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Research Article

Fixed Point Theorem of Half-Continuous Mappings on Topological Vector Spaces

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Some fixed point theorems of half-continuous mappings which are possibly discontinuous defined on topological vector spaces are presented. The results generalize the work of Philippe Bich (2006) and several well-known results.

1. Introduction

Almost a century ago, L. E. J. Brouwer proved a famous theorem in fixed point theory, that any continuous mapping from the closed unit ball of the Euclidean space $\mathbb{R}^n$ to itself has a fixed point. Later in 1930, J. Schauder extended Brouwer's theorem to Banach spaces (see [1]).

In 2008, Herings et al. (see [2]) proposed a new type of mapping which is possibly discontinuous. They called such mappings *locally gross direction preserving* and proved that every locally gross direction preserving mapping defined on a nonempty polytope (the convex hull of a finite subset of $\mathbb{R}^n$) has a fixed point. Their work both allows discontinuities of mappings and generalizes Brouwer's theorem.

Later, Bich (see [3]) extended the work of Herings et al. to an arbitrary nonempty compact convex subset of $\mathbb{R}^n$. Moreover, in [4], Bich established a new class of mappings which contains the class of locally gross direction preserving mappings. He called the mappings in that class *half-continuous* and proved that if C is a nonempty compact convex subset of a Banach space and $f : C \to C$ is half-continuous, then f has a fixed point. Furthermore, in the same work, Bich extended the notion of half-continuity to multivalued mappings and proved fixed point theorems which generalize several well-known results.

All vector spaces considered are *real* vector spaces. In this paper, we prove that some results of Bich (see [4]) are also valid in locally convex Hausdorff topological vector spaces and also show that several well-known theorems can be obtained from our results. The paper is organized as follows. In Section 2, some notations, terminologies, and fundamental facts are reviewed. Sections 3 and 4, the fixed point theorems are proved. Finally, in Section 5, we give some consequent results on inward and outward mappings.

2. Preliminaries

A mapping F from a set X into 2^Y (the set of nonempty subsets of a set Y) is called a *multivalued mapping* from X into Y, and the *fibers* of F at $y \in Y$ are the set $F^{-}(y) = \{x \in X : y \in F(x)\}$. A mapping $f : X \to Y$ is called a *selection* of F if $f(x) \in F(x)$ for all $x \in X$.

Let X, Y be topological spaces. A mapping $F : X \to 2^Y$ is called *upper semicontinuous* (u.s.c.) if for each $x_0 \in X$ and neighborhood V of $F(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that $F(x) \subseteq V$ for all $x \in U$. By a *neighborhood* of a point x in X, we mean any open subset of X that contains x.

Let E be a topological vector space (t.v.s.), not necessarily Hausdorff and E^* the topological dual of E. In this paper, we consider E^* equipped with the topology of compact convergence. Then E^* is a t.v.s. We say that E^**separates points* of E, if whenever x and y are distinct points of E, then $p(x) \neq p(y)$ for some $p \in E^*$. If E^* separates points of E, then a topology on E is Hausdorff. By Hahn-Banach theorem, if E is locally convex Hausdorff, then E^* separates points of E, but the converse is not true, for an example, see [5, 6].

Let $C \subseteq E$ and $F : C \to 2^E$. A mapping F is called *upper demicontinuous* (u.d.c) if for each $x_0 \in C$ and any open half-space (the set of the form $\{x \in E : p(x) > \alpha\}$, where $p \in E^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$) H in E containing $F(x_0)$, there exists a neighborhood U of x_0 in C such that $F(x) \subseteq H$ for all $x \in U$. It is clear that a u.s.c. multivalued mapping is u.d.c. but the converse is not true (see [7]). It is convenient to write $\langle p, x\rangle$ instead of $p(x)$ for $p \in E^*$ and $x \in E$. The reason for this is that often the vector x and/or the continuous linear functional p may be given in a notation already containing parentheses or other complicated form.

