Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 615274, 17 pages
doi:10.1155/2011/615274

Research Article

Hamming Star-Convexity Packing in Information Storage

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Received 8 December 2010; Accepted 16 December 2010

Academic Editor: Jen Chih Yao

A major puzzle in neural networks is understanding the information encoding principles that implement the functions of the brain systems. Population coding in neurons and plastic changes in synapses are two important subjects in attempts to explore such principles. This forms the basis of modern theory of neuroscience concerning self-organization and associative memory. Here we wish to suggest an information storage scheme based on the dynamics of evolutionary neural networks, essentially reflecting the meta-complication of the dynamical changes of neurons as well as plastic changes of synapses. The information storage scheme may lead to the development of a complete description of all the equilibrium states (fixed points) of Hopfield networks, a space-filling network that weaves the intricate structure of Hamming star-convexity, and a plasticity regime that encodes information based on algorithmic Hebbian synaptic plasticity.

1. Introduction

The study of memory includes two important components: the storage component of memory and the systems component of memory [1, 2]. The first is concerned with exploring the molecular mechanisms whereby memory is stored, whereas the second is concerned with analyzing the organizing principles that mediate brain systems to encode, store, and retrieve memory. The first neurophysiological description about the systems component of memory was proposed by Hebb [3]. His postulate reveals a principle of learning, which is often summarized as "the connections between neurons are strengthened when they fire simultaneously." The Hebbian concept stimulates an intensive effort to promote the building of associative memory models of the brain [4–9]. Also, it leads to the development of a LAMINART model matching in laminar visual cortical circuitry [10, 11], the development of an Ising model used in statistical physics [12–15], and the study of constrained optimization problems such as the famous traveling salesman problem [16].

However, since it was initiated by Kohonen and Anderson in 1972, associative memory has remained widely open in neural networks [17–21]. It generally includes questions concerning a description of collective dynamics and computing with attractors in neural networks. Hence the central question [22]: "given an arbitrary set of prototypes of 01-strings of length n, is there any recurrent network such that the set of all equilibrium states of this network is exactly the set of those prototypes?" Many attempts have been made to tackle this problem. For instance, using the method of energy minimization, Hopfield in 1982 constructed a network of nerve cells whose dynamics tend toward an equilibrium state when the retrieval operation is performed asynchronously [13]. Furthermore, to circumvent limited capacity in storage and retrieval of Hopfield networks, Personnaz et al. in 1986 investigated the behavior of neural networks designed with the projection rule, which guarantees the errorless storage and retrieval of prototypes [23, 24]. In 1987, Diederich and Opper proposed an iterative scheme to substitute a local learning rule for the projection rule when the prototypes are linearly independent [25, 26]. This sheds light on the possibility of storing correlated prototypes in neural networks with local learning rules.

In addition to the discussion on learning mechanisms for associative memory, Hopfield networks have also given a valuable impetus to basic research in combinatorial fixed point theory in neural networks. In 1992, Shrivastava et al. conducted a convergence analysis of a class of Hopfield networks and showed that all equilibrium states of these networks have a one-to-one correspondence with the maximal independent sets of certain undirected graphs [27]. Müezzinoğlu and Güzeliş in 2004 gave a further compatibility condition on the correspondence between equilibrium states and maximal independent sets, which avoids spurious stored patterns in information storage and provides attractiveness of prototypes in retrieval operation [28]. Moreover, the analytic approach of Shih and Ho [29] in 1999 as well as Shih and Dong [30] in 2005 illustrated the reverberating-circuit structure to determine equilibrium states in generalized boolean networks, leading to a solution of the boolean Markus-Yamabe problem and a proof of network perspective of the Jacobian conjecture, respectively.

More recently, we described an evolutionary neural network in which the connection strengths between neurons are highly evolved according to algorithmic Hebbian synaptic plasticity [31]. To explore the influence of synaptic plasticity on the evolutionary neural network's dynamics, a sort of driving forces from the meta-complication of the evolutionary neural network's nodal-and-coupling activities is introduced, in contrast with the explicit construction of global Lyapunov functions in neural networks [10, 13, 32, 33] and in accordance with the limitation of finding a common quadratic Lyapunov function to control a switched system's dynamics [34–36]. A mathematical proof asserts that the ongoing changes of the evolutionary network's nodal-and-coupling dynamics will eventually come to rest at equilibrium states [31]. This result reflects, in a deep mathematical sense, that plastic changes in the coupling dynamics may appear as a mechanism for associative memory.

In this respect, an information storage scheme for associative memory may be suggested as follows. It comprises three ingredients. First, based on the Hebbian learning rule, establish a primitive neural network whose equilibrium states contain the prototypes and derive a common qualitative property $\wp$ from all the domains of attraction of equilibrium states. Second, determine a merging process that merges the domains of attraction of equilibrium states such that each merging domain contains exactly one prototype and that preserves the property $\wp$. Third, based on algorithmic Hebbian synaptic plasticity, probe a plasticity regime that guides the evolution of the primitive neural network such that each vertex in the merging domain will tend toward the unique prototype underlying the dynamics of the resulting evolutionary neural network.

Our point of departure is the convexity packing lurking behind Hopfield networks. We consider the domain of attraction in which every initial state in the domain tends toward the equilibrium state asynchronously. For the asynchronous operating mode, each trajectory in the phase space can represent as one of the "connected" paths between the initial state and the equilibrium state when it is measured by the Hamming metric. It provides a clear map that all the domains of attraction in Hopfield networks are star-convexity-like and that the phase space can be filled with those star-convexity-like domains. And it applies to frame a primitive Hopfield network that might consolidate an insight of exploring a plasticity regime in the information storage scheme.

