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Research Article

Convergence of Iterative Sequences for Common Zero Points of a Family of m-Accretive Mappings in Banach Spaces

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We introduce implicit and explicit viscosity iterative algorithms for a finite family of m-accretive operators. Strong convergence theorems of the iterative algorithms are established in a reflexive Banach space which has a weakly continuous duality map.

1. Introduction

Let E be a real Banach space, and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad x \in E, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we denote a single-valued normalized duality mapping by j.

Let K be a nonempty subset of E. Recall that a mapping $f : K \to K$ is said to be a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha \|x - y\|, \quad \forall x, y \in K. \tag{1.2}$$

Recall that a mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in K. \tag{1.3}$$

A point $x \in K$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Given a real number $t \in (0,1)$ and a contraction $f : C \to C$, we define a mapping

$$T_t^f x = tf(x) + (1-t)Tx, \quad x \in K. \tag{1.4}$$

It is obviously that T_t^f is a contraction on K. In fact, for $x, y \in K$, we obtain

$$\begin{aligned} \left\| T_t^f x - T_t^f y \right\| &\leq \left\| t\left(f(x) - f(y)\right) + (1-t)\left(Tx - Ty\right) \right\| \\ &\leq \alpha t \|x - y\| + (1-t)\|Tx - Ty\| \\ &\leq \alpha t \|x - y\| + (1-t)\|x - y\| \\ &= (1 - t(1-\alpha))\|x - y\|. \end{aligned} \tag{1.5}$$

Let x_t be the unique fixed point of T_t^f, that is, x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{1.6}$$

A special case has been considered by Browder [1] in a Hilbert space as follows. Fix $u \in C$ and define a contraction S_t on K by

$$S_t x = tu + (1-t)Tx, \quad x \in K. \tag{1.7}$$

We use z_t to denote the unique fixed point of S_t, which yields that $z_t = tu + (1-t)Tz_t$. In 1967, Browder [1] proved the following theorem.

Theorem B. *In a Hilbert space, as $t \to 0$, z_t converges strongly to a fixed point of T, that is, closet to u, that is, the nearest point projection of u onto $F(T)$.*

In [2], Moudafi proposed a viscosity approximation method which was considered by many authors [2–8]. If H is a Hilbert space, $T : K \to K$ is a nonexpansive mapping and $f : K \to K$ is a contraction, he proved the following theorems.

Theorem M 1. *The sequence $\{x_n\}$ generated by the following iterative scheme:*

$$x_n = \frac{1}{1+\epsilon_n} Tx_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n) \tag{1.8}$$

converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T), \quad \text{such that } \left\langle (I-f)\overline{x}, \overline{x} - x \right\rangle \leq 0, \quad \forall x \in F(T), \tag{1.9}$$

where $\{\epsilon_n\}$ is a sequence of positive numbers tending to zero.

Theorem M 2. *With and initial* $z_0 \in C$ *defined the sequence* $\{z_n\}$ *by*

$$z_{n+1} = \frac{1}{1+\epsilon_n} T z_n + \frac{\epsilon_n}{1+\epsilon_n} f(z_n). \tag{1.10}$$

Suppose that $\lim_{n\to\infty}\epsilon_n = 0$, *and* $\sum_{n=1}^{\infty}\epsilon = \infty$ *and* $\lim_{n\to\infty}|1/\epsilon_{n+1} - 1/\epsilon| = 0$. *Then,* $\{z_n\}$ *converges strongly to the unique solution of the unique solutions of the variational inequality*

$$\overline{x} \in F(T), \quad \textit{such that } \langle (I-f)\overline{x}, \overline{x} - x\rangle \le 0, \quad \forall x \in F(T). \tag{1.11}$$

Recall that a (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in E is accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,2)$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \ge 0. \tag{1.12}$$

An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. The set of zeros of A is denoted by $N(A)$. Hence,

$$N(A) = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0). \tag{1.13}$$

For each $r > 0$, we denote by J_r the resolvent of A, that is, $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = N(A)$, for all $r > 0$. We also denote by A_r the Yosida approximation of A, that is, $A_r = (1/r)(I - J_r)$. It is known that J_r is a nonexpansive mapping from E to $\overline{D(A)}$.

