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Research Article

The Existence of Maximum and Minimum Solutions to General Variational Inequalities in the Hilbert Lattices

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We apply the variational characterization of the metric projection to prove some results about the solvability of general variational inequalities and the existence of maximum and minimum solutions to some general variational inequalities in the Hilbert lattices.

1. Introduction

The variational inequality theory and the complementarity theory have been studied by many authors and have been applied in many fields such as optimization theory, game theory, economics, and engineering [1–12]. The existence of solutions to a general variational inequality is the most important issue in the variational inequality theory. Many authors investigate the solvability of a general variational inequality by using the techniques of fixed point theory and the variational characterization of the metric projection in some linear normal spaces. Meanwhile, a certain topological continuity of the mapping involved in the considered variational inequality must be required, such as continuity and semicontinuity.

A number of authors have studied the solvability of general variational inequalities without the topological continuity of the mapping. One way to achieve this goal is to consider a linear normal space to be embedded with a partial order satisfying certain conditions, which is called a normed Riesz space. The special and most important cases of normed Riesz spaces are Hilbert lattices and Banach lattices [1, 2, 7, 13–15]. Furthermore, after the solvability has been proved for a general variational inequality, a new problem has been raised: does this general variational inequality have maximum and minimum solutions (with respect to the partial order)? (e.g., see [7]). In this paper, we study this theme and provide some results about the existence of maximum and minimum solutions to some general variational inequalities in Hilbert lattices.

This paper is organized as follows. Section 2 recalls some basic properties of Hilbert lattices, variational inequalities, and general variational inequalities. Section 3 provides some results about the existence of maximum and minimum solutions to some general variational inequalities defined on some closed, bounded, and convex subsets in Hilbert lattices. Section 4 generalizes the results of Section 3 to unbounded case.

2. Preliminaries

In this section, we recall some basic properties of Hilbert lattices and variational inequalities. For more details, the reader is referred to [1, 2, 7, 13–15].

We say that $(X;\succcurlyeq)$ is a Hilbert lattice if X is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and with the induced norm $\|\cdot\|$ and X is also a poset with the partial order $\succcurlyeq$ satisfying the following conditions:

(i) the mapping $\alpha \mathrm{id}_X + z$ is a $\succcurlyeq$-preserving self-mapping on X (this definition will be recalled later) for every $z \in X$ and positive number α, where id_X defines the identical mapping on X,

(ii) $(X;\succcurlyeq)$ is a lattice,

(iii) the norm $\|\cdot\|$ on X is compatible with the partial order $\succcurlyeq$, that is,

$$|x| \succcurlyeq |y| \text{ implies } \|x\| \geq \|y\|, \quad \text{where } |z| = (z \vee 0) + (-z \vee 0), \text{ for every } z \in X. \tag{2.1}$$

A nonempty subset K of a Hilbert lattice $(X;\succcurlyeq)$ is said to be a subcomplete $\succcurlyeq$- sublattice of X, if for any nonempty subset B of K, $\vee_X B \in K$ and $\wedge_X B \in K$. Since every bounded closed convex subset of a Hilbert space is weakly compact, as an immediate consequence of Lemma 2.3 in [7], we have the following result.

Lemma 2.1. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a bounded, closed, and convex $\succcurlyeq$-sublattice of X. Then, K is a subcomplete $\succcurlyeq$-sublattice of X.*

Now, we recall the $\succcurlyeq$-preserving properties of set-valued mappings below. A set-valued mapping $f : X \to 2^X/\{\emptyset\}$ is said to be upper $\succcurlyeq$-preserving, if $x \succcurlyeq y$, then for any $v \in f(y)$, there exists $u \in f(x)$ such that $u \succcurlyeq v$. A set-valued mapping $f : X \to 2^X/\{\emptyset\}$ is said to be lower $\succcurlyeq$-preserving, if $x \succcurlyeq y$, then for any $u \in f(x)$, there exists $v \in f(y)$ such that $u \succcurlyeq v$. f is said to be $\succcurlyeq$-preserving if it is both of upper and lower $\succcurlyeq$-preserving. Similarly, we can define that f is said to be strictly upper $\succcurlyeq$-preserving, if $x \succ y$, then for any $v \in f(y)$, there exists $u \in f(x)$ such that $u \succ v$ and f is said to be strictly (lower $\succcurlyeq$-preserving if $x \succ y$, then for any $u \in f(x)$, there exists $v \in f(y)$ such that $u \succ v$.