The following useful results are recalled to be referred.

Theorem 2.1 (Browder [8]). *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. If $\varphi : C \to E^*$ is a continuous mapping, then there exists $u_0 \in C$ such that $\langle \varphi(u_0), v - u_0\rangle \leq 0$ for all $v \in C$.*

Theorem 2.2 (Ben-El-Mechaiekh et al. [1]). *Let X be a paracompact Hausdorff space and Y a convex subset of a t.v.s. Suppose $\Phi : X \to 2^Y$ is a multivalued mapping having nonempty convex values and open fibers, then Φ has a continuous selection.*

Theorem 2.3 (see [6]). *Let A, B be disjoint nonempty convex subsets of a locally convex Hausdorff t.v.s. E. If A is compact and B is closed, then there exists $p \in E^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\langle p, x\rangle < \alpha_1 < \alpha_2 < \langle p, y\rangle$ for all $x \in A$ and $y \in B$.*

Theorem 2.4 (see [6]). *Let E be a t.v.s. whose E^* separates points. Suppose that A and B are disjoint nonempty compact convex sets in E. Then there exists $p \in E^*$ such that $\sup\{\langle p, x\rangle : x \in A\} < \inf\{\langle p, y\rangle : y \in B\}$.*

Theorem 2.5 (see [9]). *Let X be a topological space, Y a compact Hausdorff space, and $F : X \to 2^Y$ a multivalued mapping with nonempty closed values. Then F is u.s.c. if and only if the graph $\{(x, y) : x \in X, y \in F(x)\}$ of F is closed in $X \times Y$.*

3. Half-Continuous Mappings

Now, we introduce the notion of half-continuity on t.v.s., and investigate some of their properties.

Definition 3.1. Let C be a subset of a t.v.s. E. A mapping $f : C \to E$ is said to be *half-continuous* if for each $x \in C$ with $x \neq f(x)$ there exist $p \in E^*$ and a neighborhood W of x in C such that

$$\langle p, f(y) - y \rangle > 0 \tag{3.1}$$

for all $y \in W$ with $y \neq f(y)$.

By the name "half-continuous," it induces us to think that continuous mappings should be half-continuous. The following theorem tells us that if E^* separates points of E, then the statement is affirmative.

Proposition 3.2. *Let E be a t.v.s. whose E^* separates points and C a nonempty subset of E. Then every continuous mapping $f : C \to E$ is half-continuous.*

Proof. Let $x \in C$ be such that $x \neq f(x)$. Since E^* separates points on E, we may assume that, $\langle p, f(x) - x \rangle > 0$ for some $p \in E^*$. Since the mapping $z \mapsto \langle p, f(z) - z \rangle$ is continuous, there exists a neighborhood W of x in C such that $\langle p, f(y) - y \rangle > 0$ for all $y \in W$. Therefore, f is half-continuous. □

The hypothesis that E^* separates points of E cannot be relaxed as will be shown in the following examples.

Example 3.3. Let E be a nontrivial vector space. Then the topology $\{\emptyset, E\}$ makes E into a locally convex t.v.s. that is not Hausdorff and $E^* = \{0\}$ (see [10]). So E^* does not separate points on E. Consequently, every continuous self-mapping on E which is not the identity, is not half-continuous.

Example 3.4. For $0 < p < 1$, $L^p[0,1]$ is a Hausdorff t.v.s. with $(L^p[0,1])^* = \{0\}$ (see [6]).

Remark 3.5. There are some half-continuous mappings which are not continuous. For example [4], let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 & \text{if } x \in [0,1), \\ 2 & \text{otherwise.} \end{cases} \tag{3.2}$$

It is clear that f is half-continuous but not continuous.

Moreover, half-continuity is not closed under the composition, the addition, and the scalar multiplication. To see this consider a half-continuous mapping g on $\mathbb{R}$ defined by $g(x) = 3$ for $x \geq 3$ and $g(x) = 0$ for $x < 3$. It is easy to see that $g \circ f, g + f$ and $2g$ are not half-continuous. In fact, the composition of g and a homeomorphism $x \mapsto x + 1$ is not half-continuous yet.

