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Research Article

Ray's Theorem for Firmly Nonexpansive-Like Mappings and Equilibrium Problems in Banach Spaces

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We prove that every firmly nonexpansive-like mapping from a closed convex subset C of a smooth, strictly convex and reflexive Banach pace into itself has a fixed point if and only if C is bounded. We obtain a necessary and sufficient condition for the existence of solutions of an equilibrium problem and of a variational inequality problem defined in a Banach space.

1. Introduction

Let C be a subset of a Banach space E. A mapping $T : C \to E$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1965, it was proved independently by Browder [1], Göhde [2], and Kirk [3] that if C is a bounded closed convex subset of a Hilbert space and $T : C \to C$ is nonexpansive, then T has a fixed point. Combining the results above, Ray [4] obtained the following interesting result (see [5] for a simpler proof).

Theorem 1.1. *Let C be a closed and convex subset of a Hilbert space. Then the following statements are equivalent:*

(i) $\mathrm{Fix}(T) := \{x \in C : x = Tx\} \neq \varnothing$ *for every nonexpansive mapping $T : C \to C$;*

(ii) *C is bounded.*

It is well known that, for each subset C of a Hilbert space H, a mapping $T : C \to H$ is nonexpansive if and only if $S := (1/2)(I + T)$ is *firmly nonexpansive*, that is, the following inequality is satisfied by all $x, y \in C$:

$$\langle Sx - Sy, (x - Sx) - (y - Sy) \rangle \geq 0. \tag{1.1}$$

In this case, $\mathrm{Fix}(T) = \mathrm{Fix}(S)$. We can restate Ray's theorem in the following form.

Theorem 1.2. *Let C be a closed and convex subset of a Hilbert space. Then the following statements are equivalent:*

(i) $\mathrm{Fix}(S) \neq \emptyset$ *for every firmly nonexpansive mapping* $S : C \to C$*;*

(ii) C *is bounded.*

To extend this result to the framework of Banach spaces, let us recall some definitions and related known facts. The value of x^* in the dual space E^* of a Banach space E at $x \in E$ is denoted by $\langle x, x^* \rangle$. We assume from now on that a Banach space E is *smooth*, that is, the limit $\lim_{t \to 0}(1/t)(\|x + ty\| - \|x\|)$ exists for all norm one elements $x, y \in E$. This implies that the *duality mapping* J from E to 2^{E^*} defined by

$$x \longmapsto Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \tag{1.2}$$

is single-valued and we do consider the singleton Jx as an element in E^*. If E is additionally assumed to be *strictly convex*, that is, there are no distinct elements $x, y \in E$ such that $\|x\| = \|y\| = (1/2)\|x + y\| = 1$, then J is one-to-one. Let us note here that if E is a Hilbert space, then the duality mapping is just the identity mapping.

The following three generalizations of firmly nonexpansive mappings in Hilbert spaces were introduced by Aoyama et al. [6]. For a subset C of a (smooth) Banach space E, a mapping $T : C \to E$ is of

(i) *type (P)* (or *firmly nonexpansive-like*) if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \geq 0 \quad \forall x, y \in C, \tag{1.3}$$

(ii) *type (Q)* (or *firmly nonexpansive type*) if

$$\langle Tx - Ty, (Jx - JTx) - (Jy - JTy) \rangle \geq 0 \quad \forall x, y \in C, \tag{1.4}$$

(iii) *type (R)* (or *firmly generalized nonexpansive*) if

$$\langle (x - Tx) - (y - Ty), JTx - JTy \rangle \geq 0 \quad \forall x, y \in C. \tag{1.5}$$

Recently, Takahashi et al. [7] successfully proved the following theorem.

Theorem 1.3. *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space. Then the following statements are equivalent:*

(i) $\mathrm{Fix}(T) \neq \emptyset$ *for every mapping $T : C \to C$ which is of type (Q);*

(ii) *C is bounded.*

As a direct consequence of the duality theorem [8], we obtain the following result (see also [9]).

Theorem 1.4. *Let C be a closed subset of a smooth, strictly convex and reflexive Banach space such that JC is closed and convex. Then the following statements are equivalent:*

(i) $\mathrm{Fix}(T) \neq \emptyset$ *for every mapping $T : C \to C$ which is of type (R);*

(ii) *C is bounded.*

The purpose of this short paper is to prove the analogue of these results for mappings of type (P). Let us note that our result is different from the existence theorems obtained recently by Aoyama and Kohsaka [10]. We also obtain a necessary and sufficient condition for the existence of solutions of certain equilibrium problems and of variational inequality problems in Banach spaces.

2. Ray's Theorem for Mappings of Type (P) and Equilibrium Problems

The following result was proved by Aoyama et al. [6].