Observations

(1) If $f : X \to 2^X/\{\emptyset\}$ is upper $\succcurlyeq$-preserving, then $x \succcurlyeq y$ implies $\vee_X f(x) \succcurlyeq \vee_X f(y)$.

(2) If $f : X \to 2^X/\{\emptyset\}$ is lower $\succcurlyeq$-preserving, then $x \succcurlyeq y$ implies $\wedge_X f(x) \succcurlyeq \wedge_X f(y)$.

Let K be a nonempty, closed, and convex sublattice of X and $T : K \to X$ a mapping. Let us consider the following variational inequality:

$$\langle Tx, y - x \rangle \geq 0, \quad \text{for every } y \in K. \tag{2.2}$$

An element $x^* \in K$ is called a solution to the variational inequality (2.2) if, for every $y \in K, \langle Tx^*, y - x^* \rangle \geq 0$. The problem to find a solution to variational inequality (2.2) is called a variational inequality problem associated with the mapping T and the subset K, which is denoted by $\mathrm{VI}(K, T)$.

Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued mapping. The general variational inequality problem associated with the set-valued mapping Γ and the subset K, which is denoted by $\mathrm{GVI}(K, \Gamma)$, is to find $x^* \in K$, with some $y^* \in \Gamma(x^*)$, such that

$$\langle y^*, y - x^* \rangle \geq 0, \quad \text{for every } y \in K. \tag{2.3}$$

Let $\Pi_K : X \to K$ be the metric projection. Then, we have the well-known variational characterization of the metric projection (e.g., see [7, Lemma 2.5]): if K is a nonempty, closed, and convex sublattice of a Hilbert lattice $(X; \succcurlyeq)$, then an element $x^* \in K$ is a solution to $\mathrm{VI}(K, T)$ if and only if

$$x^* \in \mathrm{Fix}(\Pi_K \circ (\mathrm{id}_K - \lambda T)), \quad \text{for some function } \lambda : X \to R_{++}. \tag{2.4}$$

Similarly, we can have the representation of a solution to a $\mathrm{GVI}(K, \Gamma)$, defined by (2.3), by a fixed point as given by relation (2.4).

3. The Existence of Maximum and Minimum Solutions to Some General Variational Inequalities Defined on Closed, Bounded, and Convex Subsets in Hilbert Lattices

In this section, we apply the variational characterization of the metric projection in Hilbert spaces to study the solvability of general variational inequalities without the continuity of the mappings involved in the considered general variational inequalities. Then, we provide some results about the existence of maximum and minimum solutions to some general variational inequalities defined on some closed, bounded, and convex subsets in Hilbert lattices. Similar to the conditions used by Smithson [15], we need the following definitions.

Let K be a nonempty subset of a Hilbert lattice $(X; \succcurlyeq)$ and $f : K \to 2^X/\{\emptyset\}$ a set-valued correspondence. f is said to be upper (lower) $\succcurlyeq$-bound if there exists $y^*(y_*) \in X$, such that $\vee_X f(x) (\wedge_X f(x))$ exists and

$$y^* \succcurlyeq \vee_X f(x) \left(\wedge_X f(x) \succcurlyeq y^* \right). \tag{3.1}$$

f is said to have upper (lower) bound $\succcurlyeq$-closed values, if for all $x \in K$, we have

$$\vee_X f(x) \left(\wedge_X f(x) \right) \in f(x). \tag{3.2}$$

Remarks

Let K be a nonempty subset of a Hilbert lattice $(X;\succcurlyeq)$, $f : K \to 2^X/\{\emptyset\}$ a set-valued correspondence. Then, we have the following.

(1) If subset K is upper $\succcurlyeq$-bound $\succcurlyeq$-closed and f is upper $\succcurlyeq$-preserving, then $f(K)$ is upper $\succcurlyeq$-bound and

$$\vee_X f(K) = \vee_X f(\vee_X K). \tag{3.3}$$

(2) If subset K is lower $\succcurlyeq$-bound $\succcurlyeq$-closed and f is lower $\succcurlyeq$-preserving, then $f(K)$ is lower $\succcurlyeq$-bound and

$$\wedge_X f(K) = \wedge_X f(\wedge_X K). \tag{3.4}$$

(3) If f is strictly upper $\succcurlyeq$-preserving and has upper bound $\succcurlyeq$-closed values, then

$$x \succcurlyeq y \quad \text{iff } \vee_X f(x) \succcurlyeq \vee_X f(y). \tag{3.5}$$

(4) If f is strictly lower $\succcurlyeq$-preserving and has lower bound $\succcurlyeq$-closed values, then

$$x \succcurlyeq y \quad \text{iff } \wedge_X f(x) \succcurlyeq \wedge_X f(y). \tag{3.6}$$

Now, we state and prove the main theorem of this paper below, which provides the existence of maximum and minimum solutions to general variational inequalities in Hilbert lattices.