2. Information Storage of Hopfield Networks

Let $\{0,1\}^n$ denote the binary code consisting of all 01 strings of fixed length n, and let $X = \{x^1, x^2, \ldots, x^p\}$ be an arbitrary set of prototypes in $\{0,1\}^n$. For each positive integer k, let $\mathbb{N}_k = \{1,2,\ldots,k\}$. Using the formal neurons of McCulloch and Pitts [37], we can construct a Hopfield network of n coupled neurons, namely, $1,2,\ldots,n$, whose *synaptic strengths* are listed in an array, denoted by the matrix $A = (a_{ij})_{n\times n}$, and defined on the basis of the Hebbian learning rule, that is,

$$a_{ij} = \sum_{s=1}^{p} x_i^s x_j^s \quad \text{for every } i, j \in \mathbb{N}_n. \tag{2.1}$$

The *firing state* of each neuron i is denoted by $x_i = 1$, whereas the *quiescent state* is $x_i = 0$. The function $\mathbb{I}$ is the Heaviside function: $\mathbb{I}(u) = 1$ for $u \geq 0$, otherwise 0, which describes an instantaneous unit pulse. The dynamics of the Hopfield network is encoded by the function $F = (f_1, f_2, \ldots, f_n)$, where

$$f_i(x) = \mathbb{I}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right) \tag{2.2}$$

encodes the dynamics of neuron i, $x = (x_1, x_2, \ldots, x_n)$ is a vector of *state variables* in the phase space $\{0,1\}^n$, and $b_i \in \mathbb{R}$ is the *threshold* of neuron i for each $i \in \mathbb{N}_n$.

For every $x, y \in \{0,1\}^n$, define the *vectorial distance* between x and y [38, 39], denoted as $d(x,y)$, to be

$$d(x,y) = \begin{pmatrix} |x_1 - y_1| \\ \vdots \\ |x_n - y_n| \end{pmatrix}. \tag{2.3}$$

For every $x, y \in \{0,1\}^n$, define the order relation $x \le y$ by $x_i \le y_i$ for each $i \in \mathbb{N}_n$; the *chain interval* between x and y, denoted as $C[x, y]$, to be

$$C[x,y] = \{z \in \{0,1\}^n; d(z,y) \le d(x,y)\}. \tag{2.4}$$

Note that $C[x, y] = C[y, x]$, and the notation $C(x, y]$ means that $C[x, y] \setminus \{x\}$. The *Hamming metric* ρ_H on $\{0,1\}^n$ is defined to be

$$\rho_H(x,y) = \#\{i \in \mathbb{N}_n; x_i \neq y_i\} \tag{2.5}$$

for every $x, y \in \{0,1\}^n$ [40]. Denote by $\gamma(x, y)$ a chain joining x and y with the *minimum Hamming distance*, meaning that

$$\gamma(x,y) = \{x, u_1, u_2, \dots, u_{r-1}, y\}, \tag{2.6}$$

where $\rho_H(u_i, u_{i+1}) = 1$ for $i = 0, 1, \dots, r-1$ with $u_0 = x, u_r = y$, and $\rho_H(x, u_1) + \rho_H(u_1, u_2) + \cdots + \rho_H(u_{r-1}, y) = \rho_H(x, y)$. Then we have $C[x, y] = \bigcup \gamma(x, y)$, where the union is taken over all chains joining x and y with the minimum Hamming distance.

Denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^n$. A set of elements y^α in $\{0,1\}^n$, where α runs through some index set in I, is called *orthogonal* if $\langle y^\alpha, y^\beta \rangle = 0$ for each $\alpha, \beta \in I$ with $\alpha \neq \beta$. Two sets Y and Z in $\{0,1\}^n$ are called *mutually orthogonal* if $\langle y, z \rangle = 0$ for each $y \in Y$ and $z \in Z$. Given a set $Y = \{y^1, y^2, \dots, y^q\}$ in $\{0,1\}^n$, we define the 01-*span* of Y, denoted as 01-span(Y), to be the set consists of all elements of the form $\tau_1 y^1 + \tau_2 y^2 + \cdots + \tau_q y^q$, where $\tau_i \in \{0,1\}$ for each $i \in \mathbb{N}_q$. We assume that $x^i \neq 0$ for each $i \in \mathbb{N}_p$. For each $i \in \mathbb{N}_p$, define

$$N_{x^i}^1 = \left\{x^s \in X; \left\langle x^s, x^i \right\rangle \neq 0\right\} \tag{2.7}$$

and define recursively

$$N_{x^i}^{j+1} = \left\{x^s \in X; \left\langle x^s, x^k \right\rangle \neq 0 \text{ for some } x^k \in \mathbb{N}_{x^i}^j \right\} \tag{2.8}$$

for each $j \in \mathbb{N}$. Clearly, for each $i \in \mathbb{N}_p$ we have

$$N_{x^i}^1 \subset N_{x^i}^2 \subset N_{x^i}^3 \subset \cdots, \tag{2.9}$$

and thereby there exists a smallest positive integer, denoted as $s(i)$, such that

$$N_{x^i}^{s(i)} = N_{x^i}^{s(i)+j} \quad \text{for each } j \in \mathbb{N}. \tag{2.10}$$

It is readily seen that for each $i \in \mathbb{N}_p$ and for each $x^j \in N_{x^i}^{s(i)}$, we have

$$N_{x^i}^{s(i)} = N_{x^j}^{s(j)}, \tag{2.11}$$

and clearly, for every $i, j \in \mathbb{N}_p$, exactly one of the following conditions holds:

$$N_{x^i}^{s(i)} = N_{x^j}^{s(j)} \quad \text{or} \quad N_{x^i}^{s(i)} \cap N_{x^j}^{s(j)} = \emptyset. \tag{2.12}$$

According to (2.8) and (2.12), we can pick all distinct sets $N_1, N_2, \ldots, N_q$ from $\{N_{x^1}^{s(1)}, N_{x^2}^{s(2)}, \ldots, N_{x^p}^{s(p)}\}$ and obtain the *orthogonal partition* of X, that is, N_i and N_j are mutually orthogonal for every $i \neq j$ and $X = \bigcup_{i \in \mathbb{N}_q} N_i$. For each $k \in \mathbb{N}_q$, define

$$\xi^k = \begin{pmatrix} \max\{x_1^i; x^i \in N_k\} \\ \vdots \\ \max\{x_n^i; x^i \in N_k\} \end{pmatrix}. \tag{2.13}$$

Then we have the orthogonal set $\{\xi^1, \xi^2, \ldots, \xi^q\}$ generated by the orthogonal partition of X, which is denoted as $\mathrm{Gop}(X)$.

