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Research Article

On the Convergence of an Implicit Iterative Process for Generalized Asymptotically Quasi-Nonexpansive Mappings

Xiaolong Qin,[1] Shin Min Kang,[2] and Ravi P. Agarwal[3,4]

[1] *School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, China*
[2] *Department of Mathematics and the RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea*
[3] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA*
[4] *Mathematics and Statistics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*

Correspondence should be addressed to Ravi P. Agarwal, agarwal@fit.edu

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The purpose of this paper is to introduce and consider a general implicit iterative process which includes Schu's explicit iterative processes and Sun's implicit iterative processes as special cases for a finite family of generalized asymptotically quasi-nonexpansive mappings. Strong convergence of the purposed iterative process is obtained in the framework of real Banach spaces.

1. Introduction and Preliminaries

Let E be a real Banach space and $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *uniformly convex* if for any $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta. \tag{1.1}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Let C be a nonempty closed and convex subset of a Banach space E. Let $T : C \to C$ be a mapping. Denote by $F(T)$ the fixed point set of T.

Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

T is said to be *quasi-nonexpansive* if $F(T) \ne \emptyset$ and

$$\|Tx - y\| \le \|x - y\|, \quad \forall x \in C,\ y \in F(T). \tag{1.3}$$

A nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive; however, the inverse may be not true. See the following example [1].

Example 1.1. Let $E = R^1$ and define a mapping by $T : E \to E$ by

$$Tx = \begin{cases} \dfrac{x}{2}\sin\dfrac{1}{x} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.4}$$

Then T is quasi-nonexpansive but not nonexpansive.

T is said to be *asymptotically nonexpansive* if there exists a positive sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C,\ n \ge 1. \tag{1.5}$$

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive with the asymptotical sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. It is known that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive mapping on C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, a host of authors have studied weak and strong convergence problems of implicit iterative processes for such a class of mappings.

T is said to be *asymptotically quasi-nonexpansive* if $F(T) \ne \emptyset$, and there exists a positive sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - y\| \le k_n \|x - y\|, \quad \forall x \in C,\ y \in F(T),\ n \ge 1. \tag{1.6}$$

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n \to \infty} \sup_{x, y \in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0. \tag{1.7}$$

Putting $\xi_n = \max\{0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|)\}$, we see that $\xi_n \to 0$ as $n \to \infty$. Then (1.7) is reduced to the following:

$$\|T^n x - T^n y\| \le \|x - y\| + \xi_n, \quad \forall x, y \in C,\ n \ge 1. \tag{1.8}$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Kirk [3] (see also Bruck et al. [4]) as a generalization of the class of asymptotically nonexpansive mappings. It is known that if C is a nonempty closed convex and bounded subset of a real Hilbert space, then every asymptotically nonexpansive self-mapping in the intermediate sense has a fixed point; see [5] more details.

T is said to be *asymptotically quasi-nonexpansive in the intermediate sense* if it is continuous, $F(T) \neq \emptyset$, and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x\in C, y\in F(T)} \left(\|T^n x - y\| - \|x - y\|\right) \leq 0. \tag{1.9}$$

Putting $\xi_n = \max\{0, \sup_{x\in C, y\in F(T)}(\|T^n x - y\| - \|x - y\|)\}$, we see that $\xi_n \to 0$ as $n \to \infty$. Then (1.9) is reduced to the following:

$$\|T^n x - y\| \leq \|x - y\| + \xi_n, \quad \forall x \in C,\ y \in F(T),\ n \geq 1. \tag{1.10}$$

T is said to be *generalized asymptotically nonexpansive* if there exist two positive sequences $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + \xi_n, \quad \forall x, y \in C,\ n \geq 1. \tag{1.11}$$

It is easy to see that the class of generalized asymptotically nonexpansive includes the class of asymptotically nonexpansive as a special case.

T is said to be *generalized asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exist two positive sequences $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - y\| \leq k_n \|x - y\| + \xi_n, \quad \forall x \in C,\ y \in F(T),\ n \geq 1. \tag{1.12}$$

The class of generalized asymptotically quasi-nonexpansive was considered by Shahzad and Zegeye [6]; see [6, 7] for more details.

Recall that the modified Mann iteration which was introduced by Schu [8] generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1, \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$ and $T : C \to C$ is an asymptotically nonexpansive mapping.

In 1991, Schu [8] obtained the following results.

Theorem Schu 1. *Let E be a uniformly convex Banach space, $\emptyset \neq C \subset E$ closed bounded and convex, and $T : C \to C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\} \in [0, 1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in* (1.13). *Then* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Theorem Schu 2. *Let E be a uniformly convex Banach space, $\emptyset \neq C \subset E$ closed bounded and convex, and $T : C \to C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\} \in [0,1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in* (1.13). *Suppose that T^m is compact for some positive integer $m \geq 1$. Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.*

Theorem Schu 3. *Let E be a uniformly convex Banach space, $\emptyset \neq C \subset E$ closed bounded and convex, and $T : C \to C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\} \in [0,1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in* (1.13). *Suppose that there exists a nonempty compact and convex subset K of E and $\lambda \in (0,1)$ such that*

$$d(Tx, K) \leq \lambda d(x, K), \quad \forall x \in C. \tag{1.14}$$

Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.