Recently, Kim and Xu [9] and Xu [10] studied the sequence generated by the following iterative algorithm:

$$x_0 \in K, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \ge 0, \tag{1.14}$$

where $\{\alpha_n\}$ is a real sequence $[0,1]$ and $J_{r_n} = (I+rA)^{-1}$. They obtained the strong convergence of the iterative algorithm in the framework of uniformly smooth Banach spaces and reflexive Banach space, respectively. Xu [10] also studied the following iterative algorithm by viscosity approximation method

$$x_0 \in K, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T x_n, \quad n \ge 0, \tag{1.15}$$

where $\{\alpha_n\}$ is a real sequence $[0,1]$, $f : K \to K$ is a contractive mapping, and $T : K \to K$ is a nonexpansive mapping with a fixed point. Strong convergence theorems of fixed points are obtained in a uniformly smooth Banach space; see [10] for more details.

Very recently, Zegeye and Shahzad [11] studied the common zero problem of a family of m-accretive mappings. To be more precise, they proved the following result.

Theorem ZS. *Let* E *be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm,* K *a nonempty, closed, convex subset of* E*, and* $A_i : K \to E$ $(i = 1, 2, \ldots, r)$

a family of m*-accretive mappings with* $\bigcap_{i=1}^{r} N(A_i) \neq \emptyset$*. For any* $u, x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) S_r x_n, \quad n \geq 0, \tag{1.16}$$

where $\{\alpha_n\}$*is a real sequence which satisfies the following conditions:* $\lim_{n\to\infty}\alpha_n = 0$*;* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;* $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty}(|\alpha_n - \alpha_{n-1}|/\alpha_n) = 0$ *and* $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ *with* $J_{A_i} := (I + A_i)^{-1}$ *for* $0 < a_i < 1$ *for* $i = 0, 1, 2, \ldots, r$ *and* $\sum_{i=0}^{r} a_i = 1$*. If every nonempty, closed, bounded convex subset of* E *has the fixed point property for a nonexpansive mapping, then* $\{x_n\}$ *converges strongly to a common solution of the equations* $A_i x = 0$ *for* $i = 1, 2, \ldots, r$*.*

In this paper, motivated by the recent work announced in [3, 5, 9, 11–20], we consider the following implicit and explicit iterative algorithms by the viscosity approximation method for a finite family of m-accretive operators $\{A_1, A_2, \ldots, A_r\}$. The algorithms are as following:

$$x_0 \in K, \quad x_n = \alpha_n f(x_n) + (1 - \alpha_n) S_r x_n, \quad n \geq 0, \tag{1.17}$$

$$x_0 \in K, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_r x_n, \quad n \geq 0, \tag{1.18}$$

where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $0 < a_i < 1$ for $i = 0, 1, 2, \ldots, r$, $\sum_{i=0}^{r} a_i = 1$ and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. It is proved that the sequence $\{x_n\}$ generated in the iterative algorithms (1.17) and (1.18) converges strongly to a common zero point of a finite family of m-accretive mappings in reflexive Banach spaces, respectively.

2. Preliminaries

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$.

A Banach space E is said to be *strictly convex* if, for $a_i \in (0, 1)$, $i = 1, 2, \ldots, r$, such that $\sum_{i=1}^{r} a_i = 1$,

$$\|a_1 x_1 + a_2 x_2 + \cdots + a_r x_r\| < 1, \quad \forall x_i \in E, \ i = 1, 2, \ldots, r, \tag{2.2}$$

with $\|x_i\| = 1$, $i = 1, 2, \ldots, r$, and $x_i \neq x_j$ for some $i \neq j$. In a strictly convex Banach space E, we have that, if

$$\|x_1\| = \|x_2\| = \cdots = \|x_r\| = \|a_1 x_1 + a_2 x_2 + \cdots + a_r x_r\| \tag{2.3}$$

for $x_i \in E$, $a_i \in (0, 1)$, $i = 1, 2, \ldots, r$, where $\sum_{i=1}^{r} a_i = 1$, then $x_1 = x_2 = \cdots = x_r$ (see [21]).

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : E \to E^*$ defined by

$$J_\varphi(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\right\}, \quad x \in E. \tag{2.4}$$

Following Browder [22], we say that a Banach space E has a *weakly continuous duality map* if there exists a gauge φ for which the duality map $J_\varphi(x)$ is single valued and *weak*-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to $J_\varphi x$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$ with the gauge $\varphi(t) = t^{p-1}$. In the case where $\varphi(t) = t$ for all $t > 0$, we write the associated duality map as J and call it the (normalized) duality map. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \geq 0, \tag{2.5}$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E, \tag{2.6}$$

where ∂ denotes the subdifferential in the sense of convex analysis. It also follows from (2.5) that Φ is convex and $\Phi(0) = 0$.

