| \longrightarrow 0. \tag{3.17}$$

Let $z \in H$ be a weak limit point of $\{z_{n_k}\}$. Then there exists a subsequence of $\{z_{n_k}\}$, still denoted by $\{z_{n_k}\}$ which weakly converges to z . Noting that $\lambda_n \rightarrow 0$, we also have

$$y_{n_k} = z_{n_k} - \lambda_{n_k} Az_{n_k} \longrightarrow z \text{ weakly.} \tag{3.18}$$

Combining (3.1) and (3.17), we have

$$\begin{aligned}
\|Ty_{n_k} - y_{n_k}\| &= \frac{1}{\alpha_{n_k}} \|y_{n_k} - x_{n_k+1}\| \\
&= \frac{1}{\alpha_{n_k}} \|x_{n_k+1} - z_{n_k} + \lambda_{n_k} Az_{n_k}\| \\
&\leq \|x_{n_k+1} - z_{n_k}\| + \lambda_{n_k} \|Az_{n_k}\| \\
&\rightarrow 0.
\end{aligned} \tag{3.19}$$

Since T is demiclosed, then we obtain $z \in \text{Fix}(T)$.

Next we show that $z \in \Omega$. Since $z_n = S_r x_n$, we derive

$$\Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r} \langle K'(z_n) - K'(x_n), \eta(x, z_n) \rangle \geq 0, \quad \forall x \in C. \tag{3.20}$$

From the monotonicity of Θ , we have

$$\frac{1}{r} \langle K'(z_n) - K'(x_n), \eta(x, z_n) \rangle + \varphi(x) - \varphi(z_n) \geq -\Theta(z_n, x) \geq \Theta(x, z_n), \tag{3.21}$$

and hence

$$\left\langle \frac{K'(z_{n_k}) - K'(x_{n_k})}{r}, \eta(x, z_{n_k}) \right\rangle + \varphi(x) - \varphi(z_{n_k}) \geq \Theta(x, z_{n_k}). \tag{3.22}$$

Since $(K'(z_{n_k}) - K'(x_{n_k}))/r \rightarrow 0$ and $z_{n_k} \rightarrow z$ weakly, from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y , we have

$$\Theta(x, z) + \varphi(z) - \varphi(x) \leq 0, \tag{3.23}$$

for all $x \in C$. For $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)z$. Since $x \in C$ and $z \in C$, we have $x_t \in C$ and hence $\Theta(x_t, z) + \varphi(z) - \varphi(x_t) \leq 0$. From the convexity of equilibrium bifunction $\Theta(x, y)$ in the second variable y , we have

$$\begin{aligned}
0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\
&\leq t\Theta(x_t, x) + (1-t)\Theta(x_t, z) + t\varphi(x) + (1-t)\varphi(z) - \varphi(x_t) \\
&\leq t[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)],
\end{aligned} \tag{3.24}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \geq 0$. Then, we have

$$\Theta(z, x) + \varphi(x) - \varphi(z) \geq 0 \tag{3.25}$$

for all $x \in C$ and hence $z \in \Omega$.

Therefore, we have

$$z \in \text{Fix}(T) \cap \Omega. \quad (3.26)$$

Thus, if x^* is a solution of problem (2.8), we have

$$\liminf_{k \rightarrow \infty} \langle z_{n_k} - x^*, Ax^* \rangle = \langle z - x^*, Ax^* \rangle \geq 0. \quad (3.27)$$

Suppose that there exists another subsequence $\{z_{n_i}\}$ which weakly converges to z' . It is easily checked that $z' \in \text{Fix}(T) \cap \Omega$ and

$$\liminf_{i \rightarrow \infty} \langle z_{n_i} - x^*, Ax^* \rangle = \langle z' - x^*, Ax^* \rangle \geq 0. \quad (3.28)$$

Therefore, we have

$$\liminf_{n \rightarrow \infty} \langle z_n - x^*, Ax^* \rangle \geq 0. \quad (3.29)$$

Since A is δ -strongly monotone, we have

$$\langle x_{n+1} - x^*, Az_n \rangle \geq \delta \|z_n - x^*\|^2 + \langle z_n - x^*, Ax^* \rangle + \langle x_{n+1} - z_n, Az_n \rangle. \quad (3.30)$$

By (3.17)–(3.30), we get

$$\liminf_{n \rightarrow \infty} \langle x_{n+1} - x^*, Az_n \rangle \geq \delta d^2. \quad (3.31)$$

From (3.12), for $0 < \epsilon < \delta d^2$, we deduce that there exists a positive integer number n_0 large enough, when $n \geq n_0$,

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \leq -2\lambda_n (\delta d^2 - \epsilon). \quad (3.32)$$

This implies that

$$\|x_{n+1} - x^*\|^2 - \|x_{n_0} - x^*\|^2 \leq -2(\delta d^2 - \epsilon) \sum_{k=n_0}^n \lambda_k. \quad (3.33)$$