In 2007, Shahzad and Zegeye [6] considered the following implicit iterative process for a finite family of generalized asymptotically quasi-nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1},\\
&\vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1},\\
&\vdots,
\end{aligned} \tag{1.15}$$

where x_0 is the initial value and $\{\alpha_n\}$ is a sequence $(0,1)$. Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer, and $h(n) \to \infty$ as $n \to \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + \alpha_n T_{i(n)}^{h(n)} x_n, \quad \forall n \geq 1. \tag{1.16}$$

We remark that the implicit iterative process (1.16) was first considered by Sun [9]; see [9] for more details.

Shahzad and Zegeye [6] obtained the following results.

Theorem SZ 1. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i : i \in J\}$, where $J = \{1, 2, \ldots, N\}$, be N uniformly Lipschitz, generalized asymptotically quasi-nonexpansive self-mappings of C with $\{k_{in}\} \subset [1, \infty)$, $\{\xi_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{in}-1) < \infty$ and $\sum_{n=1}^{\infty}\xi_{in} < \infty$ for all $i \in J$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i : i \in J\}$ which is either semicompact or satisfies condition $(\overline{C})$. Let $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.16). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Theorem SZ 2. *Let E be a real uniformly convex Banach space and C a nonemptyclosed convex subset of E. Let $\{T_i : i \in J\}$, where $J = \{1, 2, \ldots, N\}$, be N generalized asymptotically quasi-nonexpansive self-mappings of C with $\{k_{in}\} \subset [1, \infty)$, $\{\xi_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{in} - 1) < \infty$ and $\sum_{n=1}^{\infty}\xi_{in} < \infty$ for all $i \in J$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ is closed. Let $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.16). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

In this paper, motivated by the above results, we consider the following implicit iterative process for two finite families of generalized asymptotically quasi-nonexpansive mappings $\{S_1, S_2, \ldots, S_N\}$ and $\{T_1, T_2, \ldots, T_N\}$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1 + \delta_1 u_1,\\
x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_1 + \gamma_2 T_2 x_2 + \delta_2 u_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + \beta_N S_N x_{N-1} + \gamma_N T_N x_N + \delta_N u_N,\\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_1^2 x_N + \gamma_{N+1} T_1^2 x_{N+1} + \delta_{N+1} u_{N+1},\\
&\vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} S_N^2 x_{2N-1} + \gamma_{2N} T_N^2 x_{2N} + \delta_{2N} u_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} S_1^3 x_{2N} + \gamma_{2N+1} T_1^3 x_{2N+1} + \delta_{2N+1} u_{2N+1},\\
&\vdots,
\end{aligned} \tag{1.17}$$

where x_0 is the initial value, $\{u_n\}$ is a bounded sequence in C, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad \forall n \geq 1. \tag{1.18}$$

We remark that our implicit iterative process (1.18) which includes the explicit iterative process (1.13) and the implicit iterative process (1.16) as special cases is general.

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then the implicit iterative process (1.18) is reduced to the following implicit iterative process:

$$x_n = (\alpha_n + \beta_n)x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad \forall n \geq 1. \tag{1.19}$$

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then the implicit iterative process (1.18) is reduced to the following explicit iterative process:

$$x_n = \frac{\alpha_n}{1-\gamma_n}x_{n-1} + \frac{\beta_n}{1-\gamma_n}S_{i(n)}^{h(n)} x_{n-1} + \frac{\delta_n}{1-\gamma_n}u_n, \quad \forall n \geq 1. \tag{1.20}$$

The purpose of this paper is to study the convergence of the implicit iteration process (1.18) for two finite families of generalized asymptotically quasi-nonexpansive mappings. Strong convergence theorems are obtained in the framework of real Banach spaces. The results presented in this paper improve and extend the corresponding results in Shahzad and Zegeye [6], Sun [9], Chang et al. [10], Chidume and Shahzad [11], Guo and Cho [12], Kim et al. [13], Qin et al. [14], Thianwan and Suantai [15], Xu and Ori [16], and Zhou and Chang [17].

In order to prove our main results, we also need the following lemmas.

Lemma 1.2 (see [18]). *Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \geq n_0, \tag{1.21}$$

where n_0 is some positive integer. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 1.3 (see [19]). *Let E be a real uniformly convex Banach space, $s > 0$ a positive number, and $B_s(0)$ a closed ball of E. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|ax + by + cz + dw\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x - y\|) \tag{1.22}$$

for all $x, y, z, w \in B_s(0) = \{x \in E : \|x\| \leq s\}$ and $a, b, c, d \in [0, 1]$ such that $a + b + c + d = 1$.

2. Main Results

Lemma 2.1. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$ and $S_i : C \to C$ a uniformly $L_{s,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F =$*

$\cap_{i=1}^N F(T_i) \bigcap \cap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \le i \le N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \le i \le N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \le i \le N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \ge 1$. Let $\{x_n\}$ be a sequence generated in (1.18)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants $a, b, c, d \in (0,1)$ such that $a \le \alpha_n$, $b \le \beta_n$, and $c \le \gamma_n \le d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \le i \le N\}$, for all $n \ge 1$;*

(b) $\sum_{n=1}^\infty \delta_n < \infty$.

Then

$$\lim_{n\to\infty} \|x_n - T_r x_n\| = \lim_{n\to\infty} \|x_n - S_r x_n\| = 0, \quad \forall r \in \{1,2,\ldots,N\}. \tag{2.1}$$

Proof. First, we show that the sequence $\{x_n\}$ generated in (1.18) is well defined. For each $n \ge 1$, define a mapping $C_n : C \to C$ as follows:

$$C_n x = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x + \delta_n u_n, \quad \forall x \in C. \tag{2.2}$$

Notice that

$$\begin{aligned} \|C_n x - C_n y\| &\le \gamma_n \left\| T_{i(n)}^{h(n)} x - T_{i(n)}^{h(n)} y \right\| \\ &\le d L_t \|x - y\|, \quad \forall x, y \in C. \end{aligned} \tag{2.3}$$

From the restriction (a), we see that C_n is a contraction for each $n \ge 1$. From Banach contraction mapping principle, we can prove that the sequence $\{x_n\}$ generated in (1.18) is well defined.