In order to prove our main results, we also need the following lemmas.

The first part of the next lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [23].

Lemma 2.1. *Assume that E has a weakly continuous duality map J_φ with the gauge φ.*

(i) *For all $x, y \in E$ and $j_\varphi(x+y) \in J_\varphi(x+y)$, there holds the inequality*

$$\Phi\left(\|x+y\|\right) \leq \Phi(\|x\|) + \left\langle y, j_\varphi(x+y)\right\rangle. \tag{2.7}$$

In particular, for $x, y \in E$ and $j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.8}$$

(ii) *For $\lambda \in \mathbb{R}$ and for nonzero $x \in E$,*

$$J_\varphi(\lambda x) = \operatorname{sgn}(\lambda)\left(\frac{\varphi(|\lambda|/\|x\|)}{\|x\|}\right)J(x). \tag{2.9}$$

Lemma 2.2 (see [24]). *Let E be a Banach space satisfying a weakly continuous duality map, let K be a nonempty, closed, convex subset of E, and let $T : K \to K$ be a nonexpansive mapping with a fixed point. Then, $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in K which converges weakly to x and if the sequence $\{(I-T)x_n\}$ converges strongly to zero, then $x = Tx$.*

Lemma 2.3 (see [11]). *Let K be a nonempty, closed, convex subset of a strictly convex Banach space E. Let $A_i : K \to E$, $i = 1, 2, \dots, r$, be a family of m-accretive mappings such that $\bigcap_{i=1}^{r} N(A_i) \neq \emptyset$. Let $a_0, a_1, a_2, \dots, a_r$ be real numbers in $(0,1)$ such that $\sum_{i=0}^{r} a_i = 1$ and $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, where $J_{A_i} := (I + A_i)^{-1}$. Then, S_r is nonexpansive and $F(S_r) = \bigcap_{i=1}^{r} N(A_i)$.*

Lemma 2.4 (see [25]). *Let $\sum_{n=0}^{\infty}\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \sigma_n, \quad n \geq 0, \tag{2.10}$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

3. Main Results

Theorem 3.1. *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ with the gauge φ. Lek K be a nonempty, closed, convex subset of E and $f : K \to K$ a contractive mapping with the coefficient α $(0 < \alpha < 1)$. Let $\{A_i\}_{i=1}^{r} : K \to E$ be a family of m-accretive mappings with $\bigcap_{i=1}^{r} N(A_i) \neq \emptyset$. Let $J_{A_i} := (I + A_i)^{-1}$, for each $i = 1, 2, \dots, r$. For any $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm (1.17), where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $0 < a_i < 1$ for $i = 0, 1, 2, \dots, r$, $\sum_{i=0}^{r} a_i = 1$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\lim_{n\to\infty}\|x_n - S_r x_n\| = 0$, then $\{x_n\}$ converges strongly to a common solution x^* of the equations $A_i x = 0$ for $i = 1, 2, \dots, r$, which solves the following variational inequality:*

$$\langle (I - f)x^*, J(p - x^*) \rangle \geq 0, \quad p \in F(S_r). \tag{3.1}$$

Proof. From Lemma 2.3, we see that S_r is a nonexpansive mapping and

$$F(S_r) = \bigcap_{i=1}^{r} N(A_i) \neq \emptyset. \tag{3.2}$$

Notice that Φ is convex. From Lemma 2.1, for any fixed $p \in F(S_r) = \bigcap_{i=1}^{r} N(A_i)$, we have

$$\begin{aligned} \Phi(\|x_n - p\|) &= \Phi(\|\alpha_n(f(x_n) - f(p)) + \alpha_n(f(p) - p) + (1 - \alpha_n)(S_r x_n - p)\|) \\ &\leq \Phi(\|\alpha_n(f(x_n) - f(p)) + (1 - \alpha_n)(S_r x_n - p)\|) + \alpha_n\langle f(p) - p, J_\varphi(x_n - p)\rangle \\ &\leq [1 - \alpha_n(1 - \alpha)]\Phi(\|x_n - p\|) + \alpha_n\langle f(p) - p, J_\varphi(x_n - p)\rangle, \end{aligned} \tag{3.3}$$

which in turn implies that

$$\Phi(\|x_n - p\|) \leq \frac{1}{1 - \alpha}\langle f(p) - p, J_\varphi(x_n - p)\rangle. \tag{3.4}$$