Since $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\{x_n\}$ is bounded, hence the last inequality is a contraction. Therefore, $d = 0$, that is to say, $x_n \rightarrow x^*$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Set $\Gamma_n = \|x_n - x^*\|^2$ and let $\tau : N \rightarrow N$ be a mapping for all $n \geq n_0$ by

$$\tau(n) = \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}. \quad (3.34)$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.14), we have

$$\|x_{\tau(n)+1} - z_{\tau(n)}\|^2 + \|x_{\tau(n)} - z_{\tau(n)}\|^2 \leq 2M\lambda_{\tau(n)} \rightarrow 0, \quad (3.35)$$

thus

$$\|x_{\tau(n)+1} - z_{\tau(n)}\| \rightarrow 0, \quad \|x_{\tau(n)} - z_{\tau(n)}\| \rightarrow 0. \quad (3.36)$$

Therefore,

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \rightarrow 0. \quad (3.37)$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for all $n \geq n_0$, from (3.12), we get

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 + \|x_{\tau(n)+1} - z_{\tau(n)}\|^2 + \|x_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq -2\lambda_{\tau(n)} \langle x_{\tau(n)+1} - x^*, Az_{\tau(n)} \rangle, \end{aligned} \quad (3.38)$$

which implies that

$$\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)} \rangle \leq 0 \quad \forall n \geq n_0. \quad (3.39)$$

Since $\{z_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{z_{\tau(n)}\}$, still denoted by $\{z_{\tau(n)}\}$ which converges weakly to $q \in H$. It is easily checked that $q \in \text{Fix}(T) \cap \Omega$. Furthermore, we observe that

$$\begin{aligned} \delta \|z_{\tau(n)} - x^*\|^2 &\leq \langle z_{\tau(n)} - x^*, Az_{\tau(n)} - Ax^* \rangle \\ &= \langle x_{\tau(n)+1} - x^*, Az_{\tau(n)} \rangle + \langle z_{\tau(n)} - x_{\tau(n)+1}, Az_{\tau(n)} \rangle - \langle z_{\tau(n)} - x^*, Ax^* \rangle. \end{aligned} \quad (3.40)$$

Hence, for all $n \geq n_0$,

$$\delta \|z_{\tau(n)} - x^*\|^2 \leq \langle z_{\tau(n)} - x_{\tau(n)+1}, Az_{\tau(n)} \rangle - \langle z_{\tau(n)} - x^*, Ax^* \rangle. \quad (3.41)$$

Therefore

$$\limsup_{n \rightarrow \infty} \|z_{\tau(n)} - x^*\|^2 \leq -\frac{1}{\delta} \langle q - x^*, Ax^* \rangle \leq 0, \quad (3.42)$$

which implies that

$$\lim_{n \rightarrow \infty} \|z_{\tau(n)} - x^*\| = 0. \quad (3.43)$$

Thus,

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - x^*\| = 0. \quad (3.44)$$

It is immediate that

$$\lim_{n \rightarrow \infty} \Gamma_{\tau(n)} = \lim_{n \rightarrow \infty} \Gamma_{\tau(n)+1} = 0. \quad (3.45)$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e., $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}. \quad (3.46)$$

Hence $\lim_{n \rightarrow \infty} \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to x^* . Consequently, it is easy to prove that $\{y_n\}$ and $\{z_n\}$ converge strongly to x^* . This completes the proof. \square

Remark 3.3. The advantages of these results in this paper are that less restrictions on the parameters $\{\lambda_n\}$ are imposed.

As direct consequence of Theorem 3.2, we obtain the following.

Corollary 3.4. *Let H be a real Hilbert space. Let $\varphi : H \rightarrow \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \rightarrow \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : H \rightarrow H$ be an L -Lipschitz continuous and δ -strongly monotone mapping and $T : H \rightarrow H$ be a nonexpansive mapping such that $\text{Fix}(T) \cap \Omega \neq \emptyset$. Assume what follows.*

(i) $\eta : H \times H \rightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that;

- (a) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in H,$
- (b) $\eta(\cdot, \cdot)$ is affine in the first variable,
- (c) for each fixed $y \in H, x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.

(ii) $K : H \rightarrow \mathbf{R}$ is η -strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$.

(iii) For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z_x, y) \rangle < 0. \quad (3.47)$$

(iv) $\alpha_n \in [\gamma, (1 - k)/2]$ for some $\gamma > 0, \lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the sequences $\{x_n\}, \{y_n\},$ and $\{z_n\}$ generated by (3.1) converge strongly to x^* which solves the problem (2.8) provided S_r is firmly nonexpansive.

Corollary 3.5. *Let H be a real Hilbert space. Let $\varphi : H \rightarrow \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \rightarrow \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : H \rightarrow H$ be an L -Lipschitz continuous and δ -strongly monotone mapping and $T : H \rightarrow H$ be a strictly pseudo-contractive mapping such that $\text{Fix}(T) \cap \Omega \neq \emptyset$. Assume what follows.*

- (i) $\eta : H \times H \rightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that
- (a) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in H,$
 - (b) $\eta(\cdot, \cdot)$ is affine in the first variable,
 - (c) for each fixed $y \in H, x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.
- (ii) $K : H \rightarrow \mathbf{R}$ is η -strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$.
- (iii) For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x,$

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), \eta(z_x, y) \rangle < 0. \quad (3.48)$$

- (iv) $\alpha_n \in [\gamma, (1 - k)/2]$ for some $\gamma > 0, \lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by (3.1) converge strongly to x^* which solves the problem (2.8) provided S_r is firmly nonexpansive.

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