Proposition 3.6. *Let C be a nonempty subset of a t.v.s. E and $f : C \to E$. Then f is half-continuous if and only if for any $\beta \in \mathbb{R}$, the mapping $x \mapsto (1-\beta)x + \beta f(x)$ is half-continuous.*

Proof. The sufficiency is clear. To prove the necessity, let $\beta \in \mathbb{R}$ and let $g : C \to E$ be defined by $g(x) = (1-\beta)x + \beta f(x)$ for all $x \in C$. Let $x \in C$ be such that $x \neq g(x)$. Then $x \neq f(x)$ and hence there exist $p \in E^*$ and a neighborhood W of x in C such that $\langle p, f(y) - y\rangle > 0$ for all $y \in W$ with $y \neq f(y)$. Then for each $y \in W$ with $y \neq g(y)$,

$$\langle p, g(y) - y\rangle = \langle p, (1-\beta)y + \beta f(y) - y\rangle = \beta\langle p, f(y) - y\rangle. \tag{3.3}$$

If $\beta > 0$, then done. Otherwise, consider $-p$ instead of p. □

Next, we give a sufficient condition for mappings on t.v.s. to be half-continuous.

Proposition 3.7. *Let C be a nonempty subset of a t.v.s. E and $f : C \to E$. Suppose that for each $x \in C$ with $x \neq f(x)$, there exist $p \in E^*$ such that $\langle p, f(x) - x\rangle > 0[\langle p, f(x) - x\rangle < 0]$ and $p \circ f$ is lower [upper] semicontinuous at x. Then f is half-continuous.*

Proof. Let $x \in C$ be such that $x \neq f(x)$. Then there exists $p \in E^*$ such that $\langle p, f(x) - x\rangle > 0$ and $p \circ f$ is lower semicontinuous at x. Let $\alpha \in \mathbb{R}$ be such that $\langle p, f(x) - x\rangle > \alpha > 0$. Since p is continuous at x, there exists a neighborhood V of x in E such that $|\langle p, x - z\rangle| < \alpha$ for all $z \in V$. This implies that

$$\beta := \inf_{z \in V}\langle p, x - z\rangle + \langle p, f(x) - x\rangle > \inf_{z \in V}\langle p, x - z\rangle + \alpha \geq 0. \tag{3.4}$$

By lower semicontinuity of $p \circ f$, there exists a neighborhood U of x in C such that

$$\langle p, f(y)\rangle > \langle p, f(x)\rangle - \beta \tag{3.5}$$

for all $y \in U$. Then, for each $y \in U \cap V$ with $y \neq f(y)$, we have from (3.4) and (3.5) that

$$\langle p, f(y) - y\rangle > \langle p, f(x)\rangle - \beta + \langle p, -y\rangle \geq \langle p, f(x) - x\rangle - \beta + \inf_{z \in V}\langle p, x - z\rangle = 0. \tag{3.6}$$

Therefore, f is half-continuous.

The latter case follows from the fact that f is upper semicontinuous if and only if $-f$ is lower semicontinuous. □

Remark 3.8. If E is a Banach space, then Proposition 3.7 is Proposition 2.4 in [4]. By considering the mapping f in Remark 3.5, we note that the converse of Proposition 3.7 is not true (see [4]).

Let X and Y be sets. Let f and g be mappings from X to Y. The set $\mathcal{C}(f, g) = \{x \in X : f(x) = g(x)\}$ is said to be the *coincidence set* of f and g. The next result is inspired by the idea of [4, Theorem 3.1].