Using the "orthogonal partition," we can give a complete description of the equilibrium states of the Hopfield network encoded by (2.1) and (2.2) with ultra-low thresholds.

Theorem 2.1. *Let X be a set consisting of nonzero vectors in $\{0,1\}^n$, and let the function F be defined by* (2.1) *and* (2.2) *with $0 < b_i \leq 1$ for each $i \in \mathbb{N}_n$. Then,* $\mathrm{Fix}(F) = 01\text{-span}(\mathrm{Gop}(X))$.

Proof. Let $X = \{x^1, x^2, \ldots, x^p\}$ and let $\mathrm{Gop}(X) = \{\xi^1, \xi^2, \ldots, \xi^q\}$. By orthogonality of $\mathrm{Gop}(X)$, $1 - \sum_{i=1}^{q} \xi_j^i$ is 0 or 1 for each $j \in \mathbb{N}_n$. Thus the point $\mathbb{I}(\mathrm{Gop}(X))$, defined by

$$\mathbb{I}(\mathrm{Gop}(X)) = \left(1 - \sum_{i=1}^{q} \xi_1^i, 1 - \sum_{i=1}^{q} \xi_2^i, \ldots, 1 - \sum_{i=1}^{q} \xi_n^i\right), \tag{2.14}$$

lies in $\{0,1\}^n$. Let $U_0 = C[0, \mathbb{I}(\mathrm{Gop}(X))]$ and $U_i = C(0, \xi^i]$ for each $i \in \mathbb{N}_q$. Note that the sets U_i and U_j are mutually orthogonal for every $i \neq j$. Let $\xi = \sum_{i=1}^{q} \alpha_i \xi^i$ for $\alpha_i \in \{0,1\}$ and $i \in \mathbb{N}_q$. We prove now that $F(\xi) = \xi$ by showing that

$$F(x) \in C[x, \xi] \quad \text{for each } x \in U_0 + \sum_{i=1}^{q} \alpha_i U_i. \tag{2.15}$$

Let $x = u^0 + \sum_{i=1}^{q} \alpha_i u^i$ where $u^i \in U_i$ for $i = 0, 1, \ldots, q$. Since $X \cap C[0, \xi^k] = N_k$ for each $k \in \mathbb{N}_q$, we have

$$
\begin{aligned}
F(x) &= \mathbb{I}\left(\sum_{i=1}^{p}\left(x^{i^T}u^0\right)x^i + \sum_{j=1}^{q}\sum_{i=1}^{p}\left(\alpha_j\left(x^{i^T}u^j\right)x^i\right) - b\right) \\
&= \mathbb{I}\left(\sum_{j=1}^{q}\sum_{x^i\in N_j}\left(\alpha_j\left(x^{i^T}u^j\right)x^i\right) - b\right) \\
&\leq \sum_{j=1}^{q}\alpha_j\xi^j.
\end{aligned}
\tag{2.16}
$$

Thus we need only consider the case $F(x)_\nu = 0$ and $\xi_\nu = 1$ for some $\nu \in \mathbb{N}_n$. Under the case, there exists $r \in \mathbb{N}_q$ such that $\alpha_r = 1$ and $\xi_\nu^r = 1$, so that

$$
\begin{aligned}
F(x)_\nu &\geq \mathbb{I}\left(\sum_{x^i\in N_r}\left(x^{i^T}u^r\right)x_\nu^i - b_\nu\right) \\
&\geq \mathbb{I}\left(\sum_{x^i\in N_r}u_\nu^r\left(x_\nu^i\right)^2 - b_\nu\right).
\end{aligned}
\tag{2.17}
$$

Since $F(x)_\nu = 0$, we have $x_\nu = u_\nu^r = 0$. This implies that $d(F(x),\xi) \leq d(x,\xi)$, that is, $F(x) \in C[x,\xi]$.

We turn now to prove that $F(x) \neq x$ for each $x \notin$ 01-span(Gop(X)). To accomplish this, we first show that

$$
\{0,1\}^n = \bigcup_{\alpha_i\in\{0,1\},\ i\in\mathbb{N}_q} U_0 + \sum_{i=1}^{q}\alpha_i U_i. \tag{2.18}
$$

Let $x \in \{0,1\}^n$. We associate to each $i \in \mathbb{N}_q$ a point

$$
z^i = \left(x_1\xi_1^i, x_2\xi_2^i, \ldots, x_n\xi_n^i\right) \tag{2.19}
$$

and put $z^0 = x - \sum_{i=1}^{q} z^i$. Then for each $i \in \mathbb{N}_q$, there exist $\alpha_i \in \{0,1\}$ such that $z^i \in \alpha_i U_i$. Since for each $k \in \mathbb{N}_n$

$$
z_k^0 = x_k - \sum_{i=1}^{q} x_k\xi_k^i \leq 1 - \sum_{i=1}^{q}\xi_k^i, \tag{2.20}
$$

we have $z^0 \in U_0$, proving (2.18). Thus each $x \notin$ 01-span(Gop(X)) can be written as $x = u^0 + \sum_{i=1}^{q}\alpha_i u^i$, where $\alpha_i \in \{0,1\}$, $u^i \in U_i$ for $i = 0,1,\ldots,q$ and, further, we have either $u^0 \neq 0$ or there exists $r \in \mathbb{N}_q$ such that $\alpha_r = 1$ and $u^r \neq \xi^r$.

Case 1 ($u^0 \neq 0$). Then there exists $\nu \in \mathbb{N}_n$ such that $u^0_\nu = 1$ and $x^k_\nu = 0$ for each $k \in \mathbb{N}_p$. This implies that

$$x_\nu = u^0_\nu + \sum_{i=1}^{q} \alpha_i u^i_\nu = 1,$$
$$F(x)_\nu = \mathbb{1}\left(\sum_{j=1}^{q} \sum_{x^i \in N_j} \left(\alpha_j \left(x^{i^T} u^j \right) x^i_\nu \right) - b_\nu \right) = 0, \tag{2.21}$$

proving $F(x) \neq x$.

