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Research Article

A Poincaré Formula for the Fixed Point Indices of the Iterates of Arbitrary Planar Homeomorphisms

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Let $U \subset \mathbb{R}^2$ be an open subset and $f : U \to \mathbb{R}^2$ be an arbitrary local homeomorphism with $\mathrm{Fix}(f) = \{p\}$. We compute the fixed point indices of the iterates of f at p, $i_{\mathbb{R}^2}(f^k, p)$, and we identify these indices in dynamical terms. Therefore, we obtain a sort of Poincaré index formula without differentiability assumptions. Our techniques apply equally to both orientation preserving and orientation reversing homeomorphisms. We present some new results, especially in the orientation reversing case.

1. Introduction

There is abundant literature about the fixed point index of a homeomorphism f, in a neighborhood of an isolated fixed point and the local dynamical behavior of f. There are results in both directions, that is, bounds (or explicit computation) for the fixed point index from dynamical properties of f and conversely how the knowledge of the fixed point index is used to describe the dynamics locally.

One can notice that due to the systematic use of Brouwer's translation arcs theorem (see [1] or [2]), most of the known results are limited to orientation preserving homeomorphisms.

It is well known that the classical Poincaré index formula relates the index of a planar vector field with the elliptic and hyperbolic regions in a neighborhood of a critical point. Such a formula, for the iterates of an arbitrary homeomorphism, will give a geometric interpretation of the fixed point indices of the iterates, it could help to attack some open problems and it will provide simple proofs of many of the strongest theorems in the subject. This is the main goal of this article.

The Ulam's problem about the existence of minimal homeomorphisms in the multipunctured plane was solved completely in the negative by Le Calvez and Yoccoz in [3]. The main technique in the proof of their theorem is the computation of the fixed point index of all iterates of an orientation preserving homeomorphism in a neighborhood of a fixed point p which is an isolated invariant set, neither an attractor nor a repeller. Given an orientation preserving local homeomorphism $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, they carry out a detailed local study, near the fixed point p. Then they prove the existence of integers $r, q \geq 1$ such that

$$i_{\mathbb{R}^2}\left(f^k, p\right) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}. \end{cases} \tag{1.1}$$

The authors, in [4], using Conley index ideas, gave, in a quite simple way, a general theorem extending the above result to arbitrary local homeomorphisms. In particular, if f reverses the orientation, there are integers $\delta \in \{0, 1, 2\}$ and q such that

$$i_{\mathbb{R}^2}\left(f^k, p\right) = \begin{cases} 1 - \delta & \text{if } k \text{ odd}, \\ 1 - \delta - 2q & \text{if } k \text{ even}. \end{cases} \tag{1.2}$$

Later, Le Calvez extended his theorem with Yoccoz to arbitrary isolated fixed points of orientation preserving planar homeomorphisms. Again the fixed point indices of the iterations of the homeomorphism have periodical behavior. Le Calvez, in [5], uses in a very clever way the nice Carathéodory's prime ends theory (see [6, 7]). The idea of applying the compactification of Carathéodory to study planar dynamical problems is not new. It was introduced by Pérez-Marco in [8] and it was used more recently by the first author, in [9], to prove that the index of arbitrary stable planar fixed points is equal to 1.

On the other hand, Baldwin and Slaminka, in [10], dealt with the problem of relating the fixed point index of an orientation and area preserving homeomorphism around an isolated fixed point p and the number of branches in which the stable/unstable "manifold" of p decomposes. The results of Baldwin and Slaminka were improved by Le Roux, in [11], where the fixed point index is used not only to detect stable/unstable branches but also Leau-Fatou petals around p. The authors, in [12], gave a stable/unstable "manifold" theorem for arbitrary planar homeomorphisms near a fixed point admitting *nice* filtration pairs.

There are some papers dedicated to the study of the analogous problem in dimension 3. See [13–16] and its references.

The computation of the fixed point index of any iteration of any planar homeomorphism at an isolated fixed point laying in an isolated invariant compactum was done by the authors in [4, 12]. As we said above, when p does not belong to any isolated invariant compactum and the homeomorphism is orientation preserving, Le Calvez improved a result of Brown, see [17], showing that the sequence of indices is periodic. We will find with our methods the same formula for orientation preserving homeomorphisms and we shall solve the problem also for orientation reversing homeomorphisms. The main fact to obtain our results is the existence of special classes of filtration pairs in the Carathéodory's prime ends compactification that will allow us to by-pass the technical problem that occurs if the fixed point does not lay in an isolated invariant compactum.

Roughly speaking, if a fixed point p does not lay in arbitrary small isolated compacta, we can consider any disc J containing p in its interior and take K_p, the component containing

p of the maximal invariant set contained in J. By using the Carathéodory's compactification of $S^2 \setminus K_p$, we work in a disc and we can compute the index at p from the local indices (in semidiscs) of the fixed prime ends that now will admit isolating blocks. The existence of such isolating blocks around the fixed prime ends not only provides a simple technique to compute the index of the iterations of arbitrary homeomorphisms but also allows to identify such indices in a geometrical way. Given a disc J the existence of isolating blocks, around the fixed points that appear in the compactification, allows to find dynamical objects (generalized stable/unstable branches and generalized attracting/repelling petals whose definitions we will precise later) which are the keys for the computations of the indices.

Essentially, the index of the homeomorphism at p only provides "optimal" dynamical information if p admits isolating blocks. Otherwise, the set of indices of the induced homeomorphism in the Carathéodory's compactification of $S^2 \setminus K_p$ at the new fixed points provides much more information than the index at p.

The main goals of this paper are the following:

(a) The first goal is to provide a general geometrical method to compute the fixed point index of the iterations of an arbitrary local homeomorphism at an isolated fixed point;

(b) Given any Jordan domain J, $\mathrm{Inv}(\mathrm{cl}(J), f) \cap \partial(J) \neq \emptyset$ and an isolating block, N, is a neighborhood that isolates the fixed (or periodical) prime ends of the component of $\mathrm{Inv}(\mathrm{cl}(J), f)$ containing p, to prove that J and N determine canonically a number of generalized unstable (stable) branches and generalized repelling (attracting) petals around the fixed point (see Definition 2.6). Their number depends on J and N but their difference depends just on the germ of f;

(c) The third goal is to provide some dynamical consequences. We shall give new and short proofs of some known results and new theorems in the orientation reversing framework.

The paper is organized as follows: in Section 2 we start with some preliminary definitions. We will dedicate subsections to recall the results we will need in the special case where the fixed point is an isolated invariant set and to give a brief presentation of the Carathéodory's prime ends theory. At the end of the section, we give the statement of the main results. Section 3 is devoted to the computation of the fixed point indices of the iterations of arbitrary planar homeomorphisms at an isolated fixed point. In Section 4, we will give the proof of the main theorems and the dynamical meaning of the indices. First we shall study the case where the homeomorphism has a finite number of periodic prime ends. The general case follows easily from this previous simpler case (see Remark 2.12). Finally Section 5 contains the proofs of a number of corollaries of our techniques.

2. Preliminary Definitions and Results. The Main Construction and the Statement of the Principal Results

2.1. Preliminary Definitions

Given $A \subset B \subset N$, $\mathrm{cl}(A)$, $\mathrm{cl}_B(A)$, $\mathrm{int}(A)$, $\mathrm{int}_B(A)$, $\partial(A)$ and $\partial_B(A)$ will denote the closure of A, the closure of A in B, the interior of A, the interior of A in B, the boundary of A, and the boundary of A in B, respectively.

Let $U \subset X$ be an open set. By a *(local) semidynamical system*, we mean a local homeomorphism $f : U \to X$. The *invariant part* of N, $\mathrm{Inv}(N, f)$, is defined as the set of all $x \in N$ such that there is a full orbit γ with $x \in \gamma \subset N$.

$\mathrm{Inv}^+(N, f)$ (resp., $\mathrm{Inv}^-(N, f)$) will denote the set of all $x \in N$ such that $f^j(x) \in N$ for every $j \in \mathbb{N}$ (resp., $f^{-j}(x)$ is well defined and belongs to N for every $j \in \mathbb{N}$).

A compact set $S \subset X$ is *invariant* if $f(S) = S$. A compact invariant set S is *isolated with respect to* f if there exists a compact neighborhood N of S such that $\mathrm{Inv}(N, f) = S$. The neighborhood N is called an *isolating neighborhood of* S.

An *isolating block* N is a compactum such that $\mathrm{cl}(\mathrm{int}(N)) = N$ and $f^{-1}(N) \cap N \cap f(N) \subset \mathrm{int}(N)$. Isolating blocks are a special class of isolating neighborhoods.

We consider the *exit set of* N to be defined as

$$N^- = \{x \in N : f(x) \notin \mathrm{int}(N)\}. \tag{2.1}$$

If X is a locally compact ANR (absolute neighborhood retract for metric spaces), $i_X(f, S)$ will denote the fixed point index of f in a small enough neighborhood of S. The reader is referred to the text of [18–22] for information about the fixed point index theory.

An isolated fixed point p is said to be *indifferent* if for every small enough disc D such that $p \in \mathrm{int}(D)$, $\mathrm{Inv}(D, f) \cap \partial(D) \neq \emptyset$.

An isolated fixed point p is *accumulated* if $p \in \mathrm{cl}(\mathrm{Per}(f|_V) \setminus \{p\})$ for every neighborhood V of p.

2.2. Strong Filtration Pairs

The next definition is based on the notion of *filtration* introduced by Franks and Richeson, in [23]. It is the key for the direct computation of the fixed point index of any iteration of any homeomorphism of the plane.

Definition 2.1. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Suppose that $L \subset N$ is a compact pair contained in the interior of U. The pair (N, L) is said to be a *strong filtration pair* for f provided N and L are each the closure of their interiors and

(1) N and $\partial(N \setminus L)$ are homeomorphic to a disc and S^1, respectively.

(2) $\mathrm{cl}(N \setminus L)$ is an isolating neighborhood.

(3) $f(\mathrm{cl}(N \setminus L)) \subset \mathrm{int}(N)$ (i.e., L is a neighborhood of N^- in N).

(4) For any component L_i of L, $\partial_N(L_i)$ is an arc and there exists a topological disc B_i such that $\partial_N(L_i) \subset B_i \subset L_i$, $B_i \cap N^- \neq \emptyset$, and $f(B_i) \cap \mathrm{cl}(N \setminus L) = \emptyset$.

Theorem 2.2 (see [4, 12]). *Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be a homeomorphism. Suppose that there exists a strong filtration pair, (N, L), for f and let $K = \mathrm{Inv}(\mathrm{cl}(N \setminus L), f)$. Then, there are an absolute retract for metric spaces, D_0, containing a neighborhood $V \subset \mathbb{R}^2$ of K, a finite subset $\{q_1, \ldots, q_m\} \subset D_0$, and a map $\overline{f} : D_0 \to D_0$ such that $\overline{f}|_V = f|_V$ and for every $k \in \mathbb{N}$, $\mathrm{Fix}((\overline{f})^k) \subset K \cup \{q_1, \ldots, q_m\}$.*

Moreover,

(a) *if f preserves the orientation, then*

$$i_{\mathbb{R}^2}\left(f^k, K\right) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}, \end{cases} \tag{2.2}$$

where $k \in \mathbb{N}$, q is the number of periodic orbits of $\overline{f}$ in $\{q_1, \dots, q_m\}$, and r is their period;

(b) *if f reverses the orientation, then*

$$i_{\mathbb{R}^2}\left(f^k, K\right) = \begin{cases} 1 - \delta & \text{if } k \text{ odd}, \\ 1 - \delta - 2q & \text{if } k \text{ even}, \end{cases} \tag{2.3}$$

where $\delta \in \{0, 1, 2\}$ and q are the number of fixed points and period two orbits of $\overline{f}$ in $\{q_1, \dots, q_m\}$, respectively.

Definition 2.3. Under the setting of the above theorem, the integer r ($r = 2$ if f is orientation reversing) is called the *period of the strong filtration pair* (N, L).

We conclude this subsection with the next theorem that resumes the main results of [12]. We will construct a family of *branches* of the *stable* and *unstable "manifolds"* associated to a fixed point p which admits a strong filtration pair (N, L). The minimum number of elements of these families depends on the fixed point index $i_{\mathbb{R}^2}(f^r, p)$ with r being the period of the strong filtration pair (N, L). In order to make the paper as self-contained as possible, we will sketch the proof which contains some ingredients we will need in the future.