Theorem 3.1. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a nonempty closed bounded and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Then, one has*

(1) *if $id_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving with upper bound $\succcurlyeq$-closed values for some function $\lambda : X \to R_{++}$, then the problem $GVI(K;\Gamma)$ is solvable and there exists a $\succcurlyeq$-maximum solution to $GVI(K;\Gamma)$,*

(2) *if $id_K - \lambda\Gamma$ is lower $\succcurlyeq$-preserving with lower bound $\succcurlyeq$-closed values for some function $\lambda : X \to R_{++}$, then the problem $GVI(K;\Gamma)$ is solvable and there exists a $\succcurlyeq$-minimum solution to $GVI(K;\Gamma)$,*

(3) *if $id_K - \lambda\Gamma$ is $\succcurlyeq$-preserving with both of upper and lower bounds $\succcurlyeq$-closed values for some function $\lambda : X \to R_{++}$, then the problem $GVI(K;\Gamma)$ is solvable and there exist both of $\succcurlyeq$-minimum and $\succcurlyeq$-maximum solutions to $GVI(K;\Gamma)$.*

Proof of Theorem 3.1. *Part (1)*

From (2.4), the representations of the solutions to $\text{GVI}(K;\Gamma)$ by fixed points of a projection $\Pi_K \circ (\text{id}_K - \lambda\Gamma)$, we have that x is a solution to $\text{GVI}(K;\Gamma)$ if, and only if, there exists $y \in (\text{id}_K - \lambda\Gamma)(x)$ such that

$$x = \Pi_K(y), \quad \text{that is, } x \in \Pi_K \circ (\text{id}_K - \lambda\Gamma)(x). \tag{3.7}$$

Lemma 2.4 in [7] shows that the projection Π_K is $\succcurlyeq$-preserving. As a composition of upper $\succcurlyeq$-preserving mappings, so $\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)$ is also an upper $\succcurlyeq$-preserving mapping. From Corollary 1.8 in Smithson [15] and the variational characterization of the metric projection (3.7), we have that the problem $\mathrm{GVI}(K;\Gamma)$ is solvable. Let $S(K;\Gamma)$ denote the set of solutions to the problem $\mathrm{GVI}(K;\Gamma)$. Then, $S(K;\Gamma) \neq \emptyset$. Since K is a nonempty closed bounded and convex $\succcurlyeq$-sublattice of a Hilbert lattice X, it is weakly compact. From Corollary 2.3 in [7], K is a subcomplete $\succcurlyeq$-sublattice of X. Hence, $\vee_X S(K;\Gamma) \in K$. Denote

$$x^* = \vee_X S(K;\Gamma). \tag{3.8}$$

Let

$$x_1 = \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x^*). \tag{3.9}$$

Then, from (3.8) and (3.9), we have

$$\begin{aligned} x_1 &= \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(\vee_X S(K;\Gamma)) \\ &\succcurlyeq \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma)) \\ &\succcurlyeq \vee_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma)) \\ &\succcurlyeq \vee_X S(K;\Gamma) \\ &= x^*. \end{aligned} \tag{3.10}$$

The first $\succcurlyeq$-inequality in (3.10) is based on $\vee_X S(K;\Gamma) \succcurlyeq S(K;\Gamma)$ and the property that the correspondence $\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)$ is upper $\succcurlyeq$-preserving. The second $\succcurlyeq$-inequality in (3.10) follows from $\vee_X (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma)) \succcurlyeq (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma)$ and the fact that Π_K is upper $\succcurlyeq$-preserving. The third $\succcurlyeq$-inequality in (3.10) follows from the fact that $S(K;\Gamma) \subseteq \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma))$. Then, we define

$$x_2 = \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x_1). \tag{3.11}$$

From (3.10), $x_1 \succcurlyeq x^*$, applying the upper $\succcurlyeq$-preserving property of the mapping $\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)$ again, we get

$$\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x_1) \succcurlyeq \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x^*), \tag{3.12}$$

that is, $x_2 \succcurlyeq x_1$. Denote

$$\Sigma = \{x \in K : x \succcurlyeq x^*,\ \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x) \succcurlyeq x\}. \tag{3.13}$$