Fixing $p \in F$, we see that

$$\begin{aligned} \|x_n - p\| &\le \alpha_n \|x_{n-1} - p\| + \beta_n \left\| S_{i(n)}^{h(n)} x_{n-1} - p \right\| + \gamma_n \left\| T_{i(n)}^{h(n)} x_n - p \right\| + \delta_n \|u_n - p\| \\ &\le \alpha_n \|x_{n-1} - p\| + \beta_n \left(k_{h(n)} \|x_{n-1} - p\| + \xi_{h(n)} \right) + \gamma_n \left(k_{h(n)} \|x_n - p\| + \xi_{h(n)} \right) \\ &\quad + \delta_n \|u_n - p\| \\ &\le \left(\alpha_n + \beta_n k_{h(n)} \right) \|x_{n-1} - p\| + (1 - \alpha_n - \beta_n) k_{h(n)} \|x_n - p\| + 2\xi_{h(n)} \\ &\quad + \delta_n \|u_n - p\|. \end{aligned} \tag{2.4}$$

Notice that $\sum_{n=1}^\infty (k_n - 1) < \infty$. We see from the restrictions (a) and (b) that there exists a positive integer n_0 such that

$$(1 - \alpha_n - \beta_n) k_{h(n)} \le R < 1, \quad \forall n \ge n_0, \tag{2.5}$$

where $R = (1-(a+b))(1+(a+b)/(2-2(a+b)))$. It follows from (2.4) that

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n + \beta_n k_{h(n)}}{1-(1-\alpha_n-\beta_n)k_{h(n)}}\|x_{n-1}-p\| + \frac{\delta_n}{1-(1-\alpha_n-\beta_n)k_{h(n)}}\|u_n - p\| \\ &\quad + \frac{2\xi_{h(n)}}{1-(1-\alpha_n-\beta_n)k_{h(n)}} \\ &\le \left(1+\frac{k_{h(n)}-1}{1-R}\right)\|x_{n-1}-p\| + \frac{\delta_n}{1-R}\|u_n-p\| + \frac{2\xi_{h(n)}}{1-R} \\ &\le \left(1+\frac{k_{h(n)}-1}{1-R}\right)\|x_{n-1}-p\| + M_1(\delta_n + \xi_{h(n)}), \quad \forall n \ge n_0,\end{aligned} \tag{2.6}$$

where M_1 is an appropriate constant such that $M_1 = \max\{\sup_{n\ge1}\{\|u_n - p\|/(1-R)\}, 2/(1-R)\}$. In view of the restrictions (a) and (b), we obtain from Lemma 1.2 that $\lim_{n\to\infty}\|x_n - p\|$ exists. It follows that the sequence $\{x_n\}$ is bounded. In view of Lemma 1.3, we see that

$$\begin{aligned}\|x_n - p\|^2 &\le \alpha_n\|x_{n-1}-p\|^2 + \beta_n\left\|S_{i(n)}^{h(n)}x_{n-1}-p\right\|^2 + \gamma_n\left\|T_{i(n)}^{h(n)}x_n - p\right\|^2 \\ &\quad + \delta_n\|u_n-p\|^2 - \alpha_n\beta_n g\left(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\right) \\ &\le \alpha_n\|x_{n-1}-p\|^2 + \beta_n(k_{h(n)}\|x_{n-1}-p\| + \xi_{h(n)})^2 + \gamma_n(k_{h(n)}\|x_n-p\| + \xi_{h(n)})^2 \\ &\quad + \delta_n\|u_n-p\|^2 - \alpha_n\beta_n g\left(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\right) \\ &\le \alpha_n\|x_{n-1}-p\|^2 + \beta_n\left(k_{h(n)}^2\|x_{n-1}-p\|^2 + \xi_{h(n)}^2 + 2k_{h(n)}\xi_{h(n)}\|x_{n-1}-p\|\right) \\ &\quad + \gamma_n\left(k_{h(n)}^2\|x_n-p\|^2 + \xi_{h(n)}^2 + 2k_{h(n)}\xi_{h(n)}\|x_n-p\|\right) \\ &\quad + \delta_n\|u_n-p\|^2 - \alpha_n\beta_n g\left(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\right) \\ &\le \left(\alpha_n + \beta_n k_{h(n)}^2\right)\|x_{n-1}-p\|^2 + \gamma_n k_{h(n)}^2\|x_n-p\|^2 + 2\xi_{h(n)}^2 \\ &\quad + 2k_{h(n)}\xi_{h(n)}M_2 + \delta_n M_3 - \alpha_n\beta_n g\left(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\right),\end{aligned} \tag{2.7}$$

where M_2 and M_3 are appropriate constants such that $M_2 = \sup_{n\ge1}\{\|x_n - p\| + \|x_{n-1}-p\|\}$ and $M_3 = \sup_{n\ge1}\{\|u_n - p\|^2\}$. This implies that

$$\begin{aligned}&\alpha_n\beta_n g\left(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\right) \\ &\quad \le \left(\alpha_n + \beta_n k_{h(n)}^2\right)\left(\|x_{n-1}-p\|^2 - \|x_n-p\|^2\right) + \left(k_{h(n)}^2 - 1\right)\|x_n-p\|^2 \\ &\qquad + 2\xi_{h(n)}^2 + 2k_{h(n)}\xi_{h(n)}M_2 + \delta_n M_3.\end{aligned} \tag{2.8}$$