Theorem 2.4. *Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be a homeomorphism with p being an isolated fixed point of f, and let us assume that there is a strong filtration pair of period r, (N, L), such that $p \in \operatorname{int}(N \setminus L)$, $L \neq \emptyset$, $f^j(\operatorname{cl}(N \setminus L)) \subset U$ for $j \in \{1, \dots, r\}$ and $\operatorname{Fix}(f^r) \cap \operatorname{cl}(N \setminus L) = \{p\}$. Let us suppose that the connected component of $K = \operatorname{Inv}(\operatorname{cl}(N \setminus L), f)$ which contains p is $K_p = \{p\}$. Then there exist trivial shape continua $S_1, \dots, S_s, U_1, \dots, U_s$ in $\operatorname{cl}(N \setminus L)$, with $s = 1 - i_{\mathbb{R}^2}(f^r, p)$, such that:*

(1) *$\bigcup_{i=1}^s S_i \subset K_p^+$ and $\bigcup_{i=1}^s U_i \subset K_p^-$, with K_p^+ and K_p^- being the connected components of $K^+ = \operatorname{Inv}^+(\operatorname{cl}(N \setminus L), f)$ and $K^- = \operatorname{Inv}^-(\operatorname{cl}(N \setminus L), f)$ which contain p;*

(2) *$S_i \cap S_j = U_i \cap U_j = \{p\}$ for all $i \neq j$ and $S_i \cap U_j = \{p\}$ for all $i, j \in \{1, \dots, s\}$;*

(3) *$f^r(S_i) \subset S_i$, $f^{-r}(U_i) \subset U_i$, and $\bigcap_{n \in \mathbb{N}} f^{-nr}(U_i) = \bigcap_{n \in \mathbb{N}} f^{nr}(S_i) = \{p\}$ for every $i \in \{1, \dots, s\}$;*

(4) *the sets $S_i \cap \partial(\operatorname{cl}(N \setminus L))$ and $U_i \cap \partial(\operatorname{cl}(N \setminus L))$ alternate in $\partial(\operatorname{cl}(N \setminus L))$.*

Proof. If $L = L_1 \cup \cdots \cup L_m$, let us consider the quotient space N_L obtained from $\operatorname{cl}(N \setminus L)$ by identifying each $\partial_N(L_i)$ to a point q_i for $i = 1, \dots, m$. Take the projection map $\pi : \operatorname{cl}(N \setminus L) \to N_L$ and the retraction $r : N \to \operatorname{cl}(N \setminus L)$. The map

$$f' = \pi \circ r \circ f \circ \pi^{-1} : N_L \setminus \{q_1, \dots, q_m\} \longrightarrow N_L \tag{2.4}$$

induces in a natural way a continuous map $\overline{f} : N_L \to N_L$. It is easy to see that $\overline{f}(\{q_1, \ldots, q_m\}) \subset \{q_1, \ldots, q_m\}$. Let $\theta = \{p_1, \ldots, p_s\}$ be the biggest subset of $\{q_1, \ldots, q_m\}$ on which $\overline{f}$ acts as a permutation. It is clear that θ is an attractor for $\overline{f}$ (is locally constant for every $p_i \in \theta$). Let A be the region of attraction of θ,

$$A = \left\{x \in N_L : \text{there is } n_0 \text{ such that } \left(\overline{f}\right)^{n_0}(x) \in \theta\right\} \tag{2.5}$$

and let $A(p_j)$ be the component of A containing $p_j \in \theta$. Let us observe that K_p^- and K_p^+ are trivial shape continua such that $\lim_{k\to\infty} f^{-k}(x) = p$ for every $x \in K_p^-$ and $\lim_{k\to\infty} f^k(x) = p$ for every $x \in K_p^+$ (see the Main Theorem in [12] for a proof). Then it is not difficult to see that $p \in \mathrm{cl}(A(p_j))$ for all $j = 1, \ldots, s$.

Let $K_i = \bigcap_{n\in\mathbb{N}} (\overline{f})^{nr}(\mathrm{cl}(A(p_i))$ for $i \in \{1, \ldots, s\}$. Since $(\overline{f})^r(\mathrm{cl}(A(p_i)) \subset \mathrm{cl}(A(p_i)))$, it is clear that K_i is a continuum with $\{p, p_i\} \subset K_i = (\overline{f})^r(K_i) \subset \mathrm{cl}(A(p_i))$. Then we have that $\bigcup_{i\in\{1,\ldots,s\}}(K_i \setminus \{p_i\}) \subset K_p^-$, then $\partial_N(L_i) \cap K_p^- \neq \emptyset$ for all $i = 1, \ldots, s$.

Let us define the continuum $U_i = \pi^{-1}(K_i) \cap K_p^-$. We have that U_i is negatively invariant for f^r and contains p.

On the other hand, $U_i \cap K = \{p\}$. In fact, since $\bigcap_{n\in\mathbb{N}} f^{-nr}(U_i)$ is an invariant continuum for f^r which contains p, then $\bigcap_{n\in\mathbb{N}} f^{-nr}(U_i) \subset K_p = \{p\}$. If $x \in U_i \cap K$, then $x \in \bigcap_{n\in\mathbb{N}} f^{-nr}(U_i) \subset K_p = \{p\}$.

Let us see that U_i has a trivial shape. In fact, if U_i has a hole V, then there are $a \in V$ and $n_0 \in \mathbb{N}$ such that $f^{rn_0}(a) \in \mathrm{int}(L_i)$ and $f^{rn}(a) \in \mathrm{cl}(N \setminus L)$ for all $n \in \mathbb{Z}$, $n < n_0$. Then it is immediate that $a \in U_i$ which is a contradiction.

Let us prove that $U_i \subset \pi^{-1}(A(p_i)) \cup \{p\}$. If $x \in U_i \setminus \{p\}$, then there exists $n_0 \in \mathbb{N}$ such that $f^{nr}(x) \in \mathrm{cl}(N \setminus L)$ for all integer $n < n_0$ and $f^{n_0 r}(x) \in \mathrm{int}(L_i)$ (if this is not true, $x \in K$ and we have $x = p$). Then it follows that $x \in \pi^{-1}(A(p_i))$. As a corollary, we obtain that $U_i = (\pi^{-1}(A(p_i)) \cup \{p\}) \cap K_p^-$.

It is obvious that $U_i \cap \partial(\mathrm{cl}(N \setminus L)) \subset \partial_N(L_i)$.

If we repeat this construction for $i \in \{1, \ldots, s\}$, we obtain $U_1, \ldots, U_s$ with $U_i \cap U_j = \{p\}$ for every $i \neq j$.

Let us construct the sets $S_1, \ldots, S_s$. Let us consider the set $\theta = \{p_1, \ldots, p_s\}$ with p_{i-1} adjacent to p_i (there is an arc $\gamma \subset \pi(\partial(N \setminus L))$ joining p_{i-1} with p_i such that $\gamma \cap \theta = \{p_{i-1}, p_i\}$). If $\overline{p_{i-1}p_i}$ is the arc in $\pi(\partial(\mathrm{cl}(N \setminus L))$ which makes adjacent p_{i-1} and p_i, we have that there is a component $K_{p_i} \subset K_p^+$ of $\partial(A(p_i))$ separating p_i from $\theta \setminus p_i$ (see the Main Theorem in [12]) with $K_{p_i} \cap \overline{p_{i-1}p_i} \neq \emptyset$.

Let B_i be the connected component of $\mathrm{cl}(N \setminus L) \setminus (U_{i-1} \cup U_i)$ which contains $\pi^{-1}(K_{p_i} \cap \overline{p_{i-1}p_i})$. Then we define $S_i = (B_i \cup \{p\}) \cap K_p^+$. Following the steps given with the family $\{U_i\}$, it is easy to prove the analogous properties for the family $\{S_i\}$. □

2.3. Carathéodory's Prime Ends

Let $B \subset \mathbb{C}$ be the unit open disc and let $f : B \to G \subset \mathbb{C} \cup \{\infty\}$ be an onto and conformal mapping. The problem whether f admits an extension to $\mathrm{cl}(B) = B \cup S^1$, by defining $f(z) = \lim_{x\to z} f(x)$ for $z \in S^1$, has a topological answer. Indeed, f admits that an extension if and only if $\partial(G)$ is locally connected. The problem whether f has an injective extension has also an answer of topological nature: f has an injective extension if and only if $\partial(G)$ is a Jordan curve (Carathéodory's Theorem, see [24]). If $\partial(G)$ is locally connected but not a

Jordan curve, there are points of $\partial(G)$ that have several preimages. The situation becomes much more complicated if $\partial(G)$ is not locally connected. Carathéodory introduced the notion of prime end to describe this setting. The points $z \in S^1$ correspond one-to-one to the prime ends of G and the limit $f(z)$ exists if and only if the prime end has only one point (Prime End Theorem, see [24]).

Let $D \subset \mathbb{R}^2$ be a simply connected open domain containing the point at infinity such that $\partial(D)$ contains more than one point. Then $\partial(D)$ is bounded. A *cross-cut* is a simple arc, C, lying in D, except in the end points, with different extremities. If C is a cross-cut of D then $D \setminus C$ has exactly two components A_1 and A_2 such that $D \cap \partial(A_1) = D \cap \partial(A_2) = C \setminus \{\text{end points}\}$.

A sequence $\{C_n\}$ of mutually disjoint cross-cuts and such that each C_n separates C_{n-1} and C_{n+1} is called a *chain*. A chain of cross-cuts induces a nested chain of domains (bounded by each C_n) $\cdots D_{n+1} \subset D_n \cdots$. Each chain of cross-cuts defines an *end*. Two chains of cross-cuts, $\{C_n\}$ and $\{C'_n\}$, are *equivalent* if for any $n \in \mathbb{N}$ there is $m(n)$ such that $D_m \subset D'_n$ and $D'_m \subset D_n$ for every $m > m(n)$. Equivalent chains of cross-cuts are said to induce the same end. If P and Q are ends represented by chains of cross-cuts $\{C(P)_n\}$ and $\{C(Q)_n\}$ such that for every n, $D(P)_m \subset D(Q)_n$ for $m > m(n)$, we say that P *divides* Q. A *prime end* P is an end which cannot be divided by any other.

Let P be a prime end. The *set of points of* P is the intersection $E = \bigcap_{n\in\mathbb{N}} \mathrm{cl}(D(P)_n)$ where $\{D(P)_n\}$ is the sequence of domains bounded by any sequence of cross-cuts representing P. A *principal point* of P is a limit point of a chain of cross-cuts representing P tending to a point. The set $H_P \subset E$ of principal points of a prime end P is a continuum (compact connected set) (see [6] or [7] for details).

Each chain of cross-cuts inducing a prime end P determines a basis of neighborhoods of P. We obtain in this way a topology in the set of prime ends of D. More precisely, if $\mathbb{P}$ is the set of prime ends of D and D^* is the disjoint union of D and $\mathbb{P}$, we can introduce a topology in D^* in such a way that it becomes homeomorphic to the closed disk and the boundary being composed by the prime ends. It is enough to define a basis of neighborhoods of a prime end $P \in \mathbb{P}$. Given the sequence of domains $\{D(P)_n\}$, we produce a basis of neighborhoods $\{U_n\}$ of P in D^*. Each U_n is composed by the points in $D(P)_n$ and by the prime ends Q such that $D(Q)_m \subset D(P)_n$ for m large enough.

If S^2 is the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$ and $\infty \in D \subset S^2$ is a simply connected open domain, the natural compactification, due to Carathéodory, see [6], of D obtained by attaching to D a set homeomorphic to the one-dimensional sphere S^1 is called *the prime ends compactification of* D. We identify $\mathbb{R}^2 = \mathbb{C}$ and we consider a conformal homeomorphism $g : D \to S^2 \setminus B$ (where B is the disc $B = \{z \in \mathbb{C} : |z| \leq 1\}$). Now a one-dimensional sphere S^1 is attached to D using g. Each point of S^1 corresponds to a prime end of D.

2.4. The Main Construction

Let $f : U \to W$ be a local homeomorphism with $U, W \subset \mathbb{R}^2$ open subsets and let p be a nonaccumulated and indifferent fixed point in a small enough Jordan domain J with $\{p\}$ being the unique periodic orbit contained in $\mathrm{cl}(J)$ and such that $K_p \cap \partial(J) \neq \emptyset$ for K_p being the connected component of $K = \mathrm{Inv}(\mathrm{cl}(J), f)$ which contains p. We will suppose that $p \in \partial(K_p)$ (e.g., if p is not stable and J is small enough, then $p \in \partial(K_p)$).

Remark 2.5. Let us observe that, given p being a non-accumulated and indifferent fixed point, if $i_{\mathbb{R}^2}(f^k, p) \neq 1$ for some $k \in \mathbb{N}$, then we can select a Jordan domain J, as above, with $p \in \partial(K_p)$.

In fact, if $p \in \operatorname{int}(K_p)$ for every small enough Jordan domain J, then p is stable for f^k and $i_{\mathbb{R}^2}(f^k, p) = 1$ (see [9, 25]).

It is easy to see that the set $K_p \subset \operatorname{cl}(J)$ has a trivial shape, that is, K_p and $\mathbb{R}^2 \setminus K_p$ are connected.

We follow with some of the most important notions of the paper: the generalized stable/unstable branches and generalized attracting/repelling petals. The first ones are essentially branches, in a classical sense, for the map that our homeomorphism f induces in the compactification of $\mathbb{R}^2 \setminus K_p$ at a fixed prime end.

Let $p \in J$ be an indifferent and non-accumulated fixed point for f in the above conditions. Given the open domain $S^2 \setminus K_p$, for each open arc $\gamma \subset J$ with end-points $a, b \in K_p$ (we do not exclude the possibility $a = b$) such that $\gamma \cap K_p = \emptyset$, we call D_γ the bounded connected component of $\mathbb{R}^2 \setminus (\gamma \cup K_p)$. The set D_γ is an open ball contained in J.

Definition 2.6. A continuum $U_p \subset \operatorname{cl}(J)$ is a *generalized unstable branch* for f at p if:

(i) $U_p \cap K_p$ is an invariant continuum contained in $\partial(K_p)$ such that $p \in U_p \cap K_p$ and $U_p \setminus K_p \subset J$ is nonempty and has trivial shape components;

(ii) $f^{-1}(U_p) \subset U_p$ and $\bigcap_{n\in\mathbb{N}} f^{-n}(U_p) = U_p \cap K_p$;

(iii) there exists an open ball D_γ associated to an open arc γ, as above, with $U_p \subset \operatorname{cl}(D_\gamma)$, $U_p \cap \gamma$ a compact set, and such that:

  (a) the set U_p is locally maximal, that is, if $U'_p \subset \operatorname{cl}(D_\gamma)$ satisfies conditions (i) and (ii), then $U'_p \subset U_p$;

  (b) for every open neighborhood V of U_p, there exists $x \in D_\gamma \cap V$ with $f^{-n_x}(x) \notin \operatorname{cl}(D_\gamma)$ for some $n_x \in \mathbb{N}$.

In an analogous way, we define *generalized stable branches* S_p for f at p. We only have to replace f by f^{-1} in (ii) and (iii).

A set R_p is a *generalized repelling petal* for f at p if:

(i) $R_p = \operatorname{cl}(D_\gamma) \subset \operatorname{cl}(J)$ with D_γ being an open ball associated to an open arc γ, as above, such that $cl(\gamma) = \gamma \cup \{q_1, q_2\}$ with $p \notin \{q_1, q_2\}$;

(ii) $f^{-1}(R_p) \subset R_p$ and $\bigcap_{n\in\mathbb{N}} f^{-n}(R_p) \subset \partial(K_p)$ is an invariant continuum for f which contains p.

In an analogous way, we define *generalized attracting petals* for f at p. We only have to replace f by f^{-1} in (ii).