Case 2. There exists $r \in \mathbb{N}_q$ such that $\alpha_r = 1$ and $u^r \neq \xi^r$. Then

$$C[0, \xi^r] \cap (X \setminus C[0, u^r]) \cap (X \setminus C[0, \xi^r - u^r]) \neq \emptyset. \tag{2.22}$$

Indeed, if the left hand side of (2.22) is empty, then for every $x^i \in N_r = X \cap C[0, \xi^r]$, exactly one of the following conditions holds:

$$x^i \in C[0, u^r] \quad \text{or} \quad x^i \in C[0, \xi^r - u^r]. \tag{2.23}$$

Divide the set N_r into two subsets:

$$x^i \in M_1 \quad \text{if } x^i \in C[0, u^r],$$
$$x^i \in M_2 \quad \text{if } x^i \in C[0, \xi^r - u^r]. \tag{2.24}$$

Then, by the construction of ξ^r, we have $M_1 \neq \emptyset$ and $M_2 \neq \emptyset$. Now let $x^\sigma \in M_1$ and $x^\eta \in M_2$. Since M_1 and M_2 are mutually orthogonal, we get $N^{s(\sigma)}_{x^\sigma} \subset M_1$ and $N^{s(\eta)}_{x^\eta} \subset M_2$. This con-tradicts

$$N^{s(\sigma)}_{x^\sigma} = N^{s(\eta)}_{x^\eta} = N_r, \tag{2.25}$$

proving (2.22). Therefore, there exist

$$x^k \in C[0, \xi^r] \cap (X \setminus C[0, u^r]) \cap (X \setminus C[0, \xi^r - u^r]) \tag{2.26}$$

and $k_1, k_2 \in \mathbb{N}_n$ with $u^r_{k_1} = 1$ and $(\xi^r - u^r)_{k_2} = 1$ such that $x^k_{k_1} = x^k_{k_2} = 1$. Since $(\xi^r - u^r)_{k_2} = 1$,

$u^i_{k_2} = 0$ for $i = 0, 1, \ldots, q$ and $x^i_{k_2} = 0$ for each $x^i \notin N_r$. This implies that

$$x_{k_2} = u^0_{k_2} + \sum_{i=1}^{q} \alpha_i u^i_{k_2} = 0,$$
$$F(x)_{k_2} \geq \mathbb{1}\left(\sum_{x^i \in N_r} \left({x^i}^T u^r \right) x^i_{k_2} - b_{k_2} \right) \geq \mathbb{1}\left(x^k_{k_1} u^r_{k_1} x^k_{k_2} - b_{k_2} \right) = 1, \tag{2.27}$$

revealing $F(x) \neq x$, proving Theorem 2.1. □

3. Domains of Attraction and Hamming Star-Convex Building Blocks

By analogy with the notion of star-convexity in vector spaces, a set U in $\{0,1\}^n$ is said to be *Hamming star-convex* if there exists a point $y \in U$ such that $C[x, y] \subset U$ for each $x \in U$. We call y a *star-center* of U.

Let X be a set in $\{0,1\}^n$, and let Λ_X denote the collection of all 01-span(Y), where Y is an orthogonal set consisting of nonzero vectors in $\{0,1\}^n$, such that $X \subset$ 01-span(Y). Then $\Lambda_X \neq \emptyset$. Indeed, if the order "$\leq$" on Λ_X is defined by $A \leq B$ if and only if $A \subset B$, then $(\Lambda_X, \leq)$ becomes a partially ordered set and there exists an orthogonal set Y such that 01-span(Y) is minimal in Λ_X. We call such Y the *kernel* of X. A labeling procedure for establishing the kernel Y of X is described as follows. Let $X = \{x^1, x^2, \ldots, x^p\}$ in $\{0,1\}^n$. If $X = \{0\}$, then $Y = \{y\}$, where $y \neq 0$, is the kernel of X. Otherwise, define the labelings

$$\lambda^i = \left(x^1_i, x^2_i, \ldots, x^p_i \right) \quad \text{for each } i \in \mathbb{N}_n \tag{3.1}$$

and pick all distinct nonzero labelings $v^1, v^2, \ldots, v^q$ from $\lambda^1, \lambda^2, \ldots, \lambda^n$. Then the orthogonal set $Y = \{y^1, y^2, \ldots, y^q\}$, given by $y^i_j = 1$ if $\lambda^j = v^i$, otherwise $y^i_j = 0$ for each $i \in \mathbb{N}_q$ and $j \in \mathbb{N}_n$, is the kernel of X (see Figure 1). Note that since the computation of the kernel can be implemented by radix sort, its computational complexity is in $\Theta(pn)$.

Let $Y = \{y^1, y^2, \ldots, y^q\}$ be the kernel of X. We associate to each $y^k \in Y$ an integer $n(k) \in \mathbb{N}$, two sets of nodes

$$V_k = \{v_{k,l};\ l \in \mathbb{N}_{n(k)}\}, \qquad W_k = \left\{ w_{k,j};\ y^k_j = 1, j \in \mathbb{N}_n \right\}, \tag{3.2}$$

and a set of edges E_k such that $G_k = (V_k \cup W_k, E_k)$ is a simple, connected, and bipartite graph with color classes V_k and W_k. (The graph-theoretic notion and terminologies can be found in [41]). For each $j \in \mathbb{N}_n$, put $u^{k,l}_j = 1$ if $v_{k,l}$ and $w_{k,j}$ are adjacent, otherwise 0. Let $G = \{G_1, G_2, \ldots, G_q\}$ and denote by Bip(Y, G) the collection of all vectors $u^{k,l}$ constructed by the bipartite graphs in G (see Figure 1).