Note that (3.4) actually holds for all duality maps J_φ; in particular, if we take the normalized duality J (in which case, we have $\Phi(r) = (1/2)r^2$), then we get

$$\|x_n - p\|^2 \leq \frac{2}{1-\alpha}\langle f(p) - p, J(x_n - p)\rangle \tag{3.5}$$

that is,

$$\|x_n - p\| \leq \frac{2}{1-\alpha}\|f(p) - p\|. \tag{3.6}$$

This implies that the sequence $\{x_n\}$ is bounded. Now assume that x^* is a weak limit point of $\{x_n\}$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to x^*. Then, by Lemma 2.2, we see that x^* is a fixed point of S_r. Hence, $x^* \in \bigcap_{i=1}^r N(A_i)$. In (3.4), replacing x_n with x_{n_j} and p with x^*, respectively, and taking the limit as $j \to \infty$, we obtain from the weak continuity of the duality map J_φ that

$$\lim_{j\to\infty}\Phi\left(\left\|x_{n_j} - x^*\right\|\right) \leq 0. \tag{3.7}$$

Hence, we have $x_{n_j} \to x^*$.

Next, we show that x^* solves the variation inequality (3.1). For $p \in F(S_r) = \bigcap_{i=1}^r N(A_i)$, we obtain

$$\begin{aligned}\Phi(\|x_n - p\|) &= \Phi(\|\alpha_n(f(x_n) - x_n) + \alpha_n(x_n - p) + (1-\alpha_n)(S_r x_n - p)\|)\\ &\leq \Phi(\|\alpha_n(x_n - p) + (1-\alpha_n)(S_r x_n - p)\|) + \alpha_n\langle f(x_n) - x_n, J_\varphi(x_n - p)\rangle\\ &\leq \Phi(\|x_n - p\|) + \alpha_n\langle f(x_n) - x_n, J_\varphi(x_n - p)\rangle,\end{aligned} \tag{3.8}$$

which implies that

$$\langle x_n - f(x_n), J_\varphi(x_n - p)\rangle \leq 0. \tag{3.9}$$

Replacing x_n with x_{n_j} in (3.9) and passing through the limit as $j \to \infty$, we conclude that

$$\langle x^* - f(x^*), J_\varphi(x^* - p)\rangle = \lim_{j\to\infty}\left\langle x_{n_j} - f\left(x_{n_j}\right), J_\varphi\left(x_{n_j} - p\right)\right\rangle \leq 0. \tag{3.10}$$

It follows from Lemma 2.1 that $J(x^* - p)$ is a positive-scalar multiple of $J_\varphi(x^* - p)$. We, therefore, obtain that x^* is a solution to (3.1).

Finally, we prove that the full sequence $\{x_n\}$ actually converges strongly to x^*. It suffices to prove that the variational inequality (3.1) can have only one solution. This is an easy consequence of the contractivity of f. Indeed, assume that both $u \in F(S_r) = \bigcap_{i=1}^r N(A_i)$ and $v \in F(S_r) = \bigcap_{i=1}^r N(A_i)$ are solutions to (3.1). Then, we see that

$$\langle (I-f)u, J(u-v)\rangle \leq 0, \qquad \langle (I-f)v, J(v-u)\rangle \leq 0. \tag{3.11}$$

Adding them yields that

$$\langle (I-f)u-(I-f)v, J(u-v)\rangle \le 0. \tag{3.12}$$

This implies that

$$0 \ge \langle (I-f)u-(I-f)v, J(u-v)\rangle \ge (1-\alpha)\|u-v\|^2 \ge 0, \tag{3.13}$$

which guarantees $u = v$. So, (3.1) can have at most one solution. This completes the proof. □

Next, we shall consider the explicit algorithm (1.18) which is rephrased below, the initial guess $z_0 \in K$ is arbitrary and

$$z_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)S_r z_n, \quad n \ge 0. \tag{3.14}$$

We need the strong convergence of the implicit algorithm (1.17) to prove the strong convergence of the explicit algorithm (3.14).