Remark 2.7. The stable and unstable branches in the classical sense associated to f at p and constructed in the proof of Theorem 2.4, are, of course, particular cases of generalized unstable and stable branches if we consider the map $f' = f^r$ and $K_p = \{p\}$. It is easy to obtain an adequate arc $\gamma_j \subset \operatorname{cl}(N \setminus L)$ for each unstable (stable) branch U_j.

Let U' be a Jordan domain such that $\operatorname{cl}(J) \subset U' \subset U \subset S^2$ and let $\overline{f} : S^2 \to S^2$ be a homeomorphism such that $\overline{f}|_{U'} = f$. The Carathéodory's compactification of $S^2 \setminus K_p$ is a disc (obtained by gluing S^1 to $S^2 \setminus K_p$) which we call D. The homeomorphism $\overline{f}|_{S^2\setminus K_p} : S^2 \setminus K_p \to S^2 \setminus K_p$ can be extended to a homeomorphism $\widehat{f} : D \to D$. Let us denote $D \setminus (S^2 \setminus K_p) = \partial(D)$

and let us consider the set of prime ends obtained from the accessible points $K_p \cap \partial(J)$ (by arcs on $U \setminus \mathrm{cl}(J)$) and which we call $\mathcal{P}(K_p \cap \partial(J)) \subset \partial(D)$.

If $\widehat{f}$ is orientation preserving and there exist periodic orbits for $\widehat{f}|_{\partial(D)}$, then all of them have the same period r. If $\widehat{f}$ is orientation reversing, then $\widehat{f}|_{\partial(D)}$ has exactly two fixed points and period two periodic orbits.

Let us see that the compact sets $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ and $\mathcal{P}(K_p \cap \partial(J))$ are disjoint. Let $\mathcal{P}_0$ be a prime end in $\mathcal{P}(K_p \cap \partial(J))$ associated with a point $p_0 \in K_p \cap \partial(J)$. Then $\mathcal{P}_0 \notin \mathrm{Per}(\widehat{f}|_{\partial(D)})$. In fact, if this is not true, $\mathcal{P}_0$ is a fixed prime end for $\widehat{f}^r$ ($r = 2$ if $\widehat{f}$ is orientation reversing) and, since p_0 is accessible by an arc $\gamma_{p_0} \subset U \setminus \mathrm{cl}(J)$ such that $\mathrm{cl}(\gamma_{p_0}) \setminus \gamma_{p_0} = \{p_0\}$, then the principal points of the fixed prime end $\mathcal{P}_0$ are the continuum, invariant for f^r, $H_{\mathcal{P}_0} = \{p_0\}$ ($H_{\mathcal{P}_0} \subset \mathrm{cl}(\gamma_{p_0}) \setminus \gamma_{p_0} = \{p_0\}$). Then, p_0 must be a fixed point for f^r. But this is a contradiction.

Remark 2.8. Note that both $\widehat{f}$ and the set of fixed prime ends of $\widehat{f}$ depend on the Jordan domain J such that $\mathrm{Inv}(\mathrm{cl}(J), f) \cap \partial(J) \neq \emptyset$. See Example 2.9.

Example 2.9. Let us consider the dynamical system of Figure 1, which gives us a homeomorphism f of $\mathbb{R}^2$ with p being a non-accumulated and indifferent fixed point.

The Jordan domains J_1 and J_2 of Figure 1 are such that $\mathrm{Inv}(\mathrm{cl}(J_1), f) = K_{1p}$ is a "petal" which contains p and such that $K_{1p} \cap \partial(J_1) \neq \emptyset$. On the other hand, $\mathrm{Inv}(\mathrm{cl}(J_2), f) = K_{2p}$ are two "petals" which contain p and such that $K_{2p} \cap \partial(J_2) \neq \emptyset$.

The maps $\widehat{f} : D \to D$ have the dynamical behavior in Figure 2 .

The map $\widehat{f}$ for J_1 has, in $\partial(D)$, a fixed prime end p_1 and the map $\widehat{f}$ for J_2 has, in $\partial(D)$, two fixed prime ends $\{p_1, p_2\}$.

Following with the main construction, there are two possible situations:

(a) $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is a finite set of n points;

(b) $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is an infinite set of points.

Let us suppose that we are in case (a). Remark 2.12 permit us to reduce case (b) to case (a) by identifications to points of adequate intervals in $\partial(D)$. If $\widehat{f}$ is an orientation preserving homeomorphism, we have that $n = qr$ for certain $q, r \in \mathbb{N}$ with r being the period of the periodic orbits of $\widehat{f}|_{\partial(D)}$ and q the number of periodic orbits. On the other hand, if $\widehat{f}$ is orientation reversing, we obtain q periodic orbits of period 2 and two fixed points in $\partial(D)$. It is obvious that $n = 2q + 2$.

Let us suppose that $D \subset S^2$ and let us denote by $\widehat{f}_s : S^2 \to S^2$ the homeomorphism obtained by pasting along $\partial(D)$ a symmetric copy of $\widehat{f} : D \to D$.

The next lemma is needed for the computation of the fixed point index $i_{\mathbb{R}^2}(f^k, p)$ by using strong filtration pairs.

Lemma 2.10. *Given a fixed point p_1 of $\widehat{f}_s^{\,k}|_{\partial(D)}$, ($k \in r\mathbb{N}$ if f is orientation preserving), there is a pair (N_1, L_1) which is in one of the following two situations.*

(a) *(N_1, L_1) is a strong filtration pair for $\widehat{f}_s^{\,k} : S^2 \to S^2$, in a neighborhood of p_1. The period of the strong filtration pair is 1 if f is orientation preserving or 2 if f reverses the orientation.*

(b) *The pair (N_1, L_1) has the properties (1), (2), and (3) of strong filtration pairs with L_1 being a disc with a hole, $\partial_{N_1}(L_1) \simeq S^1$ and $N_1 \subset \mathrm{int}(\widehat{f}_s^{\,k}(N_1))$.*

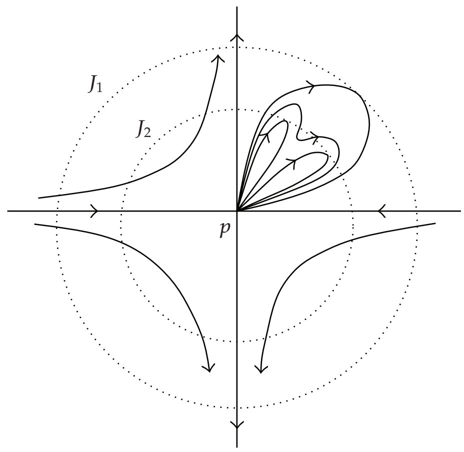


Figure 1

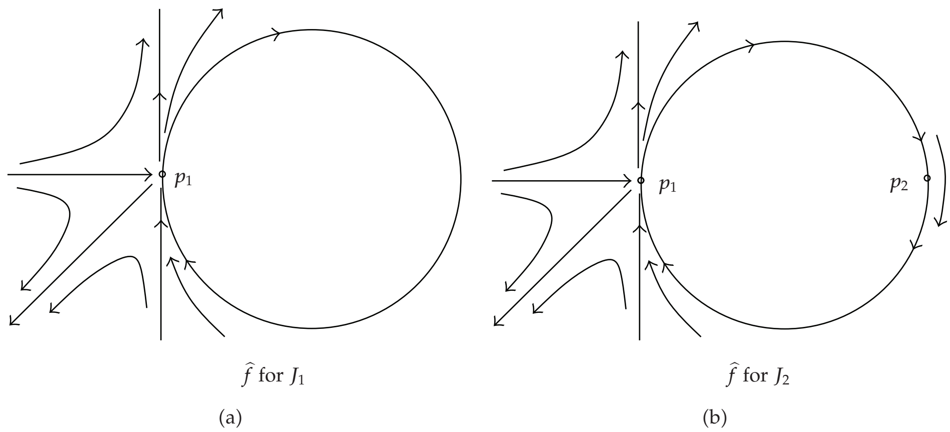


Figure 2

Proof. Given a fixed point p_1 of $\widehat{f_s}^k|_{\partial(D)}$, let us see that there exists the pair (N_1, L_1) for $\widehat{f_s}^k$ in S^2 with $p_1 \in \mathrm{Inv}(\mathrm{cl}(N_1 \setminus L_1), \widehat{f_s}^k)$.

Take a small enough arc $[a,b] \subset \partial(D)$ with $p_1 \in (a,b)$ and such that $\mathrm{Inv}([a,b], \hat{f}^k|_{\partial(D)}) = p_1$. The set $[a,b]$ is an isolating block for $\hat{f}^k|_{\partial(D)}$. Let us consider a small enough disc M in D with $M \cap \partial(D) = [a,b]$ and $\mathrm{Fix}(\hat{f}^k|_M) = \{p_1\}$. Since the space of components of $\mathrm{Inv}(M, \hat{f}^k)$ is a zero-dimensional compactum, it is easy to construct a disc $M_1 \subset M$ such that $[a,b] \subset M_1$ and $\mathrm{Inv}(M, \hat{f}^k) \cap \partial_D(M_1) = \emptyset$. If we choose the disc $N \subset S^2$ obtained by joining M_1 with its reflected disc on $\partial(D)$, M_2, we have that N is an isolating neighborhood for $\widehat{f_s}^k$.

It is not difficult to construct a disc $N_1 \subset \mathrm{int}(N)$, N_1 symmetric with respect to $\partial(D)$, and isolating block for $\widehat{f_s}^k$ (see [12, 26]), with $\partial(N_1) \cap \mathrm{Inv}(N, \widehat{f_s}^k) = \emptyset$ and $p_1 = \mathrm{Fix}(\widehat{f_s}^k|_{N_1})$.

If there is not a disc $B \subset N_1$ such that $p_1 \in \mathrm{int}(B)$ and $B \subset \mathrm{int}(\widehat{f_s}^k(B))$, then there exists a strong filtration pair (N_1, L_1) for $\widehat{f_s}^k$ with L_1 being a finite (perhaps empty) union of disjoint discs (see [4, 12]). By the symmetry property with respect to $\partial(D)$ of $\widehat{f_s}^k$, it is immediate that the period of the generalized filtration pair is 1 if $\widehat{f_s}^k$ is orientation preserving and 2 if $\widehat{f_s}^k$ is orientation reversing (see [4]). Therefore, we are in the conditions of (a).

On the other hand, it there exists the above disc B, we obtain in an easy way the pair (N_1, L_1) of the case (b). □

Definition 2.11. We are interested, for each fixed point p_i of $\widehat{f_s}^k|_{\partial(D)}$, in the pairs $(N_i \cap D,\ L_i \cap D) = (N_i', L_i')$ which we call *strong filtration pairs adapted to* D for p_i. Let us observe that the pair (N_i', L_i') has the properties of the strong filtration pairs for $\widehat{f}^k : D \to D$ at each fixed point $p_i \in \partial(D)$. We will suppose without loss of generality that each arc $\gamma_i = \partial_D(N_i')$ corresponds in J to an arc with two end points in K_p.

There are three possible cases.

(i) If $L_i = \emptyset$, then $\widehat{f}^k(N_i') \subset \operatorname{int}_D(N_i')$ and we say that N_i' is an *attracting petal* associated to $\widehat{f}^k$ at p_i.

(ii) If $\partial_{N_i}(L_i) \simeq S^1$, then $N_i' \subset \operatorname{int}_D(\widehat{f}^k(N_i'))$ and we say that N_i' is a *repelling petal* associated to $\widehat{f}^k$ at p_i.

(iii) If (N_i, L_i) is a strong filtration pair with $L_i \neq \emptyset$, given the sets of stable and unstable branches $\{S_j\}$ and $\{U_j\}$ of (N_i, L_i) associated to $\widehat{f_s}^k$ at p_i (see Theorem 2.4), we select the subsets of branches $\{S_m\}$ and $\{U_m\}$ which are contained in $(N_i' \setminus \partial(D)) \cup \{p_i\}$. We call $\{S_m\}$ and $\{U_m\}$ *stable* and *unstable branches* of (N_i', L_i') associated to $\widehat{f}^k$ at p_i.

Remark 2.12. If $\operatorname{Per}(\widehat{f}|_{\partial(D)})$ is not a finite set of points (we supposed before), we can select a finite and disjoint union $I = I_1 \cup \cdots \cup I_n$, of closed arcs of $\partial(D)$, with $\widehat{f}(I) = I$, such that $\operatorname{Per}(\widehat{f}|_{\partial(D)}) \subset I$ and $\mathcal{P}(K_p \cap \partial(J)) \cap I = \emptyset$. Let us identify each component of I to a point. We obtain a new disc which we call D again. If $\widehat{f} : D \to D$ is the new induced homeomorphism, we have that $\operatorname{Per}(\widehat{f}|_{\partial(D)})$ is a finite set and the construction of the strong filtration pairs adapted to D is also valid (see Figure 3). It is obvious that this construction depends on the choice of the set I.

Example 2.13. Let us consider the dynamical system of Figure 4. We obtain a homeomorphism f of $\mathbb{R}^2$ with p being a non-accumulated and indifferent fixed point and $\operatorname{Inv}(\operatorname{cl}(J), f) = K_p$ an infinite family of petals which contain p in their boundary.

The dynamic of the map $\widehat{f}$ in D is given in Figure 5(a). We have an infinite family of fixed prime ends (fixed points for $\widehat{f}$ in $\partial(D)$). If we consider the two invariant arcs for $\widehat{f}$, I_1 and I_2, of Figure 5(a) and make an identification of them to points p_1 and p_2, we obtain a new homeomorphism (which we call in the same way) $\widehat{f} : D \to D$. This homeomorphism has only two fixed points in $\partial(D)$ and we are in case (a); see Figure 5(b). The new map $\widehat{f}$ has a repelling point in p_2 and an unstable branch in p_1. Let us observe that the choice of the invariant intervals which contain the fixed prime ends, $I = I_1 \cup I_2$, is not unique. We can select I with an arbitrary family of intervals of this type which gives us a different dynamic for $\widehat{f}$ and a different set of fixed points in $\partial(D)$ for the identification map.