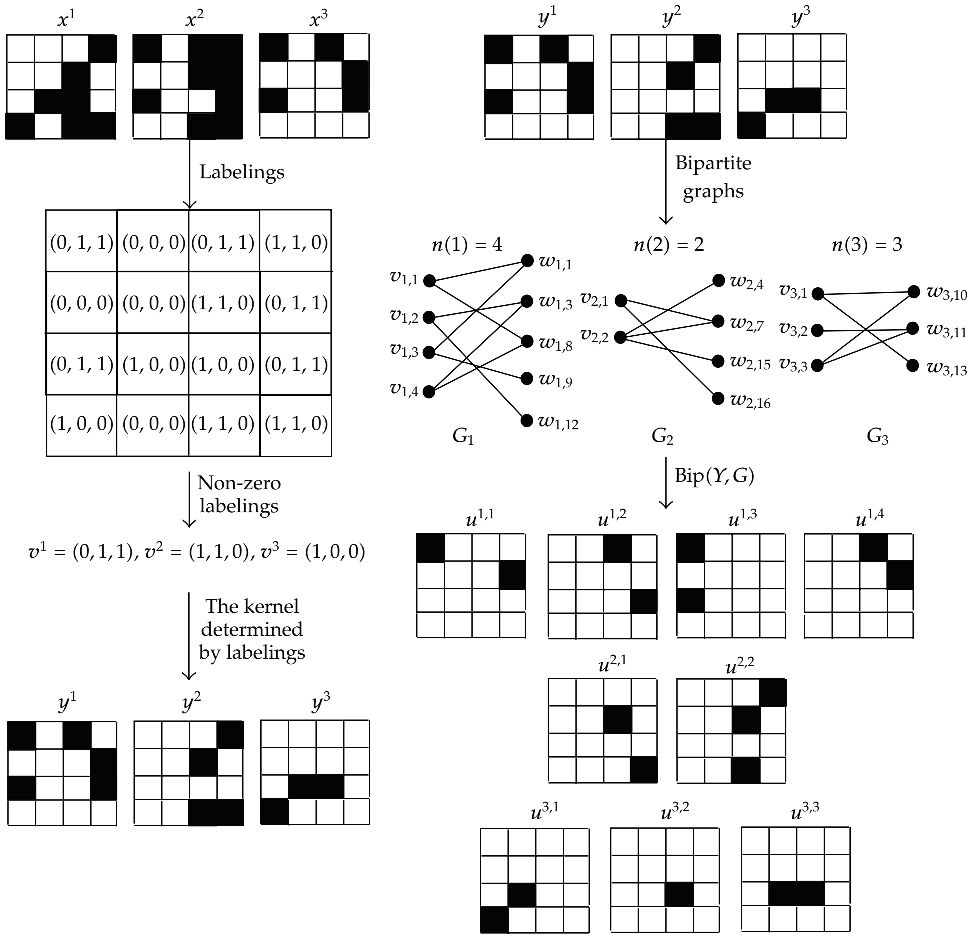

Figure 1: A schematic illustration of the generation of the kernel Y and $\mathrm{Bip}(Y, G)$.

Denote by $\mathrm{Fix}(F)$ the set of all equilibrium states (fixed points) of F and denote by $D_{\mathrm{GS}}(\xi)$ the domain of attraction of the equilibrium state ξ underlying Gauss-Seidel iteration (a particular mode of asynchronous iteration)

$$x_i(t+1) = f_i(x_1(t+1), \ldots, x_{i-1}(t+1), x_i(t), \ldots, x_n(t)) \tag{3.3}$$

for $t = 0, 1, \ldots$ and $i \in \mathbb{N}_n$.

Theorem 3.1. *Let X be a subset of $\{0,1\}^n$, and let $Y = \{y^1, y^2, \ldots, y^q\}$ be the kernel of X. Associate to each*

$$\mathrm{Bip}(Y, G) = \left\{u^{k,l};\ k \in \mathbb{N}_q, l \in \mathbb{N}_{n(k)}\right\} \tag{3.4}$$

a function F defined by (2.2) *with*

$$a_{ij} = \sum_{k=1}^{q} \sum_{l=1}^{n(k)} u_i^{k,l} u_j^{k,l} \quad \text{for each } i, j \in \mathbb{N}_n \tag{3.5}$$

and $0 < b_i \leq 1$ for each $i \in \mathbb{N}_n$. Then

(i) $X \subset \mathrm{Fix}(F)$;

(ii) *for each $\xi \in \mathrm{Fix}(F)$, the domain of attraction $D_{GS}(\xi)$ is Hamming star-convex with ξ as a star-center.*

Proof. For each $k \in \mathbb{N}_q$, since G_k is simple, connected, and bipartite with color classes V_k and W_k, we have

$$N_{u^{k,l}}^{s(k,l)} = N_{u^{k,j}}^{s(k,j)} \quad \text{for each } l, j \in \mathbb{N}_{n(k)}. \tag{3.6}$$

It follows from the orthogonality of Y that

$$\left\{u^{k,l};\ l \in \mathbb{N}_{n(k)}\right\} \subset N_{u^{k,j}}^{s(k,j)} \subset C\left(0, y^k\right] \tag{3.7}$$

for each $k \in \mathbb{N}_q$ and $j \in \mathbb{N}_{n(k)}$. Furthermore, since G_k is connected for each $k \in \mathbb{N}_q$, we have

$$\max\left\{u_j^{k,l};\ l \in \mathbb{N}_{n(k)}\right\} = y_j^k \quad \text{for each } j \in \mathbb{N}_n. \tag{3.8}$$