Theorem 3.9. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E and $f, g : C \to C$. Suppose that $g : C \to C$ is bijective continuous and for each $x \in C$ with $g(x) \neq f(x)$ there exist $p \in E^*$ and a neighborhood W of $g^{-1}(x)$ in C such that $\langle p, f(y) - g(y)\rangle > 0$ for all $y \in W$ with $g(y) \neq f(y)$. Then $\mathcal{C}(f, g)$ is nonempty.*

Proof. Suppose that $\mathcal{C}(f, g) = \emptyset$. Define $\Phi : C \to 2^{E^*}$ by

$$\begin{aligned}\Phi(x) = \Big\{p \in E^* : &\text{ there exists a neighborhood } W \text{ of } g^{-1}(x) \text{ in } C \text{ such that} \\ &\langle p, f(y) - g(y)\rangle > 0\ \forall y \in W \text{ with } g(y) \neq f(y)\Big\}\end{aligned} \tag{3.7}$$

for all $x \in C$. Clearly, $\Phi(x)$ is nonempty for all $x \in C$. Let $x \in C, p, q \in \Phi(x)$ and $\lambda \in [0, 1]$. There are neighborhoods W_1 and W_2 of $g^{-1}(x)$ in C such that

$$\begin{aligned}&\forall y \in W_1, \quad g(y) \neq f(y) \Longrightarrow \langle p, f(y) - g(y)\rangle > 0, \\ &\forall y \in W_2, \quad g(y) \neq f(y) \Longrightarrow \langle q, f(y) - g(y)\rangle > 0.\end{aligned} \tag{3.8}$$

Clearly, $\lambda p + (1-\lambda)q \in E^*$ and $W = W_1 \cap W_2$ is a neighborhood of $g^{-1}(x)$ in C. For each $y \in W$ with $g(y) \neq f(y)$,

$$\langle \lambda p + (1-\lambda)q, f(y) - g(y)\rangle = \lambda\langle p, f(y) - g(y)\rangle + (1-\lambda)\langle q, f(y) - g(y)\rangle > 0. \tag{3.9}$$

Hence, $\lambda p + (1-\lambda)q \in \Phi(x)$. This implies that $\Phi(x)$ is convex.

Next, let $p \in E^*$ and $x \in \Phi^-(p)$. There exists a neighborhood W of $g^{-1}(x)$ in C such that $\langle p, f(y) - g(y)\rangle > 0$ for all $y \in W$ with $g(y) \neq f(y)$. Then $x \in g(W) \subseteq \Phi^-(p)$. Since g is open, $\Phi^-(p)$ is open in C. From Theorems 2.1 and 2.2, there exists a continuous selection $\varphi : C \to E^*$ of Φ and $x_0 \in C$ such that for every $y \in C$,

$$\langle \varphi(x_0), y - x_0\rangle \leq 0. \tag{3.10}$$

Since g is surjective, $x_0 = g(z_0)$ for some $z_0 \in C$, and hence $\langle \varphi(g(z_0)), f(z_0) - g(z_0)\rangle \leq 0$. Also, since $\varphi(g(z_0)) \in \Phi(g(z_0))$, $\langle \varphi(g(z_0)), f(z_0) - g(z_0)\rangle > 0$, which is a contradiction. □

If g in Theorem 3.9 is the identity mapping, then the following result is immediate.

Corollary 3.10. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. If $f : C \to C$ is half-continuous, then f has a fixed point.*

Remark 3.11. If E is a Banach space, then the previous corollary is the Theorem 3.1 in [4].

The following result is obtained from Proposition 3.2 and Corollary 3.10.

Corollary 3.12 (Brouwer-Schauder-Tychonoff, see [1]). *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. Then every continuous mapping $f : C \to C$ has a fixed point.*

4. Half-Continuous Multivalued Mappings

Now, we consider half-continuity of multivalued mappings and prove that under a certain assumption they have fixed point.