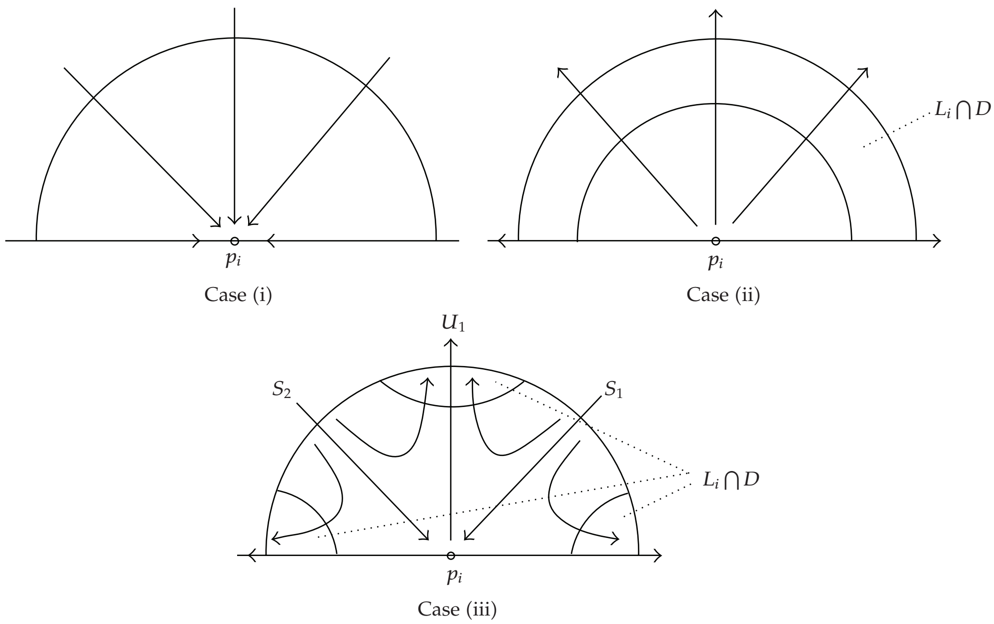


Figure 3

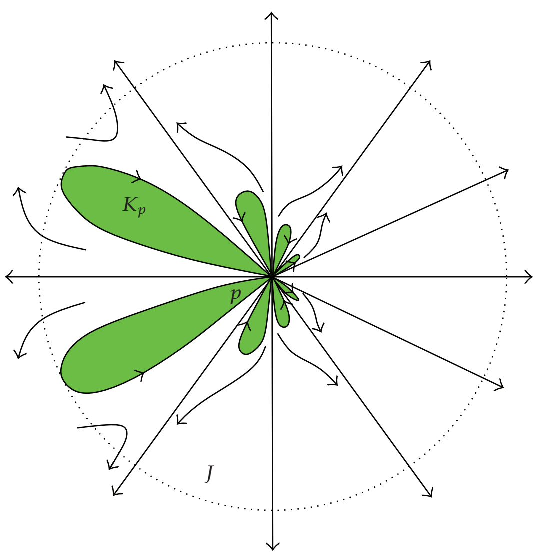
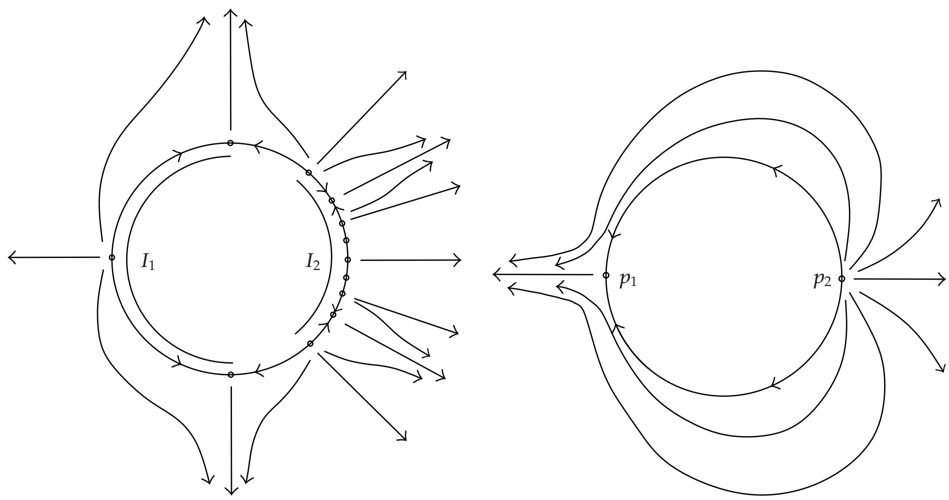


Figure 4

Definition 2.14. Given a Jordan domain J, a *set of strong filtration pairs adapted to* D is a finite collection of pairs $\{(N_i \cap D,\ L_i \cap D)\}_i$ associated to the family $\{p_i\}_i$ of fixed points of $\widehat{f}_s^{\,k}|_{\partial(D)}$.

Let us observe that this set depends on the choice of J and, if $\operatorname{Per}(\widehat{f}|_{\partial(D)})$ is not finite, on the choice of the set I such that, after an identification, it transforms $\operatorname{Per}(\widehat{f}|_{\partial(D)})$ in a finite set .



(a) (b)

Figure 5

2.5. The Statement of the Principal Results

To conclude this section, we summarize below the main results of this article.

Let $f : U \to W$ be a local homeomorphism with $U, W \subset \mathbb{R}^2$ open subsets and let p be a non-accumulated, indifferent fixed point. If p is stable, that is, if there exists a basis of neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ of p such that $f(U_n) \subset U_n$ for all $n \in \mathbb{N}$, we obtain $i_{\mathbb{R}^2}(f^k, p) = 1$ for all $k \in \mathbb{N}$ (see [9, 25]).

We are interested in the relation between the fixed point index of the iterations of f at p and the local dynamics at p, with p being a nonstable fixed point.

Main Theorem 1 (Poincaré formula: Orientation preserving case). *Let $f : U \to W$ be an orientation preserving local homeomorphism with p being an unstable, non-accumulated, and indifferent fixed point. Let us select a Jordan domain J such that $p \in J \subset \mathrm{cl}(J) \subset U$ with $K_p \cap \partial(J) \neq \emptyset$, and let $\{(N_i \cap D,\ L_i \cap D)\}_i$ be a set of strong filtration pairs adapted to D, the Carathéodory's compactification of $S^2 \setminus K_p$. Then there exist $r \in \mathbb{N}$ and $r_p, u_p, s_p, a_p \in r\mathbb{N}$ such that*

$$
\begin{aligned}
i_{\mathbb{R}^2}\left(f^k, p\right) &= \begin{cases} 1 & \text{if } k \notin r\mathbb{N}, \\ 1 - u_p + r_p = 1 - s_p + a_p & \text{if } k \in r\mathbb{N}, \end{cases} \\
&= \begin{cases} 1 & \text{if } k \notin r\mathbb{N}, \\ 1 + \dfrac{1}{2}\left(\left(r_p + a_p\right) - \left(u_p + s_p\right)\right) & \text{if } k \in r\mathbb{N}. \end{cases}
\end{aligned} \tag{2.6}
$$

We have the following dynamical interpretation: there are u_p (s_p) generalized unstable (stable) branches and r_p (a_p) generalized repelling (attracting) petals for f^r at p (see Definition 2.6). They are negatively (positively) invariant for f^r and $f^{-1}(f)$ acts as a permutation on them. Let us observe that the numbers $\{u_p, r_p, s_p, a_p\}$ depend on J and the set of strong filtration pairs but their differences depend only on the germ of f.

Remark 2.15. The last result gives us, as a corollary, a theorem due to Le Calvez (see [5]) which says that if p is a non-accumulated, indifferent fixed point, there exist $r \geq 1$ and $q \in \mathbb{Z}$ such that

$$i_{\mathbb{R}^2}\left(f^k, p\right) = \begin{cases} 1 & \text{if } k \notin r\mathbb{N}, \\ q & \text{if } k \in r\mathbb{N}. \end{cases} \tag{2.7}$$

On the other hand, if f preserves or contracts (expands) areas, then $r_p = 0$ ($a_p = 0$) and we obtain a corollary which improves a result of Simon (see [27]) about the existence of a bound for the fixed point index of the area and orientation preserving homeomorphisms at an isolated fixed point. More precisely, if f preserves or contracts areas then $i_{\mathbb{R}^2}(f, p) \leq 1$.

From the above considerations, given an orientation preserving homeomorphism $h : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ which preserves a measure supported in the open sets, such that $\mathrm{Fix}(h) = \mathrm{Per}(h) = \{0\}$ and $i_{\mathbb{R}^2}(h^k, 0) = 1$ for every $k \in \mathbb{Z}$, it is natural to ask if 0 must be a stable (in the past or in the future) fixed point. The famous example of Anosov and Katok, [28], is a counterexample to this problem. They produced a diffeomorphism of the disc which preserves natural measures and it is ergodic. This map is constructed inductively as a limit of an appropriate sequence of diffeomorphisms. In the next section (see Example 3.3), we will exhibit an explicit, very simple and geometric example of an orientation and area preserving homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{Fix}(h) = \mathrm{Per}(h) = \{0\}$, 0 is stable neither for h nor for h^{-1}, and the fixed point indices $i_{\mathbb{R}^2}(h^k, 0) = 1$ for every $k \in \mathbb{Z}$. Moreover, there will not be h-invariant subsets of positive finite measure.

For the orientation reversing case, we prove the following theorem.

Main Theorem 2 (Poincaré formula: Orientation reversing case). *Let $f : U \to W$ be an orientation reversing local homeomorphism with p being an unstable, non-accumulated, indifferent fixed point. Let us select a Jordan domain J such that $p \in J \subset \mathrm{cl}(J) \subset U$, with $K_p \cap \partial(J) \neq \emptyset$, and let $\{(N_i \cap D,\ L_i \cap D)\}_i$ be a set of strong filtration pairs adapted to D, the Carathéodory's compactification of $S^2 \setminus K_p$. Then there exist $u_p, u'_p, r_p, r'_p \in \mathbb{N}$ with $u'_p \leq u_p$, $r'_p \leq r_p$, $u'_p + r'_p \leq 2$ such that*

$$i_{\mathbb{R}^2}\left(f^k, p\right) = \begin{cases} 1 - u_p + r_p & \textit{if } k \textit{ even}, \\ 1 - u'_p - r'_p & \textit{if } k \textit{ odd}, \end{cases} \tag{2.8}$$

and with the following dynamical meaning: there are u_p generalized unstable branches for f^2 at p with $u'_p \leq 2$of them negatively invariant for f (f^{-1} sends each of the u'_p generalized unstable branches to a subset of itself). In the same way, there are r_p generalized repelling petals for f^2 at p and $r'_p \leq 2$ of them are negatively invariant for f.

As in the orientation preserving case, we have similar formulas involving generalized stable branches and generalized attracting petals.

Remark 2.16. As a corollary, $i_{\mathbb{R}^2}(f,p) \in \{-1,0,1\}$ for f an orientation reversing local homeomorphism and p a non-accumulated fixed point. This is Bonino's theorem (see [29]) when p is non-accumulated.

Remark 2.17. The Main Theorem for orientation reversing homeomorphisms says that $i_{\mathbb{R}^2}(f^{2n},p)$ is constant. Then it solves Problem 7.3.9. of [21].

Theorem 2.18. *Let $f : U \to W$ be an arbitrary local homeomorphism with $\mathrm{Fix}(f) = \{p\}$ being an indifferent fixed point, such that $i_{\mathbb{R}^2}(f^r,p) = 1-m < 1$ for some $r \in \mathbb{N}$ ($r = 2$ if f reverses orientation). Then there exist m unstable (stable) branches, in the classical sense, $\{U_i\}(\{S_i\})$ for f^r at p such that:*

(1) *f^{-1} and f act as permutations in $\{U_i\}$ and $\{S_i\}$, respectively;*

(2) *$\lim_{n\to\infty} f^{-n}(x) = p$ for every $x \in U_i$, $\lim_{n\to\infty} f^n(y) = p$ for every $y \in S_i$;*

(3) *there exists a closed disc $D_p \subset J$, with $p \in \mathrm{int}(D_p)$, $\bigcup_{i=1}^m (U_i \cup S_i) \subset D_p$, in such a way that the intersection of the stable and unstable branches with $\partial(D_p)$ alternates in $\partial(D_p)$.*

Each generalized repelling (attracting) petal contains p in its boundary. As a corollary of the Main Theorems for both orientation preserving and orientation reversing homeomorphisms, we will obtain the following result (see [11] for the orientation preserving case).

Theorem 2.19 (Petal's theorem). *Let $f : U \to W$ be an arbitrary local homeomorphism with p being a non-accumulated and isolated fixed point such that $i_{\mathbb{R}^2}(f^r,p) = 1 + m > 1$ for some $r \in \mathbb{N}$. Then there exist m generalized repelling petals $\{R_i\}$ and m generalized attracting petals $\{A_i\}$ for f^r at p such that:*

(1) *$\mathrm{int}(A_i) \cap \mathrm{int}(A_j) = \mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for all $i \neq j$, and $\mathrm{int}(A_i) \cap \mathrm{int}(R_j) = \emptyset$ for all i, j;*

(2) *the map f (f^{-1}) acts as a permutation in $\{A_i\}$ $(\{R_i\})$;*

(3) *$\lim_{n\to\infty} f^{-n}(x) = p$ for every $x \in R_i$, $\lim_{n\to\infty} f^n(y) = p$ for every $y \in A_i$;*

(4) *the sequences $\{f^{-nr}(R_i)\}_{n\in\mathbb{N}}$ and $\{f^{nr}(A_i)\}_{n\in\mathbb{N}}$ determine ends containing p and $\bigcap_{n\in\mathbb{N}} f^{-nr}(R_i)$ and $\bigcap_{n\in\mathbb{N}} f^{nr}(A_i)$ are f^r-invariant continua containing p;*

(5) *there is a Jordan curve γ around p such that γ intersects alternatively the sets $\{A_i\}$ and $\{R_i\}$, with $\gamma \cap A_i$ and $\gamma \cap R_i$ being closed arcs.*

Remark 2.20. Using the petal's theorem, one can prove the following consequences that extend a theorem due to Le Calvez (see [5]).