From the upper $\succcurlyeq$-preserving property of $\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)$, we obtain

$$\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x)) \succcurlyeq \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x), \quad \forall x \in \Sigma, \tag{3.14}$$

which implies

$$\text{if } x \in \Sigma, \text{ then } \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x) \in \Sigma. \tag{3.15}$$

From (3.9)–(3.11), it is clear that $x_1 \in \Sigma$, and therefore, $\Sigma \neq \emptyset$. Define

$$x^{**} = \vee_X \Sigma. \tag{3.16}$$

It holds that

$$x^{**} \succcurlyeq x, \quad \forall x \in \Sigma. \tag{3.17}$$

From the upper $\succcurlyeq$-preserving property of the mapping $\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)$ again, we have

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x^{**}) \succcurlyeq \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x) \succcurlyeq x, \quad \forall x \in \Sigma. \tag{3.18}$$

Applying (3.16), it implies

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x^{**}) \succcurlyeq x^{**}. \tag{3.19}$$

It is obvious that $x^{**} \succcurlyeq x^*$, so $x^{**} \in \Sigma$. From (3.15), we have

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x^{**}) \in \Sigma. \tag{3.20}$$

Then, (3.20), (3.16), and (3.19) together imply

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x^{**}) = x^{**}. \tag{3.21}$$

From the assumption that $\vee_X(\mathrm{id}_K - \lambda\Gamma)(x^{**}) \in (\mathrm{id}_K - \lambda\Gamma)(x^{**})$, we get

$$x^{**} \in \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(x^{**}). \tag{3.22}$$

Hence, $x^{**} \in S(K;\Gamma)$. Then, the relation $x^{**} \succcurlyeq x^*$ and (3.8) imply $x^{**} = x^*$. Thus,

$$\vee_X S(K;\Gamma) = x^* \in S(K;\Gamma). \tag{3.23}$$

It completes the proof of part (1) of this theorem.

Part (2)

Very similar to the proof of part (1), we can prove the second part of this theorem. Denote

$$y* = \wedge_X S(K;\Gamma). \tag{3.24}$$

From the proof of part (1), we see that $\wedge_X S(K;\Gamma) \in K$. We need to prove $y* \in S(K;\Gamma)$. Let

$$y_1 = \wedge_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(y^*). \tag{3.25}$$

Then, we have

$$\begin{aligned} y_1 &= \wedge_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(\wedge_X S(K;\Gamma)) \\ &\preccurlyeq \wedge_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma)) \\ &\preccurlyeq y^*. \end{aligned} \tag{3.26}$$

The first-order inequality in (3.26) is based on $\wedge_X S(K;\Gamma) \preccurlyeq S(K;\Gamma)$ (piecewise) and the property that the correspondence $\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)$ is lower $\succcurlyeq$-preserving, which is the composition of the $\succcurlyeq$-preserving map Π_K and a lower $\succcurlyeq$-preserving map $\mathrm{id}_K - \lambda\Gamma$ (condition 2 in this theorem). The second-order inequality in (3.26) follows from the definition of $y*$ in (3.24) and the fact that $S(K;\Gamma) \subseteq \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(S(K;\Gamma))$; it is because $S(K;\Gamma) = \mathrm{Fix}\,(\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma))$. Then, we define

$$y_2 = \wedge_X \big(\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(y_1)\big). \tag{3.27}$$

From (3.26), $y_1 \preccurlyeq y*$, the lower $\succcurlyeq$-preserving of $\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)$, and the Observation part (2) in last section, we get

$$y_2 = \wedge_X \big(\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(y_1)\big) \preccurlyeq \wedge_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(y^*), \tag{3.28}$$

that is, $y_2 \preccurlyeq y_1$. Denote

$$\Omega = \{y \in K : y \preccurlyeq y^*,\ \Pi_K \circ \wedge_X (\mathrm{id}_K - \lambda\Gamma)(y) \preccurlyeq y\}. \tag{3.29}$$