In view of the restrictions (a) and (b), we obtain that

$$\lim_{n\to\infty} g\left(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\right)=0. \tag{2.9}$$

Since $g:[0,\infty)\to[0,\infty)$ is a continuous, strictly increasing, and convex function with $g(0)=0$, we obtain that

$$\lim_{n\to\infty}\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|=0. \tag{2.10}$$

Next, we show that

$$\lim_{n\to\infty}\left\|T_{i(n)}^{h(n)}x_{n}-x_{n-1}\right\|=0. \tag{2.11}$$

From Lemma 1.3, we also see that

$$\begin{aligned}
\|x_n-p\|^2 &\le \alpha_n\|x_{n-1}-p\|^2+\beta_n\left\|S_{i(n)}^{h(n)}x_{n-1}-p\right\|^2+\gamma_n\left\|T_{i(n)}^{h(n)}x_n-p\right\|^2\\
&\quad+\delta_n\|u_n-p\|^2-\alpha_n\gamma_n g\left(\left\|T_{i(n)}^{h(n)}x_n-x_{n-1}\right\|\right)\\
&\le \alpha_n\|x_{n-1}-p\|^2+\beta_n\left(k_{h(n)}\|x_{n-1}-p\|+\xi_{h(n)}\right)^2+\gamma_n\left(k_{h(n)}\|x_n-p\|+\xi_{h(n)}\right)^2\\
&\quad+\delta_n\|u_n-p\|^2-\alpha_n\gamma_n g\left(\left\|T_{i(n)}^{h(n)}x_n-x_{n-1}\right\|\right)\\
&\le \alpha_n\|x_{n-1}-p\|^2+\beta_n\left(k_{h(n)}^2\|x_{n-1}-p\|^2+\xi_{h(n)}^2+2k_{h(n)}\xi_{h(n)}\|x_{n-1}-p\|\right)\\
&\quad+\gamma_n\left(k_{h(n)}^2\|x_n-p\|^2+\xi_{h(n)}^2+2k_{h(n)}\xi_{h(n)}\|x_n-p\|\right)\\
&\quad+\delta_n\|u_n-p\|^2-\alpha_n\gamma_n g\left(\left\|T_{i(n)}^{h(n)}x_n-x_{n-1}\right\|\right)\\
&\le\left(\alpha_n+\beta_nk_{h(n)}^2\right)\|x_{n-1}-p\|^2+\gamma_nk_{h(n)}^2\|x_n-p\|^2+2\xi_{h(n)}^2\\
&\quad+2k_{h(n)}\xi_{h(n)}M_2+\delta_nM_3-\alpha_n\gamma_n g\left(\left\|T_{i(n)}^{h(n)}x_n-x_{n-1}\right\|\right).
\end{aligned} \tag{2.12}$$

This implies that

$$\begin{aligned}
&\alpha_n\gamma_n g\left(\left\|T_{i(n)}^{h(n)}x_n-x_{n-1}\right\|\right)\\
&\quad\le\left(\alpha_n+\beta_nk_{h(n)}^2\right)\left(\|x_{n-1}-p\|^2-\|x_n-p\|^2\right)+\left(k_{h(n)}^2-1\right)\|x_n-p\|^2\\
&\qquad+2\xi_{h(n)}^2+2k_{h(n)}\xi_{h(n)}M_2+\delta_nM_3.
\end{aligned} \tag{2.13}$$

In view of the restrictions (a) and (b), we obtain that

$$\lim_{n\to\infty} g\left(\left\|T_{i(n)}^{h(n)}x_n - x_{n-1}\right\|\right) = 0. \tag{2.14}$$

Since $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, we obtain that (2.11) holds. Notice that

$$\|x_n - x_{n-1}\| \le \left\|S_{i(n)}^{h(n)}x_{n-1} - x_{n-1}\right\| + \left\|T_{i(n)}^{h(n)}x_n - x_{n-1}\right\| + \delta_n\|u_n - x_{n-1}\|. \tag{2.15}$$

In view of (2.10) and (2.11), we see from the restriction (b) that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0, \tag{2.16}$$

which implies that

$$\lim_{n\to\infty}\left\|x_n - x_{n+j}\right\| = 0, \quad \forall j \in \{1,2,\ldots,N\}. \tag{2.17}$$

Since for any positive integer $n > N$, it can be written as $n = (h(n)-1)N + i(n)$, where $i(n) \in \{1,2,\ldots,N\}$, observe that

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\le \left\|x_{n-1} - T_{i(n)}^{h(n)}x_n\right\| + \left\|T_{i(n)}^{h(n)}x_n - T_n x_n\right\| \\
&\le \left\|x_{n-1} - T_{i(n)}^{h(n)}x_n\right\| + L_t\left\|T_{i(n)}^{h(n)-1}x_n - x_n\right\| \\
&\le \left\|x_{n-1} - T_{i(n)}^{h(n)}x_n\right\| \\
&\quad + L_t\left(\left\|T_{i(n)}^{h(n)-1}x_n - T_{i(n-N)}^{h(n)-1}x_{n-N}\right\| + \left\|T_{i(n-N)}^{h(n)-1}x_{n-N} - x_{(n-N)-1}\right\|\right. \\
&\quad\quad \left.+\left\|x_{(n-N)-1} - x_n\right\|\right).
\end{aligned} \tag{2.18}$$