Definition 4.1. Let C be a subset of a t.v.s. E. A mapping $F : C \to 2^E$ is said to be *half-continuous* if for each $x \in C$ with $x \notin F(x)$ there exists $p \in E^*$ and a neighborhood W of x in C such that

$$\forall y \in W, \quad y \notin F(y) \Longrightarrow \forall z \in F(y), \quad \langle p, z - y \rangle > 0. \tag{4.1}$$

The following proposition gives a sufficient condition for a multivalued mapping to be half-continuous.

Proposition 4.2. *Let C be a nonempty subset of a locally convex Hausdorff t.v.s. E. If $F : C \to 2^E$ is a u.d.c. mapping with nonempty closed convex values, then F is half-continuous.*

Proof. Assume that $F : C \to 2^E$ is u.d.c. with nonempty closed convex values. Let $x \in C$ be such that $x \notin F(x)$. Suppose that F fails to be half-continuous. By Theorem 2.3, there exists $p \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle p, x \rangle < \alpha < \langle p, y \rangle \tag{4.2}$$

for all $y \in F(x)$. This implies that $F(x) \subseteq H := p^{-1}(\alpha, \infty)$. Since F is u.d.c., there exists a neighborhood U of x in C such that $F(y) \subseteq H$ for all $y \in U$. Set $V = U \setminus \overline{H}$. Then V is a neighborhood of x in C. Since F is not half-continuous, there exists $x_V \in V \setminus F(x_V)$ and $z_V \in F(x_V)$ such that

$$\langle p, z_V - x_V \rangle \leq 0. \tag{4.3}$$

Since $x_V \in U$, $F(x_V) \subseteq H$, so $z_V \in H$. Then, by (4.3), $\alpha < \langle p, z_V \rangle \leq \langle p, x_V \rangle$. This means that $x_V \in H$, which is a contradiction. Therefore, F is half-continuous. □

Remark 4.3. However, there are some half-continuous mappings which are not u.d.c.. To see this, consider the mapping $F : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$F(x) = \begin{cases} [-1, 1] & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0. \end{cases} \tag{4.4}$$

Then F is half-continuous but not u.d.c. at 0.

In case that E is a t.v.s. whose E^* separates points, we need more assumptions on the mapping as the following result. The proof is analogous to that of Proposition 4.2, by applying Theorem 2.4.

Proposition 4.4. *Let E be a t.v.s. whose E^* separates points and C a nonempty subset of E. If $F : C \to 2^E$ is u.d.c. with nonempty compact convex values, then F is half-continuous.*

Next, we will prove the main result which guarantees the possessing of fixed points if the multivalued mapping is half-continuous. To do this, we need the following lemma.

Lemma 4.5. *Let C be a nonempty subset of a t.v.s. E and $F : C \to 2^E$. If F is half-continuous, then F has a half-continuous selection.*

Proof. Assume that F is half-continuous. Let f be any selection of F. Define $\tilde{f} : C \to E$ by

$$\tilde{f}(x) = \begin{cases} x & \text{if } x \in F(x), \\ f(x) & \text{if } x \notin F(x). \end{cases} \tag{4.5}$$

Clearly, $\tilde{f}$ is a selection of F. To show that $\tilde{f}$ is half-continuous, let $x \in C$ be such that $x \neq \tilde{f}(x)$. Then $x \notin F(x)$ and hence there exists $p \in E^*$ and a neighborhood W of x in C such that

$$\forall y \in W, \quad y \notin F(y) \Longrightarrow \forall z \in F(y), \quad \langle p, z - y \rangle > 0. \tag{4.6}$$

It follows that $\langle p, \tilde{f}(y) - y \rangle = \langle p, f(y) - y \rangle > 0$ for every $y \in W$ with $y \neq \tilde{f}(y)$. □

Corollary 3.10 and Lemma 4.5 yield the following main result.

Theorem 4.6. *Let C be a nonempty compact subset of a locally convex Hausdorff t.v.s. E. If $F : C \to 2^C$ is half-continuous, then F has a fixed point.*

The following result is immediately obtained from Theorem 4.6 and Proposition 4.2.