This implies that $\mathrm{Gop}(\mathrm{Bip}(Y, G)) = Y$, and by Theorem 2.1, we have

$$\mathrm{Fix}(F) = 01\text{-span}\left(\mathrm{Gop}\left(\mathrm{Bip}(Y, G)\right)\right) \supset X, \tag{3.9}$$

proving (i). To prove (ii), we first show that for each $i \in \mathbb{N}_q$ and $\alpha_i \in \{0, 1\}$,

$$C[0, \mathbb{I}(Y)] + \sum_{i=1}^{q} \alpha_i C\left(0, y^i\right] \subset D_{\mathrm{GS}}\left(\sum_{i=1}^{q} \alpha_i y^i\right), \tag{3.10}$$

where $\mathbb{I}(Y) = (1 - \sum_{i=1}^{q} y_1^i, 1 - \sum_{i=1}^{q} y_2^i, \ldots, 1 - \sum_{i=1}^{q} y_n^i)$. Let U denote the set in the left hand side of (3.10), and let $x \in U$, $y = \sum_{i=1}^{q} \alpha_i y^i$, and $z \in C[x, y]$. Split z into two parts:

$$z = \left(z_1 - \sum_{i=1}^{q} z_1 y_1^i, z_2 - \sum_{i=1}^{q} z_2 y_2^i, \ldots, z_n - \sum_{i=1}^{q} z_n y_n^i\right) + \sum_{i=1}^{q} \left(z_1 y_1^i, z_2 y_2^i, \ldots, z_n y_n^i\right). \tag{3.11}$$

Then the first part of z lies in $C[0, \mathbb{I}(Y)]$, and the second part of z lies in $\sum_{i=1}^{q} \alpha_i C(0, y^i]$. This shows that U is Hamming star-convex with y as a star-center, that is,

$$C[x, y] \subset U \quad \text{for each } x \in U. \tag{3.12}$$

Combining (3.12) with (2.15) shows that

$$(x_1,\ldots,x_{i-1},f_i(x),x_{i+1},\ldots,x_n)\in C[x,y]\subset U \tag{3.13}$$

for each $i\in\mathbb{N}_n$ and $x\in U$. Since $\mathrm{Fix}(F)\cap U=\{y\}$ by Theorem 2.1, inclusion (3.10) follows immediately from (3.13). Now, by combining (3.10) with (2.18), we obtain

$$C[0,\mathbb{I}(Y)]+\sum_{i=1}^{q}\alpha_i C\left(0,y^i\right]=D_{\mathrm{GS}}\left(\sum_{i=1}^{q}\alpha_i y^i\right). \tag{3.14}$$

for each $i\in\mathbb{N}_q$ and $\alpha_i\in\{0,1\}$, proving (ii). □

4. Hamming Star-convexity Packing

Theorem 3.1 reveals how a collection of Hamming star-convex sets is generated by the dynamics of neural networks. These Hamming star-convex sets are called the *building blocks* of $\{0,1\}^n$. By merging the Hamming star-convex building blocks, we obtain the Hamming star-convexity packing as a consequence of the dynamics of neural networks (see Figure 2).

Theorem 4.1. *Let $X=\{x^1,x^2,\ldots,x^p\}$ be a subset of $\{0,1\}^n$. Then the phase space $\{0,1\}^n$ can be filled with p nonoverlapping Hamming star-convex sets with $x^1,x^2,\ldots,x^p$ as star-centers, respectively.*

Proof. Let $Y=\{y^1,y^2,\ldots,y^q\}$ be the kernel of X. According to Theorem 3.1, we can construct a neural network with a function F encoding the dynamics such that the domains of attraction $D_{\mathrm{GS}}(\xi)$, where $\xi\in$ 01-span(Y), are the Hamming star-convex building blocks of $\{0,1\}^n$. To merge these Hamming star-convex building blocks, we establish first the following.

Assertion 4.2. For every $x,y\in\{0,1\}^n$ and for every $v^1,v^2,\ldots,v^k\in C[x,y]$, there exist $u^1,u^2,\ldots,u^k\in C[x,y]$ such that

$$\begin{gathered}C[x,y]=C\left[u^1,v^1\right]\cup C\left[u^2,v^2\right]\cup\cdots\cup C\left[u^k,v^k\right],\\ C\left[u^i,v^i\right]\cap C\left[u^j,v^j\right]=\emptyset\end{gathered} \tag{4.1}$$

for every $i,j\in\mathbb{N}_k$ with $i\neq j$.

It is clear that the assertion is valid for every $x,y\in\{0,1\}^n$ with $\rho_H(x,y)=0$. Assume that the assertion is valid for every $x,y\in\{0,1\}^n$ with $\rho_H(x,y)=p<n$. Now let $x,y\in\{0,1\}^n$ with $\rho_H(x,y)=p+1$. Choose α so that $x_\alpha\neq y_\alpha$, and use the complemented notation: $\overline{0}=1$, $\overline{1}=0$. Then, we have

$$C[x,y]=C[\tilde{x}^\alpha,y]\cup C[x,\tilde{y}^\alpha], \tag{4.2}$$

$$C[\tilde{x}^\alpha,y]\cap C[x,\tilde{y}^\alpha]=\emptyset, \tag{4.3}$$

where $\tilde{x}^\alpha=(x_1,\ldots,x_{\alpha-1},\overline{x}_\alpha,x_{\alpha+1},\ldots,x_n)$ and $\tilde{y}^\alpha=(y_1,\ldots,y_{\alpha-1},\overline{y}_\alpha,y_{\alpha+1},\ldots,y_n)$.

Case 1. $\{v^1, v^2, \ldots, v^k\} \cap C[\tilde{x}^\alpha, y] = \emptyset$ or $\{v^1, v^2, \ldots, v^k\} \cap C[x, \tilde{y}^\alpha] = \emptyset$. We may assume that $v^1, v^2, \ldots, v^k \in C[\tilde{x}^\alpha, y]$. Then, by the induction hypothesis, there exist $u^1, u^2, \ldots, u^k \in C[\tilde{x}^\alpha, y]$ such that

$$C[\tilde{x}^\alpha, y] = C\left[u^1, v^1\right] \cup C\left[u^2, v^2\right] \cup \cdots \cup C\left[u^k, v^k\right], \tag{4.4}$$

$$C\left[u^i, v^i\right] \cap C\left[u^j, v^j\right] = \emptyset \tag{4.5}$$

for every $i, j \in \mathbb{N}_k$ with $i \neq j$. For each $i \in \mathbb{N}_k$, let

$$\begin{aligned} \widetilde{u^i}^\alpha &= \left(u_1^i, \ldots, u_{\alpha-1}^i, \overline{u}_\alpha^i, u_{\alpha+1}^i, \ldots, u_n^i\right), \\ \widetilde{v^i}^\alpha &= \left(v_1^i, \ldots, v_{\alpha-1}^i, \overline{v}_\alpha^i, v_{\alpha+1}^i, \ldots, v_n^i\right). \end{aligned} \tag{4.6}$$