If $f : U \to W$ is a local homeomorphism such that $\mathrm{Fix}(f) = \{p\}$ and $1 \neq i_{\mathbb{R}^2}(f^r,p) > 1-q$ for some $r \in \mathbb{N}$ ($r = 2$ if f reverses orientation), take a disc J such that $p \in \mathrm{int}(J) \subset \mathrm{cl}(J) \subset U$. We have the following two properties.

(a) If there exist q generalized stable branches for f^r at p, then there exists a domain $V_1 \subset U$ such that the domains of the sequence $\{f^n(V_1)\}_{n\in\mathbb{N}}$ are well defined and disjoint.

(b) If there exist q generalized unstable branches for f^r at p, then there exists a domain $V_2 \subset U$ such that the domains of the sequence $\{f^{-n}(V_2)\}_{n\in\mathbb{N}}$ are well defined and disjoint.

As a corollary, if $i_{\mathbb{R}^2}(f^r,p) > 1$ for some $r \in \mathbb{N}$, there exist domains $V_1, V_2 \subset U$ such that the domains of the sequences $\{f^n(V_1)\}_{n\in\mathbb{N}}$ and $\{f^{-n}(V_2)\}_{n\in\mathbb{N}}$ are well defined and disjoint.

The last remark can be applied to the following interesting situation: let M be an oriented compact 2-dimensional manifold with boundary and let $f : U \subset M \to M$ be an orientation preserving homeomorphism. Let $p \in \partial(M) \cap U$ be an isolated fixed point of f. Denote by DM the double of the manifold M and $Df : DM \to DM$ the natural map induced by f.

Then,

(a) if p is a saddle point of $f|_{\partial(M)}$ and $i_{DM}(Df, p) > 0$, then there exist domains $V_1, V_2 \subset U$ such that the domains of the sequences $\{f^n(V_1)\}_{n\in\mathbb{N}}$ and $\{f^{-n}(V_2)\}_{n\in\mathbb{N}}$ are well defined and disjoint;

(b) if p is an attractor of $f|_{\partial(M)}$ and $i_{DM}(Df, p) > -1$, then there exists a domain $V_1 \subset U$ such that the domains of the sequence $\{f^n(V_1)\}_{n\in\mathbb{N}}$ are well defined and disjoint;

(c) if p is a repeller of $f|_{\partial(M)}$ and $i_{DM}(Df, p) > -1$, then there exists a domain $V_2 \subset U$ such that the domains of the sequence $\{f^{-n}(V_2)\}_{n\in\mathbb{N}}$ are well defined and disjoint.

Note that in this particular setting, since p is isolated using Brouwer's lemma on translation arcs, it is not necessary to assume that $i_{DM}(Df, p) \neq 1$.

For orientation and area preserving homeomorphisms in surfaces, we have the following Nielsen type theorem (see [30] for the particular case where M is a disc).

Corollary 2.21. *Let M be an oriented compact 2-dimensional manifold with boundary and let $f : M \to M$ be an area and orientation preserving homeomorphism such that $f|_{\partial(M)}$ has n attracting fixed points and n repelling fixed points. Then f has, at least, $n + \Lambda(f)$ fixed points in* $\operatorname{int}(M)$ *where $\Lambda(f)$ denotes the Lefschetz number of f. As a consequence, if M is the 2-dimensional disc, we have that f has, at least, $n + 1$ fixed points in* $\operatorname{int}(M)$.

Restricting ourselves to orientation reversing homeomorphisms and using the fact that $i_{\mathbb{R}^2}(f, p) \in \{-1, 0, 1\}$, we shall produce a sharp theorem. The proof will be obtained easily by using the previous results.

Theorem 2.22. *Let $f : U \to W$ be an orientation reversing local homeomorphism with p being a non-accumulated, indifferent fixed point, and $i_{\mathbb{R}^2}(f^2, p) \neq 1$. Then there are u_p generalized unstable branches and r_p generalized repelling petals for f^2 at p such that $i_{\mathbb{R}^2}(f^2, p) = 1 - u_p + r_p$ and*

(a) *the generalized unstable (stable) branches and the generalized repelling (attracting) petals are negatively (positively) invariant for f^2;*

(b.1) *if $i_{\mathbb{R}^2}(f^2, p) = 1 + m > 1$, then $r_p \geq m$ and there are, at least, m generalized attracting petals for f^2 at p (m of the generalized attracting petals alternate with m of the generalized repelling petals around p);*

(b.2) *if $i_{\mathbb{R}^2}(f^2, p) = 1 - m < 1$, then $u_p \geq m$ and there are, at least, m generalized stable branches for f^2 at p (m of the generalized stable branches alternate with m of the generalized unstable branches around p);*

(c.1) *if $i_{\mathbb{R}^2}(f, p) = 1$, then there are neither generalized repelling petals nor generalized unstable branches for f^2 at p, negatively invariant for f. On the other hand, there are two generalized attracting petals or two generalized stable branches or a generalized stable branch and a generalized attracting petal for f^2 at p, positively invariant for f.*
The numbers u_p and r_p are even. Therefore, $i_{\mathbb{R}^2}(f^2, p)$ is odd;

(c.2) *if* $i_{\mathbb{R}^2}(f,p)=-1$, *then there are two generalized repelling petals or two generalized unstable branches or a generalized unstable branch and a generalized repelling petal for* f^2 *at* p, *negatively invariant for* f. *On the other hand, there are neither generalized attracting petals nor generalized stable branches for* f^2 *at* p, *positively invariant for* f.
The number u_p+r_p *is even and* $i_{\mathbb{R}^2}(f^2,p)$ *is odd;*

(c.3) *if* $i_{\mathbb{R}^2}(f,p)=0$, *then there are a generalized unstable branch or a generalized repelling petal for* f^2 *at* p *negatively invariant for* f. *On the other hand, there are a generalized stable branch or a generalized attracting petal for* f^2 *at* p, *positively invariant for* f.
The number u_p+r_p *is odd and* $i_{\mathbb{R}^2}(f^2,p)$ *is even.*

Corollary 2.23. *Let* $f: S^2\to S^2$ *be an orientation reversing and area preserving homeomorphism. If* f *has a fixed point, then* $|\operatorname{Per}(f)|=\infty$.

3. Computation of $i_{\mathbb{R}^2}(f^k,p)$

3.1. Orientation Preserving Case

Let $f: U\to W$ be an orientation preserving local homeomorphism with p being a non-accumulated, indifferent fixed point for f. Let J_p be a Jordan domain, with $p\in J_p$ being the unique periodic orbit contained in $\mathrm{cl}(J_p)$, $K_p\cap\partial(J_p)\neq\emptyset$, and such that $p\in\partial(K_p)$. Given $k\in\mathbb{N}$, we can select a small enough Jordan domain $J\subset J_p$ such that $\mathrm{Fix}(\overline{f}^k|_{cl(J)})=\{p\}$ (the map $\overline{f}$ is defined after Remark 2.7) and such that the above continuum K_p is also the connected component of $\mathrm{Inv}(\mathrm{cl}(J),f)$ which contains p. Then

$$i_{S^2}\left(\overline{f}^k,p\right)+i_{S^2}\left(\overline{f}^k,S^2\setminus J\right)=2. \tag{3.1}$$

We have (after identification if necessary) that $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is a finite set. Then $\widehat{f}|_{\partial(D)}$ has q periodic orbits of period r.

If $k\in r\mathbb{N}$, then

$$\mathrm{Fix}\left(\widehat{f}^k|_{\partial(D)}\right)=\{\{p_{11},\dots,p_{1r}\},\dots,\{p_{q1},\dots,p_{qr}\}\}, \tag{3.2}$$

with $\{p_{j1},\dots,p_{jr}\}$ being the periodic orbits of $\widehat{f}|_{\partial(D)}$ for $j=1,\dots,q$.

We have that $\mathrm{Fix}(\overline{f}^k|_{\partial(J)})=\emptyset$ and $\mathrm{Fix}(\widehat{f}^k|_{p(K_p\cap\partial(J))})=\emptyset$.

Let $A=(J\setminus K_p)\cup\partial(D)\subset D$. Then $i_{S^2}(\overline{f}^k,S^2\setminus J)=i_D(\widehat{f}^k,D\setminus A)$ and, since D is contractible,

$$i_D\left(\widehat{f}^k,D\setminus A\right)+r\sum_{j=1}^{q}i_D\left(\widehat{f}^k,p_{j1}\right)=1. \tag{3.3}$$

Then we have the following proposition.

Proposition 3.1. *Under the above setting,*

$$i_{\mathbb{R}^2}\left(f^k,p\right)=i_{S^2}\left(\overline{f}^k,p\right)=2-i_{S^2}\left(\overline{f}^k,S^2\setminus J\right)=2-i_D\left(\widehat{f}^k,D\setminus A\right)=1+r\sum_{j=1}^{q}i_D\left(\widehat{f}^k,p_{j1}\right). \tag{3.4}$$

For each fixed point p_{j1} of $\widehat{f}^k$, we have a strong filtration pair adapted to D, (N_{j1},L_{j1}), with $p_{j1}\in K_{j1}=\mathrm{Inv}(\mathrm{cl}(N_{j1}\setminus L_{j1}),\widehat{f}^k)$ (see Lemma 2.10).

The set L_{j1} is a finite amount of disjoint discs and it is easy to see that $i_D(\widehat{f}^k,p_{j1})=1-q_j$ with q_j being the number of components L^m_{j1} of L_{j1} such that $\widehat{f}^k(\partial_{N_{j1}}(L^m_{j1}))\subset\mathrm{int}_D(L^m_{j1})$ (see [4]). Since $\widehat{f}$ is a homeomorphism, the number q_j is the same for any other p_{jk} with $k\in\{1,\dots,r\}$.

Then, if $k\in r\mathbb{N}$,

$$i_{\mathbb{R}^2}\left(f^k,p\right)=1+r\left(q-\sum_{j=1}^{q}q_j\right). \tag{3.5}$$

If $\widehat{f}|_{\partial(D)}$ has no periodic orbits, it is obvious that $i_{\mathbb{R}^2}(f^k,p)=1$ for all $k\in\mathbb{N}$.

If $k\notin r\mathbb{N}$, then $\mathrm{Fix}(\widehat{f}^k|_{\partial(D)})=\emptyset$ and $i_{\mathbb{R}^2}(f^k,p)=1$.

Therefore, we have proved the following theorem (see [5]).

Theorem 3.2. *If $f:U\to W$ is an orientation preserving local homeomorphism with p being a non-accumulated, indifferent fixed point, then*

(a) *if* $\mathrm{Per}(\widehat{f}|_{\partial(D)})=\emptyset$,

$$i_{\mathbb{R}^2}\left(f^k,p\right)=1\quad\forall k\in\mathbb{N}. \tag{3.6}$$

(b) *if* $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ *is a nonempty finite set, then* $\widehat{f}|_{\partial(D)}$ *has q periodic orbits of period r, and*

$$i_{\mathbb{R}^2}\left(f^k,p\right)=\begin{cases}1 & \text{If } k\notin r\mathbb{N},\\ 1+r\left(q-\displaystyle\sum_{j=1}^{q}q_j\right) & \text{If } k\in r\mathbb{N},\end{cases} \tag{3.7}$$

with $q_j\in\mathbb{N}$ defined as above. Let us recall that we obtain $i_{\mathbb{R}^2}(f^k,p)$ for all $k\in\mathbb{N}$ by observing $\widehat{f}^r$.

As an application of these techniques, we shall give an explicit simple example of an area and orientation preserving homeomorphism $h:\mathbb{R}^2\to\mathbb{R}^2$ such that $\mathrm{Fix}(h)=\mathrm{Per}(h)=\{0\}$, 0 is neither stable for h nor for h^{-1} and the fixed point indices $i_{\mathbb{R}^2}(h^k,0)=1$ for every $k\in\mathbb{Z}$. Moreover, there are no h-invariant subsets of positive finite Lebesgue measure.

Example 3.3. Let $g_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation with center of the origin and angle $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Let S^1 be the unit circle and $x_0 \in S^1$. For every point of the orbit of x_0, $\{(g_\alpha)^n(x_0) : n \in \mathbb{Z}\}$, following the classical construction of Denjoy, we paste an interval I_n in each point $(g_\alpha)^n(x_0)$ for every $n \in \mathbb{Z}$ such that:

(a) $l(I_{m+1}) < l(I_m)$ and $l(I_m) = l(I_{-m})$ for every $m \in \mathbb{N}$ and $\sum_{n\in\mathbb{Z}} l(I_n) = 2\pi < \infty$ where $l(I)$ denotes the length of the interval I;

(b) $\lim_{n\to\infty} l(I_{n+1})/l(I_n) = 1$.

Extending radially to the whole plane the corresponding map of Denjoy, we obtain a homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$.

Let $Q_n = \{a \in \mathbb{R}^2 : \text{there are } \lambda \geq 0 \text{ and } b_n \in I_n \text{ such that } a = \lambda b_n\}$.

The homeomorphism h, we are looking for, will satisfy that $h_{|\mathbb{R}^2\setminus\bigcup_{n\in\mathbb{Z}} Q_n} = g$.

Let us define h in $\bigcup_{n\in\mathbb{Z}} Q_n$. Consider $n \in \mathbb{Z}$ and take an isometric copy of Q_n, denoted by $\Theta_n \subset \mathbb{R} \times [0,\infty)$ such that Θ_n is obtained by rotating Q_n in such a way that the line $x = 0$ divides Θ_n into two symmetric sectors, Θ_n^+ and Θ_n^-. We shall denote by $2\alpha_n \in [0,\pi)$ the interior angle determined by Θ_m.

It is clear that $\Theta_{m+1} \subset \Theta_m$ and $\Theta_{-m} \subset \Theta_{-m+1}$ for every $m \in \mathbb{N}$. For each $n \in \mathbb{Z}$, let us denote by $j_n : Q_n \to \Theta_n$ the obvious isometry.

