Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 493298, 6 pages
doi:10.1155/2010/493298

Research Article

A Kirk Type Characterization of Completeness for Partial Metric Spaces

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Received 1 October 2009; Accepted 25 November 2009

Academic Editor: Mohamed A. Khamsi

We extend the celebrated result of W. A. Kirk that a metric space X is complete if and only if every Caristi self-mapping for X has a fixed point, to partial metric spaces.

1. Introduction and Preliminaries

Caristi proved in [1] that if f is a selfmapping of a complete metric space (X, d) such that there is a lower semicontinuous function $\phi : X \to [0, \infty)$ satisfying

$$d(x, fx) \leq \phi(x) - \phi(fx) \tag{1.1}$$

for all $x \in X$, then f has a fixed point.

This classical result suggests the following notion. A selfmapping f of a metric space (X, d) for which there is a function $\phi : X \to [0, \infty)$ satisfying the conditions of Caristi's theorem is called a Caristi mapping for (X, d).

There exists an extensive and well-known literature on Caristi's fixed point theorem and related results (see, e.g., [2–10], etc.).

In particular, Kirk proved in [7] that a metric space (X, d) is complete if and only if every Caristi mapping for (X, d) has a fixed point. (For other characterizations of metric completeness in terms of fixed point theory see [11–14], etc., and also [15, 16] for recent contributions in this direction.)

In this paper we extend Kirk's characterization to a kind of complete partial metric spaces.

Let us recall that partial metric spaces were introduced by Matthews in [17] as a part of the study of denotational semantics of dataflow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see [18–25], etc.).

A partial metric [17] on a set X is a function $p : X \times X \to [0,\infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$; (ii) $p(x,x) \le p(x,y)$; (iii) $p(x,y) = p(y,x)$; (iv) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) where p is a partial metric on X.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Next we give some pertinent concepts and facts on completeness for partial metric spaces.

If p is a partial metric on X, then the function $p^s : X \times X \to [0,\infty)$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a metric on X.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m} p(x_n, x_m)$ ([17, Definition 5.2]).

Note that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s) (see, e.g., [17, page 194]).

A partial metric space (X,p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m} p(x_n, x_m)$ ([17, Definition 5.3]).

It is well known and easy to see that a partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

In order to give an appropriate notion of a Caristi mapping in the framework of partial metric spaces, we naturally propose the following two alternatives.

(i) A selfmapping f of a partial metric space (X,p) is called a p-Caristi mapping on X if there is a function $\phi : X \to [0,\infty)$ which is lower semicontinuous for (X,p) and satisfies $p(x,fx) \le \phi(x) - \phi(fx)$, for all $x \in X$.

(ii) A selfmapping f of a partial metric space (X,p) is called a p^s-Caristi mapping on X if there is a function $\phi : X \to [0,\infty)$ which is lower semicontinuous for (X,p^s) and satisfies $p(x,fx) \le \phi(x) - \phi(fx)$, for all $x \in X$.

It is clear that every p-Caristi mapping is p^s-Caristi but the converse is not true, in general.

In a first attempt to generalize Kirk's characterization of metric completeness to the partial metric framework, one can conjecture that a partial metric space (X,p) is complete if and only if every p-Caristi mapping on X has a fixed point.

The following easy example shows that this conjecture is false.

Example 1.1. On the set $\mathbb{N}$ of natural numbers construct the partial metric p given by

$$p(n,m) = \max\left\{\frac{1}{n}, \frac{1}{m}\right\}. \tag{1.2}$$

Note that $(\mathbb{N},p)$ is not complete, because the metric p^s induces the discrete topology on $\mathbb{N}$, and $(n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{N},p^s)$. However, there is no p-Caristi mappings on $\mathbb{N}$ as we show in the next.

Indeed, let $f : \mathbb{N} \to \mathbb{N}$ and suppose that there is a lower semicontinuous function ϕ from $(\mathbb{N}, \tau_p)$ into $[0, \infty)$ such that $p(n, fn) \le \phi(n) - \phi(fn)$ for all $n \in \mathbb{N}$. If $1 < f1$, we have $p(1, f1) = 1 = p(1,1)$, which means that $f1 \in B_p(1, \varepsilon)$ for any $\varepsilon > 0$, so $\phi(1) \le \phi(f1)$ by lower semicontinuity of ϕ, which contradicts condition $p(1, f1) \le \phi(1) - \phi(f1)$. Therefore $1 = f1$, which again contradicts condition $p(1, f1) \le \phi(1) - \phi(f1)$. We conclude that f is not a p-Caristi mapping on $\mathbb{N}$.

Unfortunately, the existence of fixed point for each p^s-Caristi mapping on a partial metric space (X, p) neither characterizes completeness of (X, p) as follows from our discussion in the next section.

2. The Main Result

In this section we characterize those partial metric spaces for which every p^s-Caristi mapping has a fixed point in the style of Kirk's characterization of metric completeness. This will be done by means of the notion of a 0-complete partial metric space which is introduced as follows.

Definition 2.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) is called 0-Cauchy if $\lim_{n,m} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Note that every 0-Cauchy sequence in (X, p) is Cauchy in (X, p^s), and that every complete partial metric space is 0-complete.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0, \infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides a paradigmatic example of a 0-complete partial metric space which is not complete.

In the proof of the "only if" part of our main result we will use ideas from [11, 26], whereas the following auxiliary result will be used in the proof of the "if" part.