Since for each $n > N$, $n = (n-N)(\operatorname{mod} N)$, on the other hand, we obtain from $n = (h(n) - 1)N + i(n)$ that $n - N = ((h(n)-1)-1)N + i(n) = (h(n-N)-1)N + i(n-N)$. That is,

$$h(n-N) = h(n) - 1, \qquad i(n-N) = i(n). \tag{2.19}$$

Notice that

$$\begin{aligned}
\left\|T_{i(n)}^{h(n)-1}x_n - T_{i(n-N)}^{h(n)-1}x_{n-N}\right\| &= \left\|T_{i(n)}^{h(n)-1}x_n - T_{i(n)}^{h(n)-1}x_{n-N}\right\| \\
&\le L_t\|x_n - x_{n-N}\|, \\
\left\|T_{i(n-N)}^{h(n)-1}x_{n-N} - x_{(n-N)-1}\right\| &= \left\|T_{i(n-N)}^{h(n-N)}x_{n-N} - x_{(n-N)-1}\right\|.
\end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.18), we arrive at

$$\begin{aligned}\|x_{n-1}-T_nx_n\| &\le \left\|x_{n-1}-T_{i(n)}^{h(n)}x_n\right\| \\ &\quad + L_t\Big(L_t\|x_n-x_{n-N}\| + \left\|T_{i(n-N)}^{h(n-N)}x_{n-N}-x_{(n-N)-1}\right\| + \left\|x_{(n-N)-1}-x_n\right\|\Big).\end{aligned} \tag{2.21}$$

In view of (2.11) and (2.17), we obtain that

$$\lim_{n\to\infty}\|x_{n-1}-T_nx_n\| = 0. \tag{2.22}$$

Notice that

$$\|x_n-T_nx_n\| \le \|x_n-x_{n-1}\| + \|x_{n-1}-T_nx_n\|. \tag{2.23}$$

It follows from (2.16) and (2.22) that

$$\lim_{n\to\infty}\|x_n-T_nx_n\| = 0. \tag{2.24}$$

Notice that

$$\begin{aligned}\left\|x_n-T_{n+j}x_n\right\| &\le \left\|x_n-x_{n+j}\right\| + \left\|x_{n+j}-T_{n+j}x_{n+j}\right\| + \left\|T_{n+j}x_{n+j}-T_{n+j}x_n\right\| \\ &\le (1+L_t)\left\|x_n-x_{n+j}\right\| + \left\|x_{n+j}-T_{n+j}x_{n+j}\right\|, \quad \forall j\in\{1,2,\ldots,N\}.\end{aligned} \tag{2.25}$$

From (2.17) and (2.24), we arrive at

$$\lim_{n\to\infty}\left\|x_n-T_{n+j}x_n\right\| = 0, \qquad \forall j\in\{1,2,\ldots,N\}. \tag{2.26}$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n\to\infty}\|x_n-T_rx_n\| = 0, \qquad \forall r\in\{1,2,\ldots,N\}. \tag{2.27}$$

Letting $L_s = \max\{L_{s,i} : 1\le i\le N\}$, we have

$$\begin{aligned}\left\|S_{i(n)}^{h(n)}x_n-x_{n-1}\right\| &\le \left\|S_{i(n)}^{h(n)}x_n-S_{i(n)}^{h(n)}x_{n-1}\right\| + \left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\| \\ &\le L_s\|x_n-x_{n-1}\| + \left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|.\end{aligned} \tag{2.28}$$

In view of (2.10) and (2.16), we see that

$$\lim_{n\to\infty}\left\|S_{i(n)}^{h(n)}x_n-x_{n-1}\right\| = 0. \tag{2.29}$$

Observe that

$$\begin{aligned}\|x_{n-1}-S_nx_{n-1}\| &\le \left\|x_{n-1}-S_{i(n)}^{h(n)}x_{n-1}\right\|+\left\|S_{i(n)}^{h(n)}x_{n-1}-S_nx_{n-1}\right\| \\ &\le \left\|x_{n-1}-S_{i(n)}^{h(n)}x_{n-1}\right\|+L_s\left\|S_{i(n)}^{h(n)-1}x_{n-1}-x_{n-1}\right\| \\ &\le \left\|x_{n-1}-S_{i(n)}^{h(n)}x_{n-1}\right\| \\ &\quad + L_s\left(\left\|S_{i(n)}^{h(n)-1}x_{n-1}-S_{i(n-N)}^{h(n)-1}x_{n-N}\right\|+\left\|S_{i(n-N)}^{h(n)-1}x_{n-N}-x_{(n-N)-1}\right\|\right. \\ &\qquad \left.+\left\|x_{(n-N)-1}-x_{n-1}\right\|\right).\end{aligned} \tag{2.30}$$

In view of

$$\begin{aligned}\left\|S_{i(n)}^{h(n)-1}x_{n-1}-S_{i(n-N)}^{h(n)-1}x_{n-N}\right\| &= \left\|S_{i(n)}^{h(n)-1}x_{n-1}-S_{i(n)}^{h(n)-1}x_{n-N}\right\| \\ &\le L_s\|x_{n-1}-x_{n-N}\|, \\ \left\|S_{i(n-N)}^{h(n)-1}x_{n-N}-x_{(n-N)-1}\right\| &= \left\|S_{i(n-N)}^{h(n-N)}x_{n-N}-x_{(n-N)-1}\right\|,\end{aligned} \tag{2.31}$$

we arrive at

$$\begin{aligned}\|x_{n-1}-S_nx_{n-1}\| \le{}& \left\|x_{n-1}-S_{i(n)}^{h(n)}x_{n-1}\right\| \\ &+L_s\left(L_s\|x_{n-1}-x_{n-N}\|+\left\|S_{i(n-N)}^{h(n-N)}x_{n-N}-x_{(n-N)-1}\right\|+\left\|x_{(n-N)-1}-x_{n-1}\right\|\right).\end{aligned} \tag{2.32}$$