Since $x_\alpha \neq y_\alpha$, it follows from (4.4) and (4.5) that

$$C[x, \tilde{y}^\alpha] = C\left[\widetilde{u^1}^\alpha, \widetilde{v^1}^\alpha\right] \cup C\left[\widetilde{u^2}^\alpha, \widetilde{v^2}^\alpha\right] \cup \cdots \cup C\left[\widetilde{u^k}^\alpha, \widetilde{v^k}^\alpha\right], \tag{4.7}$$

$$C\left[\widetilde{u^i}^\alpha, \widetilde{v^i}^\alpha\right] \cap C\left[\widetilde{u^j}^\alpha, \widetilde{v^j}^\alpha\right] = \emptyset \tag{4.8}$$

for every $i, j \in \mathbb{N}_k$ with $i \neq j$. Now combining (4.2), (4.4), and (4.7) with the property

$$C\left[u^i, \widetilde{v^i}^\alpha\right] = C\left[u^i, v^i\right] \cup C\left[\widetilde{u^i}^\alpha, \widetilde{v^i}^\alpha\right] \quad \text{for each } i \in \mathbb{N}_k, \tag{4.9}$$

we obtain

$$C[x, y] = C\left[u^1, \widetilde{v^1}^\alpha\right] \cup C\left[u^2, \widetilde{v^2}^\alpha\right] \cup \cdots \cup C\left[u^k, \widetilde{v^k}^\alpha\right]. \tag{4.10}$$

Moreover, it follows from (4.3), (4.5), and (4.8) that

$$C\left[u^i, \widetilde{v^i}^\alpha\right] \cap C\left[u^j, \widetilde{v^j}^\alpha\right] = \emptyset \tag{4.11}$$

for every $i, j \in \mathbb{N}_k$ with $i \neq j$.

Case 2. $\{v^1, v^2, \ldots, v^k\} \cap C[\tilde{x}^\alpha, y] \neq \emptyset$ and $\{v^1, v^2, \ldots, v^k\} \cap C[x, \tilde{y}^\alpha] \neq \emptyset$. We may assume that $v^1, v^2, \ldots, v^s \in C[\tilde{x}^\alpha, y]$ and $v^{s+1}, v^{s+2}, \ldots, v^k \in C[x, \tilde{y}^\alpha]$, where $s \in \mathbb{N}_k$. Then, by (4.2), (4.3), and the induction hypothesis, there exist $u^1, u^2, \ldots, u^s \in C[\tilde{x}^\alpha, y]$ and $u^{s+1}, u^{s+2}, \ldots, u^k \in C[x, \tilde{y}^\alpha]$ such that

$$\begin{gathered} C[x, y] = C\left[u^1, v^1\right] \cup C\left[u^2, v^2\right] \cup \cdots \cup C\left[u^k, v^k\right], \\ C\left[u^i, v^i\right] \cap C\left[u^j, v^j\right] = \emptyset \end{gathered} \tag{4.12}$$

for every $i, j \in \mathbb{N}_k$ with $i \neq j$, completing the inductive proof of the assertion. □

Applying now the assertion to $C[x, y] = \{0,1\}^n$ and the given points $x^1, x^2, \ldots, x^p$, we obtain $u^1, u^2, \ldots, u^r$ such that

$$\{0,1\}^n = C\left[u^1, x^1\right] \cup C\left[u^2, x^2\right] \cup \cdots \cup C[u^p, x^p], \\ C\left[u^i, x^i\right] \cap C\left[u^j, x^j\right] = \emptyset \tag{4.13}$$

for every $i, j \in \mathbb{N}_p$ with $i \neq j$. For each $k \in \mathbb{N}_p$, define

$$\Omega_k = 01\text{-span}(Y) \cap C\left[u^k, x^k\right]. \tag{4.14}$$

Then, $\Omega_k \neq \emptyset$ for each $k \in \mathbb{N}_p$, since $01\text{-span}(Y) \in \Lambda_X$.

Claim 4.3. For each $k \in \mathbb{N}_p$, the set $\bigcup_{\xi \in \Omega_k} D_{\mathrm{GS}}(\xi)$ is Hamming star-convex with x^k as a star center.

Fix $k \in \mathbb{N}_p$ and write $x^k = \sum_{i=1}^q \gamma_i y^i$, where $\gamma_i \in \{0,1\}$ for each $i \in \mathbb{N}_q$. Let $z \in \bigcup_{\xi \in \Omega_k} D_{\mathrm{GS}}(\xi)$. Then, there exists $y \in \Omega_k$ such that $z \in D_{\mathrm{GS}}(y)$. Write $y = \sum_{i=1}^q \alpha_i y^i$, where $\alpha_i \in \{0,1\}$ for each $i \in \mathbb{N}_q$. Then, by (3.14), there exist $z^0 \in C[0, \mathbb{I}(Y)]$ and $z^i \in C(0, y^i]$ for each $i \in \mathbb{N}_q$ such that $z = z^0 + \sum_{i=1}^q \alpha_i z^i$. We have to show that

$$C\left[z, x^k\right] \subset \bigcup_{\xi \in \Omega_k \cap C[y, x^k]} D_{\mathrm{GS}}(\xi). \tag{4.15}$$