Corollary 4.7. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. If $F : C \to 2^C$ is u.d.c. with nonempty closed convex values, then F has a fixed point.*

It is well known that if C is a subset of a topological space X and $F : C \to 2^X$ has closed graph, then the set of fixed points of F is closed in C. From Corollary 4.7 and Theorem 2.5, we have the following corollary.

Corollary 4.8 (Kakutani-Fan-Glicksberg, see [11, 12]). *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. If $F : C \to 2^C$ is u.s.c. with nonempty closed convex values, then the set of fixed points of F is nonempty and compact.*

5. Inward and Outward Mappings

In case that the half-continuous mapping f is a nonself-mapping on C but f has some nice property, then f still possesses a fixed point in C. We state the results in the following theorem.

Theorem 5.1. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. Suppose that $f : C \to E$ is half-continuous and for each $x \in C$ with $x \neq f(x)$ there exists $\lambda < 1$ such that $\lambda x + (1 - \lambda) f(x) \in C$, then f has a fixed point.*

Proof. Suppose that f has no fixed point. For each $x \in C$, let $\Lambda(x) = \{\lambda \in \mathbb{R} : \lambda < 1$ and $\lambda x + (1-\lambda)f(x) \in C\}$. Define $F : C \to 2^C$ by

$$F(x) = \{\lambda x + (1-\lambda)f(x) : \lambda \in \Lambda(x)\} \tag{5.1}$$

for all $x \in C$. Then $F(x) \neq \emptyset$ for every $x \in C$. It is not difficult to see that F is half-continuous. By Theorem 4.6, there exists $x_0 \in F(x_0) \cap C$ and $\alpha \in \Lambda(x_0)$ such that $x_0 = \alpha x_0 + (1-\alpha)f(x_0)$. It follows that $x_0 = f(x_0)$, which is a contradiction. □

Remark 5.2. From Theorem 5.1, for $x \in C$ with $x \neq f(x)$, if there is $\lambda < 0$ such that $z := \lambda x + (1-\lambda)f(x) \in C$, then $f(x)$, in fact, is the element in C. Indeed, by setting $\mu = \lambda/(\lambda - 1)$, then $0 < \mu < 1$ and so, by convexity of C, $f(x) = \mu x + (1-\mu)z \in C$.

Recall that the *line segment* joining vectors x and y in E is the set $[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$. As a special case of Theorem 5.1 we obtain the following corollary.

Corollary 5.3 (Fan-Kaczynski, see [1]). *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. Suppose that $f : C \to E$ is continuous and for each $x \in C$ with $x \neq f(x)$ the line segment $[x, f(x)]$ contains at least two points of C, then f has a fixed point.*

Next, we derive a generalization of a fixed point theorem due to F. E. Browder and B. R. Halpern. To do this, let us recall the definition of inward and outward mappings.

Definition 5.4 (see [1]). Let C be a subset of a vector space E. A mapping $f : C \to E$ is called *inward* (resp., *outward*) if for each $x \in C$ there exists $\lambda > 0$ (resp., $\lambda < 0$) satisfying $x + \lambda(f(x) - x) \in C$.

Theorem 5.5. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. Then every half-continuous inward (or outward) mapping $f : C \to E$ has a fixed point.*

Proof. Suppose that $f : C \to E$ is a half-continuous inward mapping. Let $x \in C$ be such that $x \neq f(x)$. There exists $\lambda > 0$ such that $x + \lambda(f(x) - x) \in C$. By letting $\beta = 1 - \lambda$ and apply Theorem 5.1, f has a fixed point.

Next, assume that f is outward. Define $g : C \to E$ by $g(x) = 2x - f(x)$ for all $x \in C$. Then g is inward and, by Proposition 3.6, g is half-continuous. Hence, there is $x_0 \in C$ such that $x_0 = g(x_0) = 2x_0 - f(x_0)$. That is $x_0 = f(x_0)$. □

Remark 5.6. In Theorem 5.5, if f is a continuous inward (or outward) mapping, then Theorem 5.5 is the theorem proved by F. E. Browder (1967) and B. R. Halpern (1968) (see [1]).