In view of (2.10), (2.17), and (2.29), we obtain that

$$\lim_{n\to\infty}\|x_{n-1}-S_nx_{n-1}\|=0. \tag{2.33}$$

Notice that

$$\begin{aligned}\|x_n-S_nx_n\| &\le \|x_n-x_{n-1}\|+\|x_{n-1}-S_nx_{n-1}\|+\|S_nx_{n-1}-S_nx_n\| \\ &\le (1+L_s)\|x_n-x_{n-1}\|+\|x_{n-1}-S_nx_{n-1}\|.\end{aligned} \tag{2.34}$$

From (2.16) and (2.33), we see that

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0. \tag{2.35}$$

On the other hand, we have

$$\begin{aligned}\left\|x_n-S_{n+j}x_n\right\| &\le \left\|x_n-x_{n+j}\right\|+\left\|x_{n+j}-S_{n+j}x_{n+j}\right\|+\left\|S_{n+j}x_{n+j}-S_{n+j}x_n\right\| \\ &\le (1+L_s)\left\|x_n-x_{n+j}\right\|+\left\|x_{n+j}-S_{n+j}x_{n+j}\right\|, \quad \forall j\in\{1,2,\ldots,N\}.\end{aligned} \tag{2.36}$$

It follows from (2.17) and (2.35) that

$$\lim_{n\to\infty} \|x_n - S_{n+j}x_n\| = 0, \quad \forall j \in \{1, 2, \dots, N\}. \tag{2.37}$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n\to\infty} \|x_n - S_r x_n\| = 0, \quad \forall r \in \{1, 2, \dots, N\}. \tag{2.38}$$

This completes the proof. □

Recall that a mapping $T : C \to C$ is said to be *semicompact* if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x \in C$.

Next, we give strong convergence theorems with the help of the semicompactness.

Theorem 2.2. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \le i \le N$, and let $S_i : C \to C$ be a uniformly $L_{s,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \le i \le N$. Assume that $F = \cap_{i=1}^{N} F(T_i) \bigcap \cap_{i=1}^{N} F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \le i \le N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \le i \le N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \le i \le N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \ge 1$. Let $\{x_n\}$ be a sequence generated in* (1.18). *Assume that the following restrictions are satisfied:*

(a) *there exist constants $a, b, c, d \in (0, 1)$ such that $a \le \alpha_n$, $b \le \beta_n$, and $c \le \gamma_n \le d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \le i \le N\}$, for all $n \ge 1$;*

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \dots, S_N\}$ or one of $\{T_1, T_2, \dots, T_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F.

Proof. Without loss of generality, we may assume that S_1 is semicompact. From (2.38), we see that there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging strongly to $x \in C$. For each $r \in \{1, 2, \dots, N\}$, we get that

$$\|x - S_r x\| \le \|x - x_{n_i}\| + \|x_{n_i} - S_r x_{n_i}\| + \|S_r x_{n_i} - S_r x\|. \tag{2.39}$$

Since S_r is Lipshcitz continuous, we obtain from (2.38) that $x \in \cap_{r=1}^{N} F(S_r)$. Notice that

$$\|x - T_r x\| \le \|x - x_{n_i}\| + \|x_{n_i} - T_r x_{n_i}\| + \|T_r x_{n_i} - T_r x\|. \tag{2.40}$$

Since T_r is Lipshcitz continuous, we obtain from (2.27) that $x \in \cap_{r=1}^{N} F(T_r)$. This means that $x \in F$. In view of Lemma 2.1, we obtain that $\lim_{n\to\infty}\|x_n - x\|$ exists. Therefore, we can obtain the desired conclusion immediately. □

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.3. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \le i \le N$. Assume that $F = \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_{n,t} = \max\{k_{n,t,i} : 1 \le i \le N\}$, and $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \ge 1$. Let $\{x_n\}$ be a sequence generated in* (1.19)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants $a, b, c \in (0, 1)$ such that $a \le \alpha_n + \beta_n$ and $b \le \gamma_n \le c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \le i \le N\}$, for all $n \ge 1$;*

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{T_1, T_2, \ldots, T_N\}$ is semicompact, then the sequence converges $\{x_n\}$ strongly to some point in F.

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.4. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $S_i : C \to C$ be a uniformly $L_{s,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \le i \le N$. Assume that $F = \cap_{i=1}^{N} F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_{n,s} = \max\{k_{n,s,i} : 1 \le i \le N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \ge 1$. Let $\{x_n\}$ be a sequence generated in* (1.20)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants $a, b, c, d \in (0, 1)$ such that $a \le \alpha_n$, $b \le \beta_n$, and $c \le \gamma_n$, for all $n \ge 1$;*

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \ldots, S_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F.