Let $v \in C[z, x^k]$. Then, by (2.18), there exist $v^0 \in C[0, \mathbb{I}(Y)]$, $v^i \in C(0, y^i]$, and $\beta_i \in \{0,1\}$ for each $i \in \mathbb{N}_q$ such that $v = v^0 + \sum_{i=1}^q \beta_i v^i$. Since $v \in C[z, x^k]$, we have

$$\begin{aligned} d\left(v, x^k\right) &= v^0 + \sum_{i=1}^q d\left(\beta_i v^i, \gamma_i y^i\right) \\ &\leq z^0 + \sum_{i=1}^q d\left(\alpha_i z^i, \gamma_i y^i\right) \\ &= d\left(z, x^k\right). \end{aligned} \tag{4.16}$$

Since Y is orthogonal, (4.16) implies that

$$d\left(\beta_i v^i, \gamma_i y^i\right) \leq d\left(\alpha_i z^i, \gamma_i y^i\right) \tag{4.17}$$

for each $i \in \mathbb{N}_q$. Moreover, we have the following inequalities:

$$\left|\beta_i - \gamma_i\right| \leq \left|\alpha_i - \gamma_i\right| \quad \text{for each } i \in \mathbb{N}_q. \tag{4.18}$$

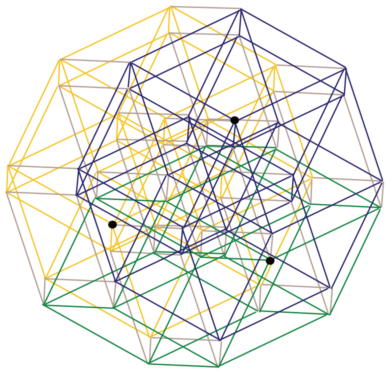

Figure 2: The proof given in **Theorem 4.1** reveals a merging process of the Hamming star-convexity packing. Here the 6-cube is filled with 3 nonoverlapping Hamming star-convex sets with star-centers spatially distributed in three vertices.

To show (4.18), fix $i \in \mathbb{N}_q$ and consider only two cases.

Case 1 ($\alpha_i = \gamma_i = 1$). Since $z^i \in C(0, y^i]$, (4.17) implies that

$$d\left(\beta_i v^i, y^i\right) \leq d\left(z^i, y^i\right) < y^i. \tag{4.19}$$

Since $v^i \in C(0, y^i]$, (4.19) implies that $\beta_i = 1$, proving (4.18).

Case 2 ($\alpha_i = \gamma_i = 0$). Then, by (4.17), we have $d(\beta_i v^i, 0) \leq 0$. Since $v^i \in C(0, y^i]$, we get $\beta_i = 0$, proving (4.18).

Now combining (4.18) with the equality

$$d\left(y, x^k\right) = \sum_{i=1}^{q} d\left(\alpha_i y^i, \gamma_i y^i\right) = \sum_{i=1}^{q} |\alpha_i - \gamma_i| y^i \tag{4.20}$$

shows that $\sum_{i=1}^{q} \beta_i y^i \in C[y, x^k]$, and hence that $\sum_{i=1}^{q} \beta_i y^i \in \Omega_k \cap C[y, x^k]$. On the other hand, by (3.14), we have $v \in D_{\mathrm{GS}}(\sum_{i=1}^{q} \beta_i y^i)$, and that (4.15) holds.

Using the fact that

$$\bigcup_{\xi \in \Omega_k \cap C[y, x^k]} D_{\mathrm{GS}}(\xi) \subset \bigcup_{\xi \in \Omega_k} D_{\mathrm{GS}}(\xi), \tag{4.21}$$

the claim follows. Combining the claim with (4.13), (4.14), and (3.14) proves the theorem.

5. Discussion

In respect of the underlying combinatorial space-filling structure of Hopfield networks, we establish an exact formula for describing all the equilibrium states of Hopfield networks with ultra-low thresholds. It provides a basis for the building of a primitive Hopfield network whose equilibrium states contain the prototypes. A common qualitative property, namely, the Hamming star-convexity, can be deduced from all those domains of attraction of equilibrium states and a merging process, which preserves the Hamming star-convexity, can also be determined. As a result, the phase space can be filled with nonoverlapping Hamming star-convex sets with all the star-centers exactly being the prototypes.

The design of the Hamming star-convexity packing can be used as a testbed for exploring the plasticity regimes that guides the evolution of the primitive Hopfield network. Indeed, we consider the evolutionary neural network whose dynamics is encoded by the nonlinear parametric equations [31]:

$$\begin{aligned} x(t+1) &= H_{A(t),s(t)}(x(t)), \quad t = 0,1,\ldots, \\ A(t+1) &= A(t) + \mathfrak{D}_{x(t)\to x(t+1)}A, \quad t = 0,1,\ldots, \end{aligned} \tag{5.1}$$

where t is time, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is the neuronal activity state at time t, $A(t) = (a_{ij}(t))_{n\times n}$ is the evolutionary coupling state at time t, $s(t) \in \{1,2,\ldots,n\}$ denotes the neuron that adjusts its activity at time t, $H_{A(t),s(t)}(x)$ is the time-and-state varying function whose ith component is defined by x_i if $i \neq s(t)$, otherwise $\mathbb{1}(\sum_{j=1}^{n} a_{ij}(t)x_j - b_i)$, and each $\mathfrak{D}_{x(t)\to x(t+1)}A$ is an n-by-n real matrix whose (i,j)-entry is a plasticity parameter representing a choice of real numbers based on algorithmic Hebbian synaptic plasticity. In [31], we have shown that for each domain $\Delta \subset \{0,1\}^n \setminus \{0\}$ and for each choice of initial neuronal activity state $x(0) \in \Delta$, there exists a plasticity regime that guides the dynamics of the evolutionary coupling states such that $x(t)$ converges and $x(t) \in \Delta$ for every $t = 0,1,\ldots$. The plasticity regime, even when insoluble in the information storage scheme by assigning $A(0)$ to be the matrix of synaptic strengths of the primitive Hopfield network given in Theorem 3.1 and Δ to be the Hamming star-convex set given in Theorem 4.1, is a guide to understand and explain the dynamism role of the Hamming star-convexity packing in storage and retrieval of associative memory.

Acknowledgment

This work was supported by the National Science Council of the Republic of China.

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