To define the required homeomorphism h, we will consider, for every $n \in \mathbb{Z}$, area and orientation preserving linear homeomorphisms $f_{n,n+1} : \Theta_n \to \Theta_{n+1}$.

Let $f_{n,n+1} : \Theta_n^+ \to \Theta_{n+1}^+$ be given by the formula

$$f_{n,n+1}(x,y) = \left(\frac{\sin\alpha_{n+1}}{\sin\alpha_n}x, y\frac{\sin\alpha_n}{\sin\alpha_{n+1}} + x\left(\frac{\cos\alpha_{n+1}}{\sin\alpha_n} - \frac{\cos\alpha_n}{\sin\alpha_{n+1}}\right)\right). \tag{3.8}$$

Since $f_{n,n+1}(0,y) = (0, y(\sin\alpha_n/\sin\alpha_{n+1}))$, we can extend $f_{n,n+1} : \Theta_n^- \to \Theta_{n+1}^-$ by the obvious symmetry.

On the other hand, $f_{n,n+1}(r\sin\alpha_n, r\cos\alpha_n) = (r\sin\alpha_{n+1}, r\cos\alpha_{n+1})$. Then, it is easy to check that $f_{n,n+1}$ is an area and orientation preserving injective map such that $f_{n,n+1}(\Theta_n) = \Theta_{n+1}$.

Moreover, $f_{n+1,n+2} \circ f_{n,n+1} = f_{n,n+2}$ and $1 \leq \|f_{n,n+1}(z)\|/\|z\| \leq \sin\alpha_n/\sin\alpha_{n+1}$ ($1 \geq \|f_{n,n+1}(z)\|/\|z\| \geq \sin\alpha_n/\sin\alpha_{n+1}$) for every $z \in \Theta_n$ and $n \geq 0$ ($n \leq 0$).

Now we are in a position to give an explicit definition of the homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

(i) $h_{|\mathbb{R}^2\setminus\bigcup_{n\in\mathbb{Z}} Q_n} = g$,

(ii) $h(z) = ((j_{n+1})^{-1} \circ f_{n,n+1} \circ j_n)(z) \in Q_{n+1}$ for $z \in Q_n$.

By the construction, it is obvious that h is a bijective and area preserving map such that $\mathrm{Fix}(h) = \mathrm{Per}(h) = \{0\}$.

(1) h is continuous in 0. Indeed, for any $\epsilon > 0$ take $\delta > 0$ such that $\epsilon = \delta\max\{\sin\alpha_m/\sin\alpha_{m+1} : m \in \mathbb{N}\}$. Then, if $B(0,s)$ denotes the open ball centered in 0 and radius s, we have that $h(B(0,\delta)) \subset B(0,\epsilon)$ and $h^{-1}(B(0,\delta)) \subset B(0,\epsilon)$.

(2) h is continuous in any $z \in \mathbb{R}^2 \setminus \{0\}$. In fact, we only have to pay attention to $z \in \mathbb{R}^2 \setminus \bigcup_{n\in\mathbb{Z}} \mathrm{int}(Q_n)$. For such points, we use polar coordinates $z = (r,\theta)$ and $g(r,\theta) = (r, g_2(\theta))$. Since $f_{n,n+1}(r\sin\alpha_n, r\cos\alpha_n) = (r\sin\alpha_{n+1}, r\cos\alpha_{n+1})$ for every $n \in \mathbb{Z}$, we have that $h(z) = h(r,\theta) = (r, g_2(\theta))$.

Consider any open neighborhood $V = (r-\epsilon, r+\epsilon) \times (g_2(\theta)-\epsilon, g_2(\theta)+\epsilon)$ of $h(z)$ and take any open neighborhood U of z such that $g(U) \subset V$ and $U \cap Q_n \neq \emptyset$ just for $|n|$ such that $\sin\alpha_n / \sin\alpha_{n+1}$ is close enough to 1. Then, if $z' \in U$ then $\|h(z')\|/\|z'\|$ is close enough to 1 and $\|h(z')\| \in (r-\epsilon, r+\epsilon)$.

(3) 0 is neither stable for h nor for h^{-1}. Indeed, take any $z \in \mathrm{int}(Q_m)$ with $m \in \mathbb{N}$ such that $j_m(z) \in \{(x,y) : x = 0,\ y \geq 0\} \subset \Theta_m$. Then, $\|f_{m,m+1}(j_m(z))\|/\|j_m(z)\| = \sin\alpha_m / \sin\alpha_{m+1}$.

Now, consider any $k \in \mathbb{N}$. There exists $k_m \in \mathbb{N}$ such that $j_{m+k_m}((h^{k_m}(z)) \in \{(x,y) : x = 0, y \geq 0\} \subset \Theta_{m+k_m}$, and $\sin\alpha_m / \sin\alpha_{m+k_m} > 2^k/\|z\|$. Then, $\|h^{k_m}(z)\| > 2^k$.

In the same way, we have that 0 is not stable for h^{-1}. The same arguments allow to prove that neither the positive semiorbit nor the negative semiorbit of $z \in \mathrm{int}(Q_n)$ are bounded. On the other hand, any h-invariant subset has null or infinite Lebesgue measure.

(4) For any closed disc, D, centered in 0, we have that $\mathrm{Inv}(D,h) = (\mathbb{R}^2 \setminus \bigcup_{n\in\mathbb{Z}} \mathrm{int}(Q_n)) \cap D$. Then, $\mathrm{Inv}(D,h)$ has no h-periodic prime ends and consequently, $i_{\mathbb{R}^2}(h^k,0) = i_{\mathbb{R}^2}(h^{-k},0) = 1$ for every k $\in \mathbb{N}$.

3.2. Orientation Reversing Case

Let $f : U \to W$ be an orientation reversing local homeomorphism with p being a non-accumulated, indifferent fixed point and let J_p and K_p be as in the orientation preserving case. Note that from a theorem of Kuperberg, see [31], $p \in \partial(K_p)$.

Given $k \in \mathbb{N}$, we can select a small enough Jordan domain $J \subset J_p$ such that $\mathrm{Fix}(\overline{f}^k|_{\mathrm{cl}(J)}) = \{p\}$ and such that K_p is the connected component of $\mathrm{Inv}(\mathrm{cl}(J), f)$ which contains p.

Since $\overline{f} : S^2 \to S^2$ is orientation reversing,

$$i_{S^2}\left(\overline{f}^k, p\right) + i_{S^2}\left(\overline{f}^k, S^2 \setminus J\right) = \begin{cases} 0 & \text{if } k \text{ odd}, \\ 2 & \text{if } k \text{ even}, \end{cases} \tag{3.9}$$

and, since $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is, after identification if necessary, a finite set, then $\widehat{f}|_{\partial(D)}$ has q periodic orbits of period 2 and two fixed points $\{p_0, p_1\}$.

Let us divide the computation of $i_{\mathbb{R}^2}(f^k, p)$ into two cases: k odd and k even.

Case 1. Let us suppose that k is odd.

Since $\overline{f}^k$ is orientation reversing,

$$i_{S^2}\left(\overline{f}^k, p\right) + i_{S^2}\left(\overline{f}^k, S^2 \setminus J\right) = 0. \tag{3.10}$$

On the other hand, since $\mathrm{Fix}(\widehat{f}^k|_{\partial(D)}) = \{p_0, p_1\}$,

$$i_D\left(\widehat{f}^k, D \setminus A\right) + i_D\left(\widehat{f}^k, p_0\right) + i_D\left(\widehat{f}^k, p_1\right) = 1. \tag{3.11}$$

Then we have the following proposition.

Proposition 3.4. *Under the above setting,*

$$i_{\mathbb{R}^2}\left(f^k,p\right)=i_D\left(\widehat{f}^k,p_0\right)+i_D\left(\widehat{f}^k,p_1\right)-1. \tag{3.12}$$

Let us compute $i_D(\widehat{f}^k,p_0)$ for k odd.

There exists a strong filtration pair adapted to D, (N_0,L_0), associated to p_0.

If q^0 is the number of components $\{L_0^j\}$ of L_0 such that $\widehat{f}^k(\partial_{N_0}(L_0^j))\subset \operatorname{int}_D(L_0^j)$, since $\widehat{f}^k$ is orientation reversing, we have that $q^0\in\{0,1\}$ (see [4]).

We obtain that

$$i_D\left(\widehat{f}^k,p_0\right)=1-q^0. \tag{3.13}$$

In the same way, we have

$$i_D\left(\widehat{f}^k,p_1\right)=1-q^1 \tag{3.14}$$

with $q^1\in\{0,1\}$ defined as q^0.

Then, for k being odd and f being an orientation reversing local homeomorphism,

$$i_{\mathbb{R}^2}\left(f^k,p\right)=\left(1-q^0\right)+\left(1-q^1\right)-1=1-q^0-q^1\in\{-1,0,1\} \tag{3.15}$$

and Case 1 is finished.

Case 2. Let us suppose that k is even.

Then $\widehat{f}^k$ is an orientation preserving homeomorphism with $\operatorname{Fix}(\widehat{f}^k|_{\partial(D)})=\{p_0,p_1,\{p_{11},p_{12}\},\dots,\{p_{q1},p_{q2}\}\}$. Following the steps of the orientation preserving case,

$$i_D\left(\widehat{f}^k,p_{j1}\right)=1-q_j\quad\text{for } j\in\{1,\dots,q\},\qquad i_D\left(\widehat{f}^k,p_i\right)=1-q^i\quad\text{for } i\in\{0,1\}. \tag{3.16}$$

Then

$$\begin{aligned} i_{\mathbb{R}^2}\left(f^k,p\right)&=2-i_D\left(\widehat{f}^k,D\setminus A\right)=2-\left[1-2\sum_{j=1}^{q}\left(1-q_j\right)-\left(1-q^0\right)-\left(1-q^1\right)\right]\\ &=3+2q-q^0-q^1-2\sum_{j=1}^{q}q_j. \end{aligned} \tag{3.17}$$

Let us observe that in this case (k even) we have not $q^i\in\{0,1\}$.

Therefore, we have the following theorem.

Theorem 3.5. *Let $f : U \to W$ be an orientation reversing local homeomorphism with p being a non-accumulated, indifferent fixed point such that $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is a finite set (two fixed points and q periodic orbits of period 2). Then*

$$i_{\mathbb{R}^2}\left(f^k,p\right)=\begin{cases}1-q^0-q^1\in\{-1,0,1\} & \text{if } k \text{ odd},\\ 3+2q-q^0-q^1-2\sum_{j=1}^{q}q_j & \text{if } k \text{ even},\end{cases}\tag{3.18}$$

with q_j, q^0, and q^1 defined as above. Let us recall that we obtain $i_{\mathbb{R}^2}(f^k,p)$, for all $k\in\mathbb{N}$, by observing $\widehat{f}$ and $\widehat{f}^2$.

4. Dynamical Meaning of $i_{\mathbb{R}^2}(f^k,p)$

Proof of Main Theorem 1 (Orientation preserving case). Let $f : U \to W$ be an orientation preserving local homeomorphism with p being a non-accumulated, indifferent fixed point for f in the conditions of the orientation preserving case of Section 2. Then $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is a finite set of q periodic orbits of period r. Let $p_{j1}\in\mathrm{Fix}(\widehat{f}^k|_{\partial(D)})$ with $k\in r\mathbb{N}$. We will relate $i_D(\widehat{f}^k,p_{j1})$ with the dynamical behavior of $\widehat{f}^k$ in the proximity of p_{j1}. This fact permits us to establish a new relation between $i_{\mathbb{R}^2}(f^k,p)$ and the dynamical meaning of f at a neighborhood of p.

Let (N_j,L_j) be a pair, as in Lemma 2.10, for $\widehat{f_s}^k$ at p_{j1}. If (N_j,L_j) is a strong filtration pair, the period of (N_j,L_j) is 1. We have then a family (perhaps empty) $\{U_1,\dots,U_s\}$ of unstable branches of (N_j,L_j) associated to $\widehat{f_s}^k$ at p_{j1} with $s=1-i_{S^2}(\widehat{f_s}^k,p_{j1})$.

If (N_{j1},L_{j1}) is a strong filtration pair adapted to D for p_{j1}, we call u_j the number of unstable branches of (N_{j1},L_{j1}) associated to $\widehat{f}^k$ at p_{j1}. If we select any other p_{jk} with $k\in\{1,\dots,r\}$, since $\widehat{f}$ is a homeomorphism, we obtain the same numbers u_j associated to p_{jk}. Let us study the relations between the numbers u_j and q_j.

Case 1. If p_{j1} is an attractor for $\widehat{f}^k|_{\partial(D)}$, then $L_{j1}\cap\partial(D)=\emptyset$ and $q_j=u_j$.

If $q_j=0$, then N_{j1} is an attracting petal associated to $\widehat{f}^k$ at p_{j1}, that is, $\widehat{f}^k(N_{j1})\subset\mathrm{int}_D(N_{j1})$.

Case 2. Let us suppose that p_{j1} is a repeller for $\widehat{f}^k|_{\partial(D)}$.

Then $q_j\geq 1$. We have two subcases.

Sub case 2.1. If $q_j=1$, N_{j1} is a repelling petal associated to $\widehat{f}^k$ at p_{j1}, that is, $N_{j1}\subset\mathrm{int}_D(\widehat{f}^k(N_{j1}))$, we have $u_j=0$.

Sub case 2.2. If $q_j>1$, we obtain $u_j=q_j-2$.

Case 3. If p_{j1} is a saddle point for $\widehat{f}^k|_{\partial(D)}$, then $q_j=u_j+1$.

Let us denote

$$
\begin{aligned}
A &= \{j \in \{1,\ldots,q\} : p_{j1} \text{ is in Case } 1\},\\
R_1 &= \{j \in \{1,\ldots,q\} : p_{j1} \text{ is in Case } 2.1\},\\
R_2 &= \{j \in \{1,\ldots,q\} : p_{j1} \text{ is in Case } 2.2\},\\
S &= \{j \in \{1,\ldots,q\} : p_{j1} \text{ is in Case } 3\}.
\end{aligned}
\tag{4.1}
$$

Since $\widehat{f}^k|_{\partial(D)}$ is orientation preserving, the sets A and $R = R_1 \cup R_2$ have the same number of elements. There are $r|A| = r|R|$ attractors (and repellers) and $r|S| = r(q - 2|A|)$ saddle points for $\widehat{f}^k|_{\partial(D)}$.