Theorem 2.1. *Let E be a smooth, strictly convex and reflexive Banach space, and let C be a bounded, closed and convex subset of E. If a mapping $T : C \to C$ is of type (P), then T has a fixed point.*

Let C be a closed and convex subset of a Banach space E. An *equilibrium problem* for a bifunction $f : C \times C \to \mathbb{R}$ is the problem of finding an element $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \quad \forall y \in C. \tag{2.1}$$

We denote the set of solutions of the equilibrium problem for f by $\mathrm{EP}(f)$. We assume that a bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions (see [11]):

(C1) $f(x, x) = 0$ for all $x \in C$;

(C2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(C3) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$;

(C4) f is *maximal monotone*, that is, for each $x \in C$ and $x^* \in E^*$,

$$f(x, y) + \langle y - x, x^* \rangle \geq 0 \quad \forall y \in C \tag{2.2}$$

whenever $\langle z - x, x^* \rangle \geq f(z, x)$ for all $z \in C$.

Remark 2.2. It is noted (see [12]) that if f satisfies conditions (C1)–(C3) and the following condition:

(C4′) $\limsup_{t\downarrow 0} f((1-t)x+tz,y) \le f(x,y)$ for all $x,y,z \in C$,

then f satisfies condition (C4).

Lemma 2.3 (see [12]). *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space E and $f : C \times C \to \mathbb{R}$ satisfy conditions* (C1)–(C4). *Then for each $x \in E$, there exists a unique element $z \in C$ such that*

$$f(z,y) + \langle y - z, J(z-x)\rangle \ge 0 \quad \forall y \in C. \tag{2.3}$$

Employing the methods in [5, 7], we obtain the following result.

Theorem 2.4. *Let E be a smooth, strictly convex and reflexive Banach space and C a closed and convex subset of E. The following statements are equivalent.*

(a) $\mathrm{Fix}(T) \ne \varnothing$ *for every mapping $T : C \to C$ which is of type (P);*

(b) $\mathrm{EP}(f) \ne \varnothing$ *for every bifunction $f : C \times C \to \mathbb{R}$ satisfying conditions* (C1)–(C4);

(c) $\mathrm{EP}(f) \ne \varnothing$ *for every bifunction $f : C\times C \to \mathbb{R}$ satisfying conditions* (C1)–(C3) *and* (C4′);

(d) *C is bounded.*

Proof. (a)⇒(b) Assume that a bifunction $f : C \times C \to \mathbb{R}$ satisfies conditions (C1)–(C4). We define $T : E \to C$ by $x \mapsto Tx = z \in C$ where z is given by Lemma 2.3. The mapping T is of type (P). In fact, for $x, x' \in E$, we have $Tx, Tx' \in C$ and hence

$$\begin{gathered} f(Tx,Tx') + \langle Tx' - Tx, J(Tx - x)\rangle \ge 0, \\ f(Tx',Tx) + \langle Tx - Tx', J(Tx' - x')\rangle \ge 0. \end{gathered} \tag{2.4}$$

By the condition (C2),

$$\langle Tx - Tx', J(x - Tx) - J(x' - Tx')\rangle \ge 0. \tag{2.5}$$

In particular, the restriction of T to the closed and convex subset C is of type (P). It then follows from (a) that $EP(f) = \mathrm{Fix}(T) \ne \varnothing$.

(b)⇒(c) It follows directly from Remark 2.2.

(c)⇒(d) We suppose that C is not bounded. By the uniform boundedness theorem, there exists an element $x^* \in E^*$ such that $\inf\{\langle x, x^*\rangle : x \in C\} = -\infty$. We define $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = \langle y - x, x^*\rangle \quad \forall x, y \in C. \tag{2.6}$$

It is clear that f satisfies conditions (C1)–(C3) and (C4′). Moreover, $EP(f) = \varnothing$ since

$$\begin{aligned} p \in EP(f) &\Longleftrightarrow \langle y - p, x^* \rangle \geq 0 \quad \forall y \in C \\ &\Longleftrightarrow -\infty = \inf\{\langle y, x^* \rangle : y \in C\} \geq \langle p, x^* \rangle. \end{aligned} \tag{2.7}$$

(d)⇒(a) This is Theorem 2.1. □

Let C be a subset of a Banach space E. We now discuss a *variational inequality problem* for a mapping $A : C \to E^*$, that is, the problem of finding an element $\widehat{x} \in C$ such that $\langle y - \widehat{x}, A\widehat{x} \rangle \geq 0$ for all $y \in C$ and the set of solutions of this problem is denoted by $\mathrm{VI}(C, A)$. Recall that a mapping $A : C \to E^*$ is said to be

(i) *monotone* if $\langle x - y, Ax - Ay \rangle \geq 0$ for all $x, y \in C$;

(ii) *hemicontinuous* if for each $x, y \in C$ the mapping $t \mapsto A((1-t)x + ty)$, where $t \in [0, 1]$, is continuous with respect to the weak∗ topology of E^*;

(iii) *demicontinuous* if $\{Ax_n\}$ converges to Ax with respect to the weak∗ topology of E^* whenever $\{x_n\}$ is a sequence in C such that it converges strongly to $x \in C$.

It is known that if C is a nonempty weakly compact and convex subset of a reflexive Banach space E and $A : C \to E^*$ is monotone and hemicontinuous, then $\mathrm{VI}(C, A) \neq \varnothing$ (see e.g., [13]).