From the lower $\succcurlyeq$-preserving property of $\Pi_K \circ \wedge_X (\mathrm{id}_K - \lambda\Gamma)$, we obtain

$$\Pi_K \circ \wedge_X (\mathrm{id}_K - \lambda\Gamma)\big(\Pi_K \circ \wedge_X (\mathrm{id}_K - \lambda\Gamma)(y)\big) \succcurlyeq \Pi_K \circ \wedge_X (\mathrm{id}_K - \lambda\Gamma)(y), \quad \forall y \in \Omega, \tag{3.30}$$

which implies

$$\text{if } y \in \Omega, \text{ then } \Pi_K \circ \wedge_X (\mathrm{id}_K - \lambda\Gamma)(y) \in \Omega. \tag{3.31}$$

From (3.24)–(3.27), it is clear that $y*, y_1 \in \Omega$, and therefore, $\Omega \neq \emptyset$. Define

$$y^{**} = \wedge_X \Omega, \tag{3.32}$$

that is,

$$y^{**} \preccurlyeq y, \quad \forall y \in \Omega. \tag{3.33}$$

From the lower $\succcurlyeq$-preserving property of the mapping $\Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)$ again, we have

$$\Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(y^{**}) \preccurlyeq \Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(y) \preccurlyeq y, \quad \forall y \in \Omega. \tag{3.34}$$

Applying (3.32), it implies

$$\Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(y^{**}) \preccurlyeq y^{**}. \tag{3.35}$$

It is obvious that $y^{**} \preccurlyeq y*$, so $y^{**} \in \Omega$. From (3.35), we have

$$\Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(y^{**}) \in \Omega. \tag{3.36}$$

Then, (3.36), (3.32), and (3.35) together imply

$$\Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(y^{**}) = y^{**}. \tag{3.37}$$

From the assumption that $\wedge_X(\mathrm{id}_K - \lambda\Gamma)(y^{**}) \in (\mathrm{id}_K - \lambda\Gamma)(y^{**})$, we get

$$y^{**} \in \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(y^{**}). \tag{3.38}$$

Hence, $y^{**} \in S(K;\Gamma)$. Then, the relation $y^{**} \preccurlyeq y^*$ and (3.24) imply $y^{**} = y^*$. Thus,

$$\wedge_X S(K;\Gamma) = y^* = y^{**} \in S(K;\Gamma). \tag{3.39}$$

It completes the proof of part (2) of this theorem. Part (3) is an immediate consequence of parts (1) and (2). It completes the proof of Theorem 3.1. □

If $\Gamma : K \to X$ is a single-valued mapping, then it can be considered as a special case of set-valued mapping with singleton values. The result below follows immediately from Theorem 3.1.

Corollary 3.2. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a nonempty closed, bounded, and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to X$ be a single-valued mapping such that $id_K - \lambda\Gamma$ is $\succcurlyeq$-preserving, for some function $\lambda : X \to R_{++}$. Then, one has*

(1) *the problem $VI(K;\Gamma)$ is solvable,*

(2) *there are both of $\succcurlyeq$-maximum and $\succcurlyeq$-minimum solutions to $VI(K;\Gamma)$.*

For a bounded and convex $\succcurlyeq$-sublattice of a Hilbert lattice X, the behavior of its maximum and minimum solutions to a problem GVI$(K;\Gamma)$ should be noticeable. The following corollary can be obtained from the proof of Theorem 3.1.

Corollary 3.3. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a nonempty, closed, bounded, and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Then, the following properties hold.*

(1) *Assume that $id_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving for some function $\lambda : X \to R_{++}$, and has upper bound $\succcurlyeq$-closed values. Let $S(K;\Gamma)$ be the set of solutions to $GVI(K;\Gamma)$, then*

$$\vee_X S(K;\Gamma) = \Pi_K \circ \vee_X (id_K - \lambda\Gamma)(\vee_X S(K;\Gamma)). \tag{3.40}$$

(2) *Assume that $id_K - \lambda\Gamma$ is lower $\succcurlyeq$-preserving for some function $\lambda : X \to R_{++}$, and has lower bound $\succcurlyeq$-closed values. Then,*

$$\wedge_X S(K;\Gamma) = \Pi_K \circ \wedge_X (id_K - \lambda\Gamma)(\wedge_X S(K;\Gamma)). \tag{3.41}$$

Proof of Corollary 3.3. *Part (1)*

In the proof of part (1) of Theorem 3.1, we have

$$x^{**} = x^*, \qquad \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x^{**}) = x^{**}. \tag{3.42}$$

It implies

$$\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(x^*) = x^*. \tag{3.43}$$

From the definition of x^* in (3.8), we get

$$\vee_X S(K;\Gamma) = \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(\vee_X S(K;\Gamma)). \tag{3.44}$$

Similar to the proof of part (2) of Theorem 3.1, we can prove Part (2) of this corollary. □

The following corollary is an immediate consequence of Corollary 3.3.