In the final part, we prove the fixed points theorem for half-continuous inward and outward multivalued mappings.

Definition 5.7 (see [7]). Let C be a subset of a vector space E. A mapping $F : C \to 2^E$ is called *inward* (resp., *outward*) if for each $x \in C$ there exists $y \in F(x)$ and $\lambda > 0$ (resp., $\lambda < 0$) satisfying $x + \lambda(y - x) \in C$.

Theorem 5.8. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. Then every half-continuous inward (or outward) mapping $F : C \to 2^E$ has a fixed point.*

Proof. Let $F : C \to 2^E$ be a half-continuous mapping. Suppose that F is inward but it has no fixed point. Define $G : C \to 2^C$ by

$$G(x) = \{u \in C : \text{ there exists } v \in F(x) \text{ and } \lambda > 0 \text{ such that } u = x + \lambda(v - x)\} \tag{5.2}$$

for all $x \in C$. We can see that $G(x)$ is nonempty for all $x \in C$ and G is half-continuous. By Theorem 4.6, there exists $x_0 \in C \cap G(x_0), v \in F(x_0)$, and $\alpha > 0$ such that $x_0 = x_0 + \alpha(v - x_0)$. That is $x_0 \in F(x_0)$, which is a contradiction.

Next, assume that F is outward. Define $H : C \to 2^E$ by $H(x) = 2x - F(x)$ for all $x \in C$. It is easy to see that H is half-continuous. Let $x \in C$ be arbitrary. There exists $y \in F(x)$ and $\lambda < 0$ satisfying $x + \lambda(y - x) \in C$. Then $x + (-\lambda)(2x - y - x) = x + \lambda(y - x) \in C$. Since $2x - y \in H(x)$, H is inward. Thus $x_0 = 2x_0 - v$ for some $x_0 \in H(x_0) \cap C$ and $v \in F(x_0)$. That is $x_0 \in F(x_0)$. □

Any selection of half-continuous inward multivalued mappings may not be inward as shown in the following example. Let $F : [0,1] \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} [x+1, \infty) & \text{if } x \in [0,1), \\ \{0,1,2\} & \text{if } x = 1. \end{cases} \tag{5.3}$$

Clearly, F is inward half-continuous but a selection $f : [0,1] \to \mathbb{R}$ of F defined by $f(x) = x+2$ if $0 \leq x < 1$ and $f(x) = 2$ if $x = 1$ is not inward.

Remark 5.9. If the half-continuity of F is replaced by upper semicontinuity, then Theorem 5.8 is the result of Halpern-Bergman (1968) (see [7]) and Fan (1969) (see [13]).

As an interesting special case of Theorem 5.8, we obtain the following corollary.

Corollary 5.10. *Let C be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. E. Suppose that $F : C \to 2^E$ is half-continuous and for each $x \in C$, $F(x) \cap C$ is nonempty, then F has a fixed point.*

6. Discussion

It is worth to notice that there exists a multivalued mapping which is not half-continuous but some of its selection is half-continuous. For example, let $F : [0,1] \to 2^{[0,1]}$ be defined by

$$F(x) = \begin{cases} \left(\frac{3}{4}, 1\right] \cup \{0\} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \left\{\frac{3}{4}\right\} & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{6.1}$$

Then F is not half-continuous since (4.1) fails for $x = 1/2$. Nevertheless, a mapping $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{3}{4} & \text{if } x \in \left(\dfrac{1}{2}, 1\right] \end{cases} \tag{6.2}$$

is a half-continuous selection of F.

From Theorem 4.6 we see that if a multivalued mapping F has a half-continuous selection, then F has a fixed point. It is interesting to investigate the condition(s) for a multivalued mapping to induce a half-continuous selection.

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