As a consequence of Theorem 2.4, we obtain a necessary and sufficient condition for the existence of solutions of a variational inequality problem.

Corollary 2.5. *Let E be a reflexive Banach space and C a nonempty, closed and convex subset of E. Then the following statements are equivalent:*

(a) $\mathrm{VI}(C, A) \neq \varnothing$ *for every monotone and hemicontinuous mapping* $A : C \to E^*$;

(b) $\mathrm{VI}(C, A) \neq \varnothing$ *for every monotone and demicontinuous mapping* $A : C \to E^*$;

(c) C *is bounded.*

Proof. (a)⇒(b) It is clear since demicontinuity implies hemicontinuity.

(b)⇒(c) To see this, let us note that there is an equivalent norm on E such that the underlying space equipped with this new norm is smooth and strictly convex (see [14, 15]). Moreover, the monotonicity and demicontinuity of any mapping $A : C \to E^*$ remain unaltered with respect to this renorming. We now assume in addition that E is smooth and strictly convex. Suppose that C is not bounded. By Theorem 2.4, there exists a fixed point-free mapping $T : C \to C$ such that it is of type (P). We define $A : C \to E^*$ by

$$Ax = J(x - Tx) \quad \forall x \in C. \tag{2.8}$$

For each $x, y \in C$, we have $\langle x - y, Ax - Ay \rangle = \langle x - y, J(x - Tx) - J(y - Ty) \rangle \geq 0$, that is, A is monotone. Moreover, it is proved in [6, Theorem 7.3] that A is demicontinuous. Therefore, $\mathrm{VI}(C, A) = \mathrm{Fix}(T) = \varnothing$.

(c)⇒(a) It is a corollary of [13, Theorem 7.1.8.] □

We finally discuss an equilibrium problem defined in the dual space of a Banach space. This problem was initiated by Takahashi and Zembayashi [16]. Let C be a closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. We assume that a bifunction $f^* : JC \times JC \to \mathbb{R}$ satisfies the following conditions:

(D1) $f^*(Jx, Jx) = 0$ for all $x \in C$;

(D2) $f^*(Jx, Jy) + f^*(Jy, Jx) \leq 0$ for all $x, y \in C$;

(D3) $f^*(Jx, \cdot)$ is convex and lower semicontinuous for all $x \in C$;

(D4) f^* is *maximal monotone* (with respect to JC), that is, for each $x \in C$ and $u \in E$,

$$f^*(Jx, Jy) + \langle u, Jy - Jx \rangle \geq 0 \quad \forall y \in C \tag{2.9}$$

whenever $\langle u, Jz - Jx \rangle \geq f^*(Jz, Jx)$ for all $z \in C$.

In [16], a bifunction is assumed to satisfy conditions (D1)–(D3) and

(D4′) $\limsup_{t \downarrow 0} f^*((1-t)Jx + tJz, Jy) \leq f^*(Jx, Jy)$ for all $x, y, z \in C$.

We are interested in the problem of finding an element $\hat{x} \in C$ such that

$$f^*(J\hat{x}, Jy) \geq 0 \quad \forall y \in C \tag{2.10}$$

and the set of solutions of this problem is denoted by $\mathrm{EP}^*(f^*)$.

The following lemma was proved by Takahashi and Zembayashi ([16], Lemma 2.10) where the bifunction satisfies conditions (D1)–(D3) and (D4′). However, it can be proved that the conclusion remains true under the conditions (D1)–(D4). We also note that the uniform smoothness assumption on a space in [16, Lemma 2.10] is a misprint.

Lemma 2.6. *Let C be a closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. Suppose that a bifunction $f^* : JC \times JC \to \mathbb{R}$ satisfies conditions* (D1)–(D4)*. Then for each $x \in E$ there exists a unique element $z \in C$ such that*

$$f^*(Jz, Jy) + \langle z - x, Jy - Jz \rangle \geq 0 \quad \forall y \in C. \tag{2.11}$$

Moreover, if $T : E \to C$ is defined by $x \mapsto Tx = z$ where z is given above, then T is of type (R).

Based on the preceding lemma and Theorem 2.4, we obtain the result whose proof is omitted.

Theorem 2.7. *Let E be a smooth, strictly convex and reflexive Banach space, and let C be a closed subset of E such that JC is closed and convex. The following statements are equivalent:*

(i) $\mathrm{Fix}(T) \neq \emptyset$ *for every mapping $T : C \to C$ which is of type (R);*

(ii) $\mathrm{EP}^*(f^*) \neq \emptyset$ *for every bifunction $f^* : JC \times JC \to \mathbb{R}$ satisfying conditions* (D1)–(D4)*;*

(iii) $\mathrm{EP}^*(f^*) \neq \emptyset$ *for every bifunction $f^* : JC \times JC \to \mathbb{R}$ satisfying conditions* (D1)–(D3) *and* (D4′)*;*

(iv) *C is bounded.*

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