Corollary 3.4. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a nonempty, closed, bounded, and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Then, the following properties hold.*

(1) *Assume that $id_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving for some function $\lambda : X \to R_{++}$, and has upper bound $\succcurlyeq$-closed value at point $\vee_X K$. If $\vee_X K$ is a solution to $GVI(K;\Gamma)$, then*

$$\vee_X K = \Pi_K \circ \vee_X (id_K - \lambda\Gamma)(\vee_X K). \tag{3.45}$$

(2) *Suppose that $id_K - \lambda\Gamma$ is lower $\succcurlyeq$-preserving for some function $\lambda : X \to R_{++}$, and has lower bound $\succcurlyeq$-closed value at point $\wedge_X K$. If $\wedge_X K$ is a solution to $GVI(K;\Gamma)$, then*

$$\wedge_X K = \Pi_K \circ \wedge_X (id_K - \lambda\Gamma)(\wedge_X K). \tag{3.46}$$

Proof of Corollary 3.4. Part (1)

If $\vee_X K$ is a solution to GVI$(K;\Gamma)$, then we must have

$$\vee_X K = \vee_X S(K;\Gamma). \tag{3.47}$$

Substituting it into part (1) of Corollary 3.3, we get

$$\vee_X K = \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(\vee_X K). \tag{3.48}$$

The first part is proved. Similarly, the second part can be proved. □

In Theorem 3.1, without the upper bound $\succcurlyeq$-closed condition for the values of the mapping $\mathrm{id}_K - \lambda\Gamma$, Theorem 3.1 may be failed, that is, if $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving that has no upper bound $\succcurlyeq$-closed values for some function $\lambda : X \to R_{++}$, then, there may not exist a $\succcurlyeq$-maximum solution to GVI$(K;\Gamma)$. The following example demonstrates this argument.

Example 3.5. Take $X = R^2$. Define the partial order $\succcurlyeq$ as follows:

$$(x_1, y_1) \succcurlyeq (x_2, y_2), \quad \text{iff } x_1 \geq x_2, y_1 \geq y_2. \tag{3.49}$$

Then, X is a Hilbert lattice with the normal inner product in R^2 and the above partial order $\succcurlyeq$.

Let K be the closed rhomb with vertexes $(0,0)$, $(1,2)$, $(2,1)$, and $(2,2)$. Then, K is a compact (of course weakly compact) and convex $\succcurlyeq$-sublattice of X.

Take $\lambda \equiv 1$ and define $\Gamma : K \to 2^X/\{\emptyset\}$ as follows:

$$\Gamma(x,y) = \{(x,-x), (-y,y)\}, \quad \text{for every } (x,y) \in K. \tag{3.50}$$

Then, Γ is a set-valued mapping with compact values. From the definitions of λ and Γ, we have

$$(\mathrm{id}_K - \lambda\Gamma)(x,y) = \{(0, x+y), (x+y, 0)\}, \quad \text{for every } (x,y) \in K. \tag{3.51}$$

We can see that $\mathrm{id}_K - \lambda\Gamma$ is an upper $\succcurlyeq$-preserving correspondence (in fact, it is both of upper $\succcurlyeq$-preserving and lower $\succcurlyeq$-preserving) and $(\mathrm{id}_K - \lambda\Gamma)(K)$ has no upper bound $\succcurlyeq$-closed values. One can check that the mapping $\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)$ has the set of fixed points below

$$\mathrm{Fix}(\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)) = \{(0,0), (1,2), (2,1)\}, \tag{3.52}$$

which is the set of solutions to GVI$(K;\Gamma)$. It is clear that

$$\vee_X\{(0,0), (1,2), (2,1)\} = (2,2). \tag{3.53}$$

But, the point $(2,2)$ is not a solutions to $\mathrm{GVI}(K;\Gamma)$, which shows that there does not exist a $\succcurlyeq$-maximum solution to this problem $\mathrm{GVI}(K;\Gamma)$.