Lemma 2.2. *Let (X, p) be a partial metric space. Then, for each $x \in X$, the function $p_x : X \to [0, \infty)$ given by $p_x(y) = p(x, y)$ is lower semicontinuous for (X, p^s).*

Proof. Assume that $\lim_n p^s(y, y_n) = 0$, then

$$p_x(y) \le p_x(y_n) + p(y_n, y) - p(y_n, y_n) = p_x(y_n) + p^s(y_n, y) - p(y_n, y) + p(y, y). \quad (2.1)$$

This yields $\liminf_n p_x(y_n) \ge p_x(y)$ because $p(y, y) \le p(y, y_n)$. □

Theorem 2.3. *A partial metric space (X, p) is 0-complete if and only if every p^s-Caristi mapping f on X has a fixed point.*

Proof. Suppose that (X, p) is 0-complete and let f be a p^s-Caristi mapping on X, then, there is a $\phi : X \to [0, \infty)$ which is lower semicontinuous function for (X, p^s) and satisfies

$$p(x, fx) \le \phi(x) - \phi(fx), \quad (2.2)$$

for all $x \in X$.

Now, for each $x \in X$ define

$$A_x := \{y \in X : p(x,y) \leq \phi(x) - \phi(y)\}. \tag{2.3}$$

Observe that $A_x \neq \phi$ because $fx \in A_x$. Moreover A_x is closed in the metric space (X, p^s) since $y \mapsto p(x,y) + \phi(y)$ is lower semicontinuous for (X, p^s).

Fix $x_0 \in X$. Take $x_1 \in A_{x_0}$ such that $\phi(x_1) < \inf_{y \in A_{x_0}} \phi(y) + 2^{-1}$. Clearly $A_{x_1} \subseteq A_{x_0}$. Hence, for each $x \in A_{x_1}$ we have

$$\begin{aligned} p(x_1, x) &\leq \phi(x_1) - \phi(x) < \inf_{y \in A_{x_0}} \phi(y) + 2^{-1} - \phi(x) \\ &\leq \phi(x) + 2^{-1} - \phi(x) = 2^{-1}. \end{aligned} \tag{2.4}$$

Following this process we construct a sequence $(x_n)_{n \in \omega}$ in X such that its associated sequence $(A_{x_n})_{n \in \omega}$ of closed subsets in (X, p^s) satisfies

(i) $A_{x_{n+1}} \subseteq A_{x_n},\ x_{n+1} \in A_{x_n}$ for all $n \in \omega$,

(ii) $p(x_n, x) < 2^{-n}$ for all $x \in A_{x_n},\ n \in \mathbb{N}$.

Since $p(x_n, x_n) \leq p(x_n, x_{n+1})$, and, by (i) and (ii), $p(x_n, x_m) < 2^{-n}$ for all $m > n$, it follows that $\lim_{n,m} p(x_n, x_m) = 0$, so $(x_n)_{n \in \omega}$ is a 0-Cauchy sequence in (X, p), and by our hypothesis, there exists $z \in X$ such that $\lim_n p(z, x_n) = p(z, z) = 0$, and thus $\lim_n p^s(z, x_n) = 0$. Therefore $z \in \bigcap_{n \in \omega} A_{x_n}$.

Finally, we show that $z = fz$. To this end, we first note that

$$\begin{aligned} p(x_n, fz) &\leq p(x_n, z) + p(z, fz) \\ &\leq \phi(x_n) - \phi(z) + \phi(z) - \phi(fz), \end{aligned} \tag{2.5}$$

for all $n \in \omega$. Consequently $fz \in \bigcap_{n \in \omega} A_{x_n}$, so by (ii), $p(x_n, fz) < 2^{-n}$ for all $n \in \mathbb{N}$. Since $p(z, fz) \leq p(z, x_n) + p(x_n, fz)$, and $\lim_n p(z, x_n) = 0$, it follows that $p(z, fz) = 0$. Hence $p^s(z, fz) = 0$ since $p^s(z, fz) \leq 2p(z, fz)$, so $z = fz$.

Conversely, suppose that there is a 0-Cauchy sequence $(x_n)_{n \in \omega}$ of distinct points in (X, p) which is not convergent in (X, p^s). Construct a subsequence $(y_n)_{n \in \omega}$ of $(x_n)_{n \in \omega}$ such that $p(y_n, y_{n+1}) < 2^{-(n+1)}$ for all $n \in \omega$.

Put $A = \{y_n : n \in \omega\}$, and define $f : X \to X$ by $fx = y_0$ if $x \in X \setminus A$, and $fy_n = y_{n+1}$ for all $n \in \omega$.

Observe that A is closed in (X, p^s).

Now define $\phi : X \to [0, \infty)$ by $\phi(x) = p(x, y_0) + 1$ if $x \in X \setminus A$, and $\phi(y_n) = 2^{-n}$ for all $n \in \omega$.

Note that $\phi(y_{n+1}) < \phi(y_n)$ for all $n \in \omega$ and that $\phi(y_0) \leq \phi(x)$ for all $x \in X \setminus A$.

From this fact and the preceding lemma we deduce that ϕ is lower semicontinuous for (X, p^s).

Moreover, for each $x \in X \setminus A$ we have

$$p(x, fx) = p(x, y_0) = \phi(x) - \phi(y_0) = \phi(x) - \phi(fx), \tag{2.6}$$

and for each $y_n \in A$ we have

$$p(y_n, fy_n) = p(y_n, y_{n+1}) < 2^{-(n+1)} = \phi(y_n) - \phi(y_{n+1}) \\ = \phi(y_n) - \phi(fy_n). \tag{2.7}$$

Therefore f is a Caristi p^s-mapping on X without fixed point, a contradiction. This concludes the proof. □

Acknowledgments

The author is very grateful to the referee for his/her useful suggestions. This work was partially supported by the Spanish Ministry of Science and Innovation, and FEDER, Grant MTM2009-12872-C02-01.

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