In 2005, Chidume and Shahzad [11] introduced the following conception. Recall that a family $\{T_i\}_{i=1}^{N} : C \to C$ with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy *Condition* (B) on C if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(m) > 0$ for all $m \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \le i \le N}\{\|x - T_i x\|\} \ge f(d(x, F)). \tag{2.41}$$

Based on Condition (B), we introduced the following conception for two finite families of mappings. Recall that two families $\{S_i\}_{i=1}^{N} : C \to C$ and $\{T_i\}_{i=1}^{N} : C \to C$ with $F = \cap_{i=1}^{N} F(S_i) \bigcap \cap_{i=1}^{N} F(T_i) \neq \emptyset$ are said to satisfy Condition (B') on C if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(m) > 0$ for all $m \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \le i \le N}\{\|x - S_i x\|\} + \max_{1 \le i \le N}\{\|x - T_i x\|\} \ge f(d(x, F)). \tag{2.42}$$

Next, we give strong convergence theorems with the help of Condition (B').

Theorem 2.5. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1,\infty)$ and $\{\xi_{n,t,i}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,t,i}-1) < \infty$ and $\sum_{n=1}^{\infty}\xi_{n,t,i} < \infty$ for each $1 \le i \le N$, and let $S_i : C \to C$ be a uniformly $L_{s,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1,\infty)$ and $\{\xi_{n,s,i}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,s,i}-1) < \infty$ and $\sum_{n=1}^{\infty}\xi_{n,s,i} < \infty$ for each $1 \le i \le N$. Assume that $F = \cap_{i=1}^{N}F(T_i)\bigcap\cap_{i=1}^{N}F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \le i \le N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \le i \le N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \le i \le N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$ for each $n \ge 1$. Let $\{x_n\}$ be a sequence generated in* (1.18). *Assume that the following restrictions are satisfied:*

(a) *there exist constants $a,b,c,d \in (0,1)$ such that $a \le \alpha_n$, $b \le \beta_n$, and $c \le \gamma_n \le d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \le i \le N\}$, for all $n \ge 1$;*

(b) $\sum_{n=1}^{\infty}\delta_n < \infty$.

If $\{S_1, S_2, \dots, S_N\}$ and $\{T_1, T_2, \dots, T_N\}$ satisfy Condition (B'), then the sequence converges strongly to some point in F.

Proof. In view of Condition (B'), we obtain from (2.27) and (2.38) that $f(d(x_n,F)) \to 0$, which implies $d(x_n,F) \to 0$. Next, we show that the sequence $\{x_n\}$ is Cauchy. In view of (2.6), for any positive integers m,n, where $m > n > n_0$, we see that

$$\|x_m - p\| \le B\|x_n - p\| + B\sum_{i=n+1}^{\infty}M_1(\delta_i + \xi_{h(i)}) + M_1(\delta_m + \xi_{h(m)}), \tag{2.43}$$

where $B = \exp\{\sum_{n=1}^{\infty}(k_{h(n)}-1)/(1-R)\}$. It follows that

$$\begin{aligned}\|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\ &\le (1+B)\|x_n - p\| + B\sum_{i=n+1}^{\infty}M_1(\delta_i + \xi_{h(i)}) + M_1(\delta_m + \xi_{h(m)}).\end{aligned} \tag{2.44}$$

It follows that $\{x_n\}$ is a Cauchy sequence in C and so $\{x_n\}$ converges strongly to some $\overline{q} \in C$. Since T_r and S_r are Lipschitz for each $r \in \{1,2,\dots,N\}$, we see that F is closed. This in turn implies that $\overline{q} \in F$. This completes the proof. □

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1,2,\dots,N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.6. *Let E be a real uniformly convex uniformly convex Banach space and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1,\infty)$ and $\{\xi_{n,t,i}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,t,i}-1) < \infty$ and $\sum_{n=1}^{\infty}\xi_{n,t,i} < \infty$ for each $1 \le i \le N$. Assume that $F = \cap_{i=1}^{N}F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_{n,t} = \max\{k_{n,t,i} : 1 \le i \le N\}$ and where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that*

$\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.19). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b, c \in (0,1)$ *such that* $a \leq \alpha_n + \beta_n$ *and* $b \leq \gamma_n \leq c < 1/L_t$, *where* $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, *for all* $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{T_1, T_2, \ldots, T_N\}$ *satisfies Condition* (B), *then the sequence* $\{x_n\}$ *converges strongly to some point in* F.

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.7. *Let* E *be a real uniformly convex Banach space and* C *a nonempty closed convex subset of* E. *Let* $S_i : C \to C$ *be a uniformly* $L_{s,i}$*-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences* $\{k_{n,s,i}\} \subset [1, \infty)$ *and* $\{\xi_{n,s,i}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,s,i} - 1) < \infty$ *and* $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ *for each* $1 \leq i \leq N$. *Assume that* $F = \cap_{i=1}^{N} F(S_i)$ *is nonempty. Let* $\{u_n\}$ *be a bounded sequence in* C, $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$, *and* $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.20). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b, c, d \in (0,1)$ *such that* $a \leq \alpha_n$, $b \leq \beta_n$ *and* $c \leq \gamma_n$, *for all* $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{S_1, S_2, \ldots, S_N\}$ *satisfies Condition* (B), *then the sequence* $\{x_n\}$ *converges strongly to some point in* F.

Finally, we give a strong convergence theorem criterion.