If we come back to the computation of $i_{\mathbb{R}^2}(f^k, p)$, then

$$
\begin{aligned}
i_{\mathbb{R}^2}\left(f^k, p\right) &= 1 + r\left(q - \sum_{j=1}^{q} q_j\right)\\
&= 1 + r\left[q - \sum_{j\in A} u_j - |R_1| - \sum_{j\in R_2}(u_j + 2) - \sum_{j\in S}(u_j + 1)\right]\\
&= 1 - r\sum_{j\in A\cup R_2\cup S} u_j + r|R_1|.
\end{aligned}
\tag{4.2}
$$

If we associate the number $u_{jm} = u_j$ to each point p_{jm}, then

$$
i_{\mathbb{R}^2}\left(f^k, p\right) = 1 - \sum_{\substack{j\in\{1,\ldots,q\}\\ m\in\{1,\ldots,r\}}} u_{jm} + r|R_1|.
\tag{4.3}
$$

The number u_{jm} is the number of unstable branches of (N_{jm}, L_{jm}) associated to $\widehat{f}^r$ at p_{jm}.

Let U_{jm} be an unstable (stable) branch of (N_{jm}, L_{jm}) associated to $\widehat{f}^r$ at p_{jm}. It is easy to see that the continuum $\mathrm{cl}_{\mathbb{R}^2}(U_{jm} \setminus p_{jm}) \subset U$ is a generalized unstable (stable) branch for f^r at p.

We can select the repelling petals N_{jm} in such a way that the arcs $\partial_D(N_{jm})$ are crosscuts of $\partial(K_p)$, that is, their end points are exactly two points in $\partial(K_p)$ (the set of elements of $\partial(D)$ which are accessible by arcs on $U \setminus K_p$ is dense in $\partial(D)$). Then, the continuum $\mathrm{cl}_{\mathbb{R}^2}(\mathrm{int}(N_{jm}))$ is a generalized repelling petal for f^r at p.

The generalized attracting petals for f^r at p are constructed in an analogous way.

We define $u_p = \sum u_{jm}$ to be the number of generalized unstable branches for f^r at p and $r_p = r|R_1|$ to be the number of generalized repelling petals for f^r at p.

We have proved that if f is an orientation preserving local homeomorphism, then $r_p, u_p \in r\mathbb{N}$ and

$$i_{\mathbb{R}^2}\left(f^k, p\right) = \begin{cases} 1 & \text{if } k \notin r\mathbb{N}, \\ 1 - u_p + r_p & \text{if } k \in r\mathbb{N}. \end{cases} \tag{4.4}$$

Let us recall that $i_{\mathbb{R}^2}(f^k, p)$ is computed by observing $\widehat{f}^r$. The numbers u_p and r_p depend on the choice of the Jordan domain J and of the set of strong filtration pairs adapted to D (if $\mathrm{Per}(\widehat{f}_{\partial(D)})$ is not a finite set). However, the difference $r_p - u_p$ does not change. □

Remark 4.1. Note that the above techniques allow us to compute $i_{\mathbb{R}^2}(f, p)$ even if p is an accumulated isolated fixed point. Using Lemma 2.10, there are no problems to construct strong filtration pairs adapted to each fixed prime end. Since it is well known that for an accumulated isolated fixed point $p \, i_{\mathbb{R}^2}(f, p) = 1$, we have that the number of generalized unstable (stable) branches and generalized repelling (attracting) petals that are negatively (positively) invariant for f coincide.

Corollary 4.2. *Let $p_{j1} \in \mathrm{Fix}(\widehat{f}^k|_{\partial(D)})$ with $k \in r\mathbb{N}$ and let (N'_{j1}, L'_{j1}) be a pair as in Lemma 2.10 and $(N_{j1}, L_{j1}) = (N'_{j1} \cap D, L'_{j1} \cap D)$ a strong filtration pair adapted to D at p_{j1}. Then*

$$i_D\left(\widehat{f}^k, N_{j1}\right) - i_{S^2}\left(\widehat{f_s}^k, N'_{j1}\right) = \begin{cases} -1 & \text{if } \partial_{N'_{j1}}\left(L'_{j1}\right) \simeq S^1, \\ u_{j1} & \text{otherwise}, \end{cases} \tag{4.5}$$

with u_{j1} being the number of unstable branches of (N_{j1}, L_{j1}) associated to $\widehat{f}^k$ at p_{j1}. Therefore,

$$\begin{gathered} \sum i_D\left(\widehat{f}^k, N_{jm}\right) - \sum i_{S^2}\left(\widehat{f_s}^k, N'_{jm}\right) = u_p - r_p, \\ \sum_{j \notin R_1} i_D\left(\widehat{f}^k, N_{jm}\right) - \sum_{j \notin R_1} i_{S^2}\left(\widehat{f_s}^k, N'_{jm}\right) = u_p. \end{gathered} \tag{4.6}$$

Proof of Main Theorem 2 (Orientation reversing case). Let $f : U \to W$ be an orientation reversing local homeomorphism and let p be a non-accumulated, indifferent fixed point for f in the conditions of the orientation reversing case of Section 2. Then, $\mathrm{Per}(\widehat{f}|_{\partial(D)})$ is a finite set with two fixed points and q periodic orbits of period two.

If k is even, we have that $\widehat{f}^k$ is orientation preserving and $\mathrm{Fix}(\widehat{f}^k|_{\partial(D)}) = \{p_0, p_1, \{p_{11}, p_{12}\}, \ldots, \{p_{q1}, p_{q2}\}\}$. Then

$$i_{\mathbb{R}^2}\left(f^k, p\right) = 1 - u_p + r_p, \tag{4.7}$$

with r_p and u_p being the number of generalized repelling petals and unstable branches for f^2 at p. The petals and branches are constructed as in the orientation preserving case.

If k is odd, then let $\mathrm{Fix}(\widehat{f}^k|_{\partial(D)}) = \{p_0, p_1\}$ with $\{(N_0, L_0), (N_1, L_1)\}$ be strong filtration pairs adapted to D for p_0 and p_1. Let u'_p and r'_p be the number of unstable branches and generalized repelling petals associated to $\widehat{f}^k$ at the fixed points of $\partial(D)$ which are negatively invariant for $\widehat{f}^k$. Since $\widehat{f}^k$ is orientation reversing, we obtain that $u'_p \leq 2$, $r'_p \leq 2$ and $r'_p + u'_p = q^0 + q^1 \leq 2$. Then

$$i_{\mathbb{R}^2}\left(f^k, p\right) = 1 - q^0 - q^1 = 1 - u'_p - r'_p \in \{-1, 0, 1\}. \tag{4.8}$$

If f is an orientation reversing local homeomorphism,

$$i_{\mathbb{R}^2}\left(f^k, p\right) = \begin{cases} 1 - u_p + r_p & \text{if } k \text{ even}, \\ 1 - u'_p - r'_p & \text{if } k \text{ odd}, \end{cases} \tag{4.9}$$

with $i_{\mathbb{R}^2}(f^k, p) \in \{-1, 0, 1\}$ if k is odd. The numbers $\{u_p, r_p\}$ and $\{u'_p, r'_p\}$ are computed by observing $\widehat{f}^2$ and $\widehat{f}$. □

Definition 4.3 (Irreducibility of branches and petals). Let $p \in J$ be a non-accumulated and indifferent fixed point with J being a Jordan domain such that $K_p \cap \partial(J) \neq \emptyset$, and let us construct the Carathéodory's compactification of $S^2 \setminus K_p$, D, and the homeomorphism $\widehat{f} : D \to D$. If $p_i \in \mathrm{Fix}(\widehat{f}^k|_{\partial(D)})$ is an isolated fixed prime end (and not an identification to a point of an interval I_i of prime ends) and it gives us a family of generalized unstable branches for f^r at p, we call them *irreducible unstable branches* for f^r at p in J. In the same way, if p_i gives us a generalized repelling petal for f^r at p, we call it *irreducible repelling petal* for f^r at p in J.

Remark 4.4. If the set of isolated fixed prime ends of $\widehat{f}^k|_{\partial(D)}$ is not finite then, given $m \in \mathbb{N}$, we can obtain another identification homeomorphism, which we call again $\widehat{f}^k : D \to D$, which gives us a number $>m$ of generalized unstable branches and a number $>m$ of generalized repelling petals at p (obviously, we have $u_p > m$ and $r_p > m$). However, the number $r_p - u_p = i_{\mathbb{R}^2}(f^k, p) - 1$ is constant and it just depends on the germ of f.

Remark 4.5. Let us observe that if f is orientation reversing, since $\mathrm{Fix}(\widehat{f}|_{\partial(D)})$ is a set of two fixed prime ends for every $\widehat{f}$, then the numbers u'_p and r'_p of $i_{\mathbb{R}^2}(f, p) = 1 - u'_p - r'_p$ are independent of the map $\widehat{f}$ considered (this fact is not true for u_p and r_p).

The following proposition is a consequence of our previous results.

Proposition 4.6. *Let us suppose that* $i_{\mathbb{R}^2}(f^r, p) \neq 1$ *for some* $r \in \mathbb{N}$ *(*$r = 2$ *if* f *reverses orientation). There exists a family of* u_p *generalized unstable branches,* $\{U_j\}$*, and a family of* r_p *generalized repelling petals,* $\{R_i\}$*, for* f^r *at* p *such that* $i_{\mathbb{R}^2}(f^r, p) = 1 - u_p + r_p$ *and*

(1) *the open repelling petals and the sets* $\{U_j \setminus K_p\}$ *are two families of mutually disjoint sets. Moreover, each set of a family is disjoint from the sets of the other family;*

(2) $\lim_{n\to\infty} f^{-rn}(x) = \{p\}$ *and* $\lim_{n\to\infty} f^{-rn}(y) = \{p\}$ *for every* $x \in U_i$ *and every* $y \in R_i$*;*

(3) $\bigcap_{n\in\mathbb{N}} f^{-nr}(U_i)$ *and* $\bigcap_{n\in\mathbb{N}} f^{-nr}(R_i)$ *are* f^r*-invariant continua containing* p *and the sequence* $\{f^{-nr}(R_i)\}_{n\in\mathbb{N}}$ *determines an end containing* p*.*

If f is orientation reversing, the numbers $u'_p \leq 2$ and $r'_p \leq 2$, of the decomposition $i_{\mathbb{R}^2}(f,p) = 1 - u'_p - r'_p$, determine the number of generalized unstable branches and generalized repelling petals of our families which are negatively invariant for f.

Proof. Let us select an adequate J such that $\mathrm{Fix}(f^r|_J) = \{p\}$. Given a fixed point p_i for $\widehat{f}^r|_{\partial(D)}$ and a strong filtration pair (N_i, L_i) adapted to D, the unstable branches $\{U_{1i}, \ldots, U_{si}\}$ for $\widehat{f}^r$ at p_i are compact sets of trivial shape. We define generalized unstable branches $\{U_j\}$ as the closure in $\mathbb{R}^2$ of the sets $\{U_{1i} \setminus \{p_i\}, \ldots, U_{si} \setminus \{p_i\}\}$ for every $p_i \in \mathrm{Fix}(\widehat{f}^r_{\partial(D)})$.

Since $i_{\mathbb{R}^2}(f^r, p) \neq 1$ and $\mathrm{Fix}(f^r|_J) = \{p\}$, it is not difficult to prove that for every $x \in cl_{\mathbb{R}^2}(U_{li} \setminus \{p_i\})$, $f^{-rn}(x) \to \{p\}$ (see [12, Proposition 2]).

Let us construct generalized repelling petals $\{R_i\}$. There are r_p generalized repelling petals $\{N_1, \ldots, N_{r_p}\}$ associated to the fixed points of $\widehat{f}^r|_{\partial(D)}$. We can select generalized repelling petals $\{N_i\}$ in such a way that each arc $\gamma_i = \partial_D(N_i)$ has two end points in $\partial(K_p)$. Each p_i has associated a union of prime ends $\{\mathcal{P}_i\}$. At least one of these prime ends, $\mathcal{P}_i$, is a fixed prime end for $\widehat{f}^r$. We call P_i the set of points of $\mathcal{P}_i$. It is not difficult to prove that $P_i \subset \partial(K_p)$ is a continuum, invariant for f^r, with $p \in P_i$.

For each p_i, we obtain a generalized repelling petal, R_i, for f^r at p

$$R_i = \mathrm{cl}_{\mathbb{R}^2}(\mathrm{int}_{S^2}(N_i)) \tag{4.10}$$

with $p \in P_i \subset \partial(R_i)$. The associated open repelling petals are disjoint and it is obvious that they are disjoint from the sets $\{U_j \setminus K_p\}$ which are also disjoint. □

Remark 4.7. If a generalized unstable branch (or a generalized repelling petal) for f^r at p, U_0, is irreducible, then $\bigcap_{n\in\mathbb{N}} f^{-rn}(U_0) \subset \partial(K_p)$ is a continuum, invariant for f^r, and it is the set of points of a fixed prime end for $\widehat{f}^r$.

Remark 4.8. Note that from our techniques one can provide reasonable notions of local hyperbolic and elliptic sectors in terms of the generalized stable/unstable branches and generalized attracting/repelling petals such that the classical Poincaré formula remains true (Question 1.16 of [11]).

5. The Remaining Proofs

Proof of Theorem 2.18. We can assume that $p \in \partial(K_p)$.

The fixed point index $i_{\mathbb{R}^2}(f^r, p) = 1 - u_p + r_p < 1$ gives us $m = u_p - r_p$. We obtain that there are $u_p \geq m$ unstable branches $\{U_1, \ldots, U_{u_p}\}$ for $\widehat{f}^r$ at the fixed points in $\partial(D)$.