Theorem 3.2. *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ with the gauge φ. Lek K be a nonempty, closed, convex subset of E and $f : K \to K$ a contractive mapping. Let $\{A_i\}_{i=1}^r : K \to E$ be a family of m-accretive mappings with $\bigcap_{i=1}^r N(A_i) \ne \emptyset$. Let $J_{A_i} := (I + A_i)^{-1}$ for each $i = 1, 2, \ldots, r$. For any $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm* (1.18)*, where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $0 < a_i < 1$ for $i = 0, 1, 2, \ldots, r$, $\sum_{i=0}^r a_i = 1$, and $\{\alpha_n\}$ is a sequence in $[0, 1]$ which satisfies the following conditions: $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$. Assume also that*

- (i) $\lim_{n\to\infty}\|z_n - S_r z_n\| = 0$,
- (ii) *$\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^r N(A_i)$, where $\{x_n\}$ is the sequence generated by the implicity algorithm* (1.17).

Then, $\{z_n\}$ converges strongly to x^*, which solves the variational inequality (3.1).

Proof. From Lemma 2.3, we obtain that S_r is a nonexpansive mapping and

$$F(S_r) = \bigcap_{i=1}^r N(A_i) \ne \emptyset. \tag{3.15}$$

We observe that $\{z_n\}_{n=0}^\infty$ is bounded. Indeed, take $p \in F(S_r) = \bigcap_{i=1}^r N(A_i)$ and notice that

$$\begin{aligned} \|z_{n+1}-p\| &= \|\alpha_n(f(z_n)-p) + (1-\alpha_n)(S_r z_n - p)\| \\ &\le \alpha_n(\|f(z_n)-f(p)\| + \|f(p)-p\|) + (1-\alpha_n)\|z_n-p\| \\ &= [1-\alpha_n(1-\alpha)]\|z_n-p\| + \alpha_n\|f(p)-p\| \\ &\le \max\left\{\|z_n-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\}. \end{aligned} \tag{3.16}$$

By simple inductions, we have

$$\|z_n - p\| \le \max\left\{\|z_0 - p\|, \frac{\|p - f(p)\|}{1-\alpha}\right\}, \tag{3.17}$$

which gives that the sequence $\{z_n\}$ is bounded, so are $\{f(z_n)\}$ and $\{S_r z_n\}$. From (1.17), we have

$$x_m - z_n = \alpha_m\left[f(x_m) - z_n\right] + (1-\alpha_m)(S_r x_m - z_n). \tag{3.18}$$

This implies that

$$\begin{aligned}
\|x_m - z_n\|^2 &\le (1-\alpha_m)^2\|S_r x_m - z_n\|^2 + 2\alpha_m\langle f(x_m) - z_n, J(x_m - z_n)\rangle \\
&= (1-\alpha_m)^2\|S_r x_m - S_r z_n + S_r z_n - z_n\|^2 + 2\alpha_m\langle f(x_m) - x_m, J(x_m - z_n)\rangle \\
&\quad + 2\alpha_m\langle x_m - z_n, J(x_m - z_n)\rangle \\
&\le (1-\alpha_m)^2(\|x_m - z_n\| + \|S_r z_n - z_n\|)^2 + 2\alpha_m\langle f(x_m) - x_m, J(x_m - z_n)\rangle \\
&\quad + 2\alpha_m\|x_m - z_n\|^2 \\
&\le \left(1+\alpha_m^2\right)\|x_m - z_n\|^2 + \|S_r z_n - z_n\|(\|S_r z_n - z_n\| + 2\|x_m - z_n\|) \\
&\quad + 2\alpha_m\langle f(x_m) - x_m, J(x_m - z_n)\rangle,
\end{aligned} \tag{3.19}$$

which in turn implies that

$$\langle f(x_m) - x_m, J(z_n - x_m)\rangle \le \alpha_m\|x_m - z_n\|^2 + \frac{\|S_r z_n - z_n\|}{\alpha_m}(\|S_r z_n - z_n\| + 2\|x_m - z_n\|). \tag{3.20}$$

It follows from $\lim_{n\to\infty}\|S_r z_n - z_n\| = 0$ that

$$\limsup_{n\to\infty}\langle f(x_m) - x_m, J(z_n - x_m)\rangle \le \limsup_{n\to\infty}\alpha_m\|x_m - z_n\|^2. \tag{3.21}$$