Similarly, in Theorem 3.1, without the lower bound $\succcurlyeq$-closed condition for the values of the mapping $\mathrm{id}_K - \lambda\Gamma$, then Theorem 3.1 (part (2)) may be failed. That is, if $\mathrm{id}_K - \lambda\Gamma$ is lower $\succcurlyeq$-preserving that has no lower bound $\succcurlyeq$-closed values for some function $\lambda : X \to R_{++}$, then there may not exist a $\succcurlyeq$-minimum solution to $\mathrm{GVI}(K;\Gamma)$. This can be demonstrated by the following example.

Example 3.6. Take $X = R^2$ as in Example 3.5. Let K be the closed rhomb with vertexes $(0,0)$, $(-1,-2)$, $(-2,-1)$, and $(-2,-2)$. Then, K is a compact (of course weakly compact) and convex $\succcurlyeq$-sublattice of X.

Take $\lambda \equiv 1$ and define $\Gamma : K \to 2^X/\{\emptyset\}$ exactly the same as that in the proof of part (1)

$$\Gamma(x,y) = \{(x,-x), (-y,y)\}, \quad \text{for every } (x,y) \in K. \tag{3.54}$$

We also have

$$\begin{gathered}(\mathrm{id}_K - \lambda\Gamma)(x,y) = \{(0,x+y), (x+y,0)\}, \quad \text{for every } (x,y) \in K, \\ \mathrm{Fix}(\Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)) = \{(0,0), (-1,-2), (-2,-1)\},\end{gathered} \tag{3.55}$$

which is the set of solutions to $\mathrm{GVI}(K;\Gamma)$. It is clear that

$$\vee_X\{(0,0), (-1,-2), (-2,-1)\} = (-2,-2). \tag{3.56}$$

But, $(-2,-2)$ is not a solutions to $\mathrm{GVI}(K;\Gamma)$, which shows that there does not exist a $\succcurlyeq$-minimum solution to this problem $\mathrm{GVI}(K;\Gamma)$.

Suppose that $\mathrm{id}_K - \lambda\Gamma$ is upper (lower) $\succcurlyeq$-preserving. The condition that $\mathrm{id}_K - \lambda\Gamma$ has upper (lower) bound $\succcurlyeq$-closed values for some function $\lambda : X \to R_{++}$, is not necessary for the problem $\mathrm{GVI}(K;\Gamma)$ to have a $\succcurlyeq$-maximum (minimum) solution to $\mathrm{GVI}(K;\Gamma)$. The following example was given by Nishimura and Ok.

Example 3.7. Take $X = R^2$ as in Example 3.5. Let $K = \{x : (0,0) \succcurlyeq x \succcurlyeq (-1,-1)\}$. Define

$$\Gamma(x,y) = \{(x,0), (0,y)\}, \quad \text{for every } (x,y) \in K. \tag{3.57}$$

Take $\lambda \equiv 1$. Then, $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving. $\mathrm{GVI}(K;\Gamma)$ has a unique solution $(0,0)$, which is also the $\succcurlyeq$-maximum solution to $\mathrm{GVI}(K;\Gamma)$. But $\mathrm{id}_K - \lambda\Gamma$ does not have upper bound $\succcurlyeq$-closed values (except at point $(0,0)$).

Example 3.7 leads us to consider some conditions on the mapping Γ that are weaker than that in Theorem 3.1 which still guarantees the existence of a $\succcurlyeq$-maximum (minimum) solution to $\mathrm{GVI}(K;\Gamma)$. To achieve this goal, we have the following notations. Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a bounded and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. An element $x \in K$ is said to be nondescending (nonascending) with respect to the mapping Γ if

$$x \preccurlyeq \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x)(x \succcurlyeq \Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(x)), \tag{3.58}$$

for some function $\lambda : X \to R_{++}$.

Applying the $\succcurlyeq$-preserving property of Π_K, for every $x \in K$, we have

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x) \succcurlyeq \vee_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)(x). \tag{3.59}$$

If $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving (lower $\succcurlyeq$-preserving), then from the upper (lower) $\succcurlyeq$-preserving property of the mapping $\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)$, we have

$$\begin{aligned}
&\text{If