Let a_p and r_p be the number of fixed points in $\partial(D)$ associated to attracting and repelling petals for $\widehat{f}^r$. If the disc N_i of a strong filtration pair adapted to D, (N_i, L_i), is not an attracting nor a repelling petal, then we say that N_i is an *unstable petal*. We call $R \geq r_p$ the number of repelling fixed points for $\widehat{f}^r|_{\partial(D)}$. R is also the number of attracting fixed points for $\widehat{f}^r|_{\partial(D)}$.

Given a point $p_i \in \mathrm{Fix}(\widehat{f}^r|_{\partial(D)})$ associated to an unstable petal N_i, there are three cases.

Case 1. If p_i is a saddle point for $\hat{f}^r|_{\partial(D)}$, then there are the same number of unstable and stable branches for $\hat{f}^r$ at p_i.

Case 2. p_i is a repelling fixed point for $\hat{f}^r|_{\partial(D)}$. If r_i is the number of unstable branches for $\hat{f}^r$ at p_i, then there are $r_i + 1$ stable branches at p_i.

Case 3. p_i is an attracting fixed point for $\hat{f}^r|_{\partial(D)}$. If r_i is the number of unstable branches for $\hat{f}^r$ at p_i, then there are $r_i - 1$ stable branches at p_i.

We have a family $\{S_1, \dots, S_{s_p}\}$ of stable branches for $\hat{f}^r$ at the fixed points in $\partial(D)$ with

$$s_p = u_p - (R - a_p) + (R - r_p) = u_p + a_p - r_p \geq u_p - r_p = m. \tag{5.1}$$

Let $\{p_i\}$ be the family of fixed points of $\hat{f}^r|_{\partial(D)}$ and let $\{N_i\}$ be the family of attracting, repelling and unstable petals of the strong filtration pairs adapted to D, $\{(N_i, L_i)\}$, associated to each fixed point. We denote $N_u = \bigcup_i N_i$ such that N_i is unstable.

Let us consider the Jordan curve contained in D,

$$\gamma = (\partial(D) \setminus N_u) \cup \partial_D(N_u). \tag{5.2}$$

Let $L = \bigcup_{i,j} L_i^j$, with L_i^j being the components of each L_i such that $L_i^j \subset \mathrm{int}_{S^2}(D)$ and $\hat{f}^r(\partial_{N_i}(L_i^j)) \subset \mathrm{int}_{S^2}(L_i^j)$. Let U_1 be an unstable branch for $\hat{f}^r$ at D with $U_1 \cap \gamma \subset l_1$, where l_1 is the connected component of $L \cap \gamma$ which intersects U_1. Two unstable branches $\{U_1, U_2\}$ are *adjacent* if there is an arc $l_{1,2}$ in γ joining the arcs l_1 and l_2, with $l_1 \cup l_2 \subset l_{1,2}$ in such a way that $l_{1,2} \cap L = l_1 \cup l_2$.

If two unstable and adjacent branches for $\hat{f}^r$, $\{U_1, U_2\}$, are contained in the same region N_1, there exists a stable branch S_1 in N_1 between U_1 and U_2.

If two unstable and adjacent branches $\{U_1, U_2\}$ are contained in disjoint regions N_1 and N_2 associated to fixed points p_1 and p_2, then we have the following two situations.

(i) If there is a stable branch S_1 which intersects $l_{1,2} \cap \partial_D(N_1 \cup N_2)$, then S_1 is a stable branch between U_1 and U_2 in $l_{1,2}$.

(ii) If there is not a stable branch which intersects $l_{1,2} \cap \partial_D(N_1 \cup N_2)$, then the points p_1 and p_2 are attractors on the right side and on the left side, respectively, for $\hat{f}^r|_{\partial(D)}$. By this observation, if $\overline{p_1 p_2} \subset \partial(D)$ is the arc induced by $l_{1,2}$ joining p_1 and p_2, we have that there exists a repelling fixed point p' for $\hat{f}^r|_{\partial(D)}$ contained in the interior of $\overline{p_1 p_2}$. The point p' has associated an unstable or repelling petal N'.

If N' is unstable, there exists in N' a stable branch S_1 (between U_1 and U_2 in $l_{1,2}$).

Since there are r_p repelling petals, we can construct, at least, $u_p - r_p = m$ stable branches $\{S_1, \dots, S_m\}$ alternating in γ with m unstable branches $\{U_1, \dots, U_m\}$.

The stable and unstable branches for $\hat{f}^r$ at D, $\{S_1, \dots, S_m\}$ and $\{U_1, \dots, U_m\}$, give us the alternating set of generalized stable and generalized unstable branches for f^r at p which we are looking for.

Let us consider a Jordan curve $\gamma_0 \subset D$ near enough $\partial(D)$ and let

$$\gamma_1 = (\gamma_0 \setminus N_u) \cup \partial_D(N_u). \tag{5.3}$$

The closed disc $D_p \subset J$ (and containing p) determined by γ_1 is the disc we are looking for. □

Proof of Theorem 2.19. Since $i_{\mathbb{R}^2}(f^r, p) = 1 + m > 1$, we have that p is indifferent. On the other hand, $p \in \partial(K_p)$ (if $p \in \mathrm{int}(K_p)$, then p is stable and $i_{\mathbb{R}^2}(f^r, p) = 1$; see [25]).

We have $i_{\mathbb{R}^2}(f^k, p) = i_{\mathbb{R}^2}(f^r, p) = 1 - u_p + r_p > 1$ for all $k \in r\mathbb{N}$ with r being the period of the periodic orbits of $\widehat{f}|_{\partial(D)}$. We obtain that there is a family of r_p generalized repelling petals (see Proposition 4.6), $\{R_i\}$, with $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \neq j$. The fixed point index $i_{\mathbb{R}^2}(f^r, p) = 1 - u_p + r_p$ gives us $m = r_p - u_p$. Since $r_p \geq m$, there are, at least, m generalized repelling petals.

Let us construct the m generalized attracting petals $\{A_i\}$. Since $\widehat{f}^r|_{\partial(D)}$ has, at least, r_{p} repelling fixed points, then there are also r_p attracting fixed points $\{p'_1, \ldots, p'_{r_p}\}$ ($\widehat{f}^r|_{\partial(D)}$ is an orientation preserving homeomorphism). From these r_p fixed points, there are no more than u_p without generalized attracting petals. Then, the remainder points (at least $r_p - u_p = m$) are points with associated generalized attracting petals N'_i. We define the generalized attracting petals A_i as

$$A_i = \mathrm{cl}_{\mathbb{R}^2}\left(\mathrm{int}_{S^2}\left(N'_i\right)\right). \tag{5.4}$$

It only remains to construct the Jordan curve γ around p. Let us consider two repelling fixed points $\{p_1, p_2\}$ of $\widehat{f}^r|_{\partial(D)}$, with repelling petals $\{N_1, N_2\}$, and adjacent in the set of fixed points $R = \{p_1, \ldots, p_{r_p}\} \subset \partial(D)$ associated to repelling petals. Given the arc $\gamma_{1,2} \subset \partial(D)$ joining the points p_1 and p_2 (with $\gamma_{1,2} \cap R = \{p_1, p_2\}$), there are λ_1 unstable branches in D associated to the fixed points of $\widehat{f}^r|_{\partial(D)}$ contained in $\gamma_{1,2}$. In the same way, we consider the arcs $\gamma_{i,i+1}$ for $\{p_i, p_{i+1}\}$ and the numbers λ_i of unstable branches of the fixed points in the arcs $\gamma_{i,i+1}$. Then

$$\sum_{i=1}^{r_p} \lambda_i = u_p = r_p - m. \tag{5.5}$$

There are, at least, m elements $\{\lambda_{i_1}, \ldots, \lambda_{i_m}\} \subset \{\lambda_1, \ldots, \lambda_{r_p}\}$ such that $\lambda_{i_1} = \cdots = \lambda_{i_m} = 0$.

Since p_{i_1} and p_{i_1+1} are repellers for $\widehat{f}^r|_{\partial(D)}$, there exists, at least, an attractor for $\widehat{f}^r|_{\partial(D)}$, p'_{i_1}, in the interior of γ_{i_1,i_1+1}. Since $\lambda_{i_1} = 0$, then p'_{i_1} is associated to an attracting petal N'_{i_1}. In the same way, we construct attracting petals $\{N'_{i_1}, \ldots, N'_{i_m}\}$ which alternate with the repelling petals $\{N_{i_1}, \ldots, N_{i_m}\}$ around $\partial(D)$. The required Jordan curve is obtained by selecting $\gamma \subset \mathrm{int}(D)$ near enough $\partial(D)$. The generalized attracting petals $\{A_{i_1}, \ldots, A_{i_m}\}$ associated to $\{N'_{i_1}, \ldots, N'_{i_m}\}$ and the generalized repelling petals $\{R_{i_1}, \ldots, R_{i_m}\}$ associated to $\{N_{i_1}, \ldots, N_{\mathrm{i}_m}\}$ alternate with respect to γ. □

Proof of Corollary 2.21. Let DM be the double of the manifold M and let $Df : DM \to DM$ be the homeomorphism induced by f.

We only have to pay attention to the case where $\mathrm{Fix}(f) \cap \mathrm{int}(M)$ is finite.

Let $p_1, \ldots, p_n \in \partial(M)$, $(q_1, \ldots, q_n \in \partial(M))$ the repellers (attractors) of $f|_{\partial(M)}$, and $r_1, \ldots, r_q$ the fixed points of f in $\mathrm{int}(M)$.

We know that the index of Df at each fixed point is ≤ 1 because there are no generalized repelling petals.

Note that the saddle points in $\partial(M)$ have index ≤ 0 because there exists, at least, a generalized unstable branch.

Then,

$$\Lambda(Df) \leq 2 \sum_{j\in\{1,\ldots,q\}} i_M(f,r_j) + \sum_{i\in\{1,\ldots,n\}} i_{DM}(Df,p_i) + \sum_{i\in\{1,\ldots,n\}} i_{DM}(Df,q_i). \tag{5.6}$$

Now, since there exist al least two generalized unstable (stable) branches with each repeller (attractor), $i_{DM}(Df,p_i) \leq -1$ and $i_{DM}(Df,q_i) \leq -1$ for every $i \in \{1,\ldots,n\}$.

Then $2\Lambda(f) = \Lambda(Df) \leq 2\sum_{j\in\{1,\ldots,q\}} i_M(f,r_j) - 2n$.

Therefore, $\Lambda(f) + n \leq \sum_{j\in\{1,\ldots,q\}} i_M(f,r_j)$ and $q \geq \Lambda(f) + n$. □

Remark 5.1. In the particular case where M is the closed 2-disc much more can be said. Indeed, if f has a fixed point in the boundary, then it has another fixed point in $\mathrm{int}(M)$. Then $Df : S^2 \to S^2$ is an area and orientation preserving homeomorphism with at least three fixed points. Therefore, using a theorem of Franks [32] (see also [5]), we have that Df has infinite periodic orbits. Consequently, f also has infinite periodic orbits.

Proof of Theorem 2.22. The proof of (a), (b.1), and (b.2) follows as in the orientation preserving case. Let us prove (c.1). Since $i_{\mathbb{R}^2}(f,p) = 1-u'_p-r'_p = 1$, we obtain that there are no generalized repelling petals and generalized unstable branches for f^2 at p, negatively invariant for f (that are associated to the two fixed points for $\widehat{f}$, $\{p_0,p_1\}$).

An easy topological argument allows us to say that p_0 and p_1 are attracting or repelling fixed points for $\widehat{f}|_{\partial(D)}$. For each one of the three cases (two attractors, two repellers or an attractor and a repeller), we obtain the three situations of the case (c.1).

Since $u'_p = r'_p = 0$ and f is orientation reversing, it is easy to see that u_p and r_p are even. This fact gives us $i_{\mathbb{R}^2}(f^2,p) = 1 - u_p + r_p$ odd.

The proofs of (c.2) and (c.3) are analogous. □

Proof of Corollary 2.23. We shall give a proof based on our results and a strong theorem of existence of periodic orbits of orientation and area preserving homeomorphisms in the 2-sphere. Note that it can be used also the results of Bonino in [33].

If $|\mathrm{Fix}(f)| \geq 3$, then $|\mathrm{Fix}(f^2)| \geq 3$ and, since f^2 is an orientation and area preserving homeomorphism, by a theorem of Franks [32] (see also [5]) we have that $|\mathrm{Per}(f^2)| = \infty$ and, therefore, $|\mathrm{Per}(f)| = \infty$.

If $1 \leq |\mathrm{Fix}(f)| \leq 2$, let us see that $|\mathrm{Per}(f)| = \infty$. If we suppose that $|\mathrm{Per}(f)| < \infty$, then each $p_j \in \mathrm{Fix}(f)$ is an isolated periodic orbit and we have $i_{S^2}(f,p_j) \leq 1$ (see [29]). If p_j is stable, the index is 1 (see [9]). If p_j is not indifferent, then the index is $1-\delta \in \{-1,0,1\}$ (see [12]). If p_j is indifferent, then the index is $1-u'_{p_j} \in \{0,1\}$. Let us observe the following two equalities:

$$\begin{aligned} 0 &= i_{S^2}\left(f,S^2\right) = \sum_{p_j\in\mathrm{Fix}(f)} i_{S^2}(f,p_j), \\ 2 &= i_{S^2}\left(f^2,S^2\right) = \sum_{p_j\in\mathrm{Fix}(f)} i_{S^2}\left(f^2,p_j\right) + \sum_{q_j\in\mathrm{Fix}(f^2)\backslash\mathrm{Fix}(f)} i_{S^2}\left(f^2,q_j\right). \end{aligned} \tag{5.7}$$

It is easy to see that $i_{S^2}(f^2,p_j) \le i_{S^2}(f,p_j)$. In fact, if p_j is stable the index for f^2 is 1. If p_j is not indifferent, the index for f^2 is $1-\delta-2q \le 1-\delta$ (see [12]) and, if p_j is indifferent, the index for f^2 is $1-u_{p_j} \le 1-u'_{p_j}$. Using the above two equalities, we have that $|\operatorname{Fix}(f^2)| \ge 3$ and we obtain a contradiction which gives us $|\operatorname{Per}(f)| = \infty$. □

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