From the assumption $x_m \to x^*$ and the weak continuity of J_φ imply that,

$$J(x_m - z_n) = \frac{\|x_m - z_n\|}{\varphi(\|x_m - z_n\|)}J_\varphi(x_m - z_n) \rightharpoonup \frac{\|x^* - z_n\|}{\varphi(\|x^* - z_n\|)}J_\varphi(x^* - z_n) = J(x^* - z_n). \tag{3.22}$$

Letting $m \to \infty$ in (3.21), we obtain that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(z_n - x^*)\rangle \le 0. \tag{3.23}$$

Finally, we show the sequence $\{z_n\}$ converges stongly to x^*. Observe that

$$z_{n+1} - x^* = \alpha_n\big(f(z_n) - x^*\big) + (1-\alpha_n)(S_r z_n - x^*). \tag{3.24}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\le (1-\alpha_n)^2\|S_r z_n - x^*\|^2 + 2\alpha_n\langle f(z_n) - x^*, J(z_{n+1} - x^*)\rangle \\
&\le (1-\alpha_n)^2\|z_n - x^*\|^2 + 2\alpha_n\langle f(z_n) - f(x^*), J(z_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, J(z_{n+1} - x^*)\rangle \\
&\le (1-\alpha_n)^2\|z_n - x^*\|^2 + \alpha_n\alpha\Big(\|z_n - x^*\|^2 + \|z_{n+1} - x^*\|^2\Big) \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, J(z_{n+1} - x^*)\rangle,
\end{aligned} \tag{3.25}$$

which yields that

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\le \frac{(1-\alpha_n)^2 + \alpha\alpha_n}{1-\alpha\alpha_n}\|z_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*) - x^*, J(z_{n+1} - x^*)\rangle \\
&\le \left[1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\right]\|z_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*) - x^*, J(z_{n+1} - x^*)\rangle + M\alpha_n^2 \\
&\le \left[1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\right]\|z_n - x^*\|^2 + \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n} \\
&\quad \times \left[\frac{1}{1-\alpha}\langle f(x^*) - x^*, J(z_{n+1} - x^*)\rangle + M\frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)}\right],
\end{aligned} \tag{3.26}$$

where M is a appropriate constant such that $M \ge \sup_{n\ge 0}\{\|z_n - x^*\|^2/(1-\alpha\alpha_n)\}$. In view of Lemma 2.4, we can obtain the desired conclusion easily. This completes the proof. ☐

As an application of Theorems 3.1 and 3.2, we have the following results for a single mapping.

Corollary 3.3. *Let E be a reflexive Banach space which has a weakly continuous duality map J_φ with the gauge φ. Lek K be a nonempty, closed, convex subset of E and $f : K \to K$ a contractive mapping with the coefficient α $(0 < \alpha < 1)$. Let $A : K \to E$ be a m-accretive mapping with $N(A) \neq \emptyset$. Let $J_A := (I + A)^{-1}$. For any $x_0 \in K$, let $\{x_n\}$ be generated by the following iterative algorithm:*

$$x_0 \in K, \quad x_n = \alpha_n f(x_n) + (1-\alpha_n)J_A x_n, \quad n \ge 0. \tag{3.27}$$

Then, $\{x_n\}$ converges strongly to a solution of the equations $Ax = 0$.

Corollary 3.4. *Let E be a reflexive Banach space which has a weakly continuous duality map J_φ with gauge φ. Let K be a nonempty, closed, convex subset of E and $f : K \to K$ a contractive mapping.*

Let $A : K \to E$ *be a* m*-accretive mappings with* $N(A) \neq \emptyset$*. Let* $J_A := (I + A)^{-1}$*. For any* $x_0 \in K$*, let* $\{x_n\}$ *be generated by the following algorithm:*

$$x_0 \in K, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_A x_n, \quad n \geq 0, \tag{3.28}$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *which satisfies the following conditions:* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Also assume that*

(i) $\lim_{n\to\infty}\|z_n - S_1 z_n\| = 0$,

(ii) $\{x_n\}$ *converges strongly to* x^**, where* $\{x_n\}$ *is the sequence generated by the implicity scheme* (3.27) *and* $x^* \in N(A)$.

Then, the sequence $\{z_n\}$ *generated by the following iterative algorithm*

$$z_{n+1} = \alpha_n f(z_n) + (1 - \alpha_n) J_A z_n, \quad n \geq 0 \tag{3.29}$$

converges strongly to a solution x^* *of the equation* $Ax = 0$*.*

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