Theorem 2.8. *Let* E *be a real Banach space and* C *a nonempty closed convex subset of* E. *Let* $T_i : C \to C$ *be a uniformly* $L_{t,i}$*-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences* $\{k_{n,t,i}\} \subset [1, \infty)$ *and* $\{\xi_{n,t,i}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,t,i} - 1) < \infty$ *and* $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ *for each* $1 \leq i \leq N$, *and let* $S_i : C \to C$ *be a uniformly* $L_{s,i}$*-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences* $\{k_{n,s,i}\} \subset [1, \infty)$ *and* $\{\xi_{n,s,i}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,s,i} - 1) < \infty$ *and* $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ *for each* $1 \leq i \leq N$. *Assume that* $F = \cap_{i=1}^{N} F(T_i) \bigcap \cap_{i=1}^{N} F(S_i)$ *is nonempty. Let* $\{u_n\}$ *be a bounded sequence in* C, $k_n = \max\{k_{n,t}, k_{n,s}\}$, *where* $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ *and* $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$ *and* $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, *where* $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$ *and* $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.18). *Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b, c, d \in (0,1)$ *such that* $a \leq \alpha_n$, $b \leq \beta_n$, *and* $c \leq \gamma_n \leq d < 1/L_t$, *where* $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, *for all* $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ *converges strongly to some point in* F *if and only if* $\liminf_{n \to \infty} d(x_n, F) = 0$.

Proof. The necessity is obvious. We only show the sufficiency. Assume that

$$\liminf_{n \to \infty} d(x_n, \mathcal{F}) = 0. \tag{2.45}$$

For each $p \in F$, we see that

$$\begin{aligned}\|x_n - p\| &\le \alpha_n \|x_{n-1} - p\| + \beta_n \left\|S_{i(n)}^{h(n)} x_{n-1} - p\right\| + \gamma_n \left\|T_{i(n)}^{h(n)} x_n - p\right\| + \delta_n \|u_n - p\| \\ &\le \alpha_n \|x_{n-1} - p\| + \beta_n \left(k_{h(n)} \|x_{n-1} - p\| + \xi_{h(n)}\right) + \gamma_n \left(k_{h(n)} \|x_n - p\| + \xi_{h(n)}\right) \\ &\quad + \delta_n \|u_n - p\| \\ &\le \left(\alpha_n + \beta_n k_{h(n)}\right) \|x_{n-1} - p\| + (1 - \alpha_n - \beta_n) k_{h(n)} \|x_n - p\| + 2\xi_{h(n)} \\ &\quad + \delta_n \|u_n - x_n\|.\end{aligned} \tag{2.46}$$

Notice that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. We see from the restrictions (a) and (b) that there exists a positive integer n_0 such that

$$(1 - \alpha_n - \beta_n) k_{h(n)} \le R < 1, \quad \forall n \ge n_0, \tag{2.47}$$

where $R = (1 - (a + b))(1 + (a + b)/(2 - 2(a + b)))$. Notice that the sequence $\{x_n\}$ is bounded. It follows from (2.46) that

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n + \beta_n k_{h(n)}}{1 - (1 - \alpha_n - \beta_n) k_{h(n)}} \|x_{n-1} - p\| + \frac{\delta_n}{1 - (1 - \alpha_n - \beta_n) k_{h(n)}} \|u_n - x_n\| \\ &\quad + \frac{2\xi_{h(n)}}{1 - (1 - \alpha_n - \beta_n) k_{h(n)}} \\ &\le \left(1 + \frac{k_{h(n)} - 1}{1 - R}\right) \|x_{n-1} - p\| + \frac{\delta_n}{1 - R} \|u_n - x_n\| + \frac{2\xi_{h(n)}}{1 - R} \\ &\le \left(1 + \frac{k_{h(n)} - 1}{1 - R}\right) \|x_{n-1} - p\| + M_4 \left(\delta_n + \xi_{h(n)}\right), \quad \forall n \ge n_0,\end{aligned} \tag{2.48}$$

where M_4 is an appropriate constant such that $M_4 = \max\{\sup_{n\ge1}\{\|u_n - x_n\|/(1 - R)\}, 2/(1 - R)\}$. In view of the restrictions (a) and (b), we obtain from Lemma 1.2 that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. This implies that

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0. \tag{2.49}$$

In view of Theorem 2.5, we can conclude the desired conclusion easily. □

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.9. *Let E be a real Banach space and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \le i \le N$. Assume that $F = \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C, $k_{n,t} = \max\{k_{n,t,i} : 1 \le i \le N\}$ and where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \le i \le N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$*

and $\{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$*. Let* $\{x_n\}$ *be a sequence generated in* (1.19)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b, c \in (0,1)$ *such that* $a \leq \alpha_n + \beta_n$ *and* $b \leq \gamma_n \leq c < 1/L_t$*, where* $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$*, for all* $n \geq 1$*;*

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ *converges strongly to some point in* F *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.10. *Let* E *be a real Banach space and* C *a nonempty closed convex subset of* E*. Let* $S_i : C \to C$ *be a uniformly* $L_{s,i}$*-Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences* $\{k_{n,s,i}\} \subset [1,\infty)$ *and* $\{\xi_{n,s,i}\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,s,i} - 1) < \infty$ *and* $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ *for each* $1 \leq i \leq N$*. Assume that* $F = \cap_{i=1}^{N} F(S_i)$ *is nonempty. Let* $\{u_n\}$ *be a bounded sequence in* C*,* $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$*, and* $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*, and* $\{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for each* $n \geq 1$*. Let* $\{x_n\}$ *be a sequence generated in* (1.20)*. Assume that the following restrictions are satisfied:*

(a) *there exist constants* $a, b, c, d \in (0,1)$ *such that* $a \leq \alpha_n$*,* $b \leq \beta_n$*, and* $c \leq \gamma_n$*, for all* $n \geq 1$*;*

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ *converges strongly to some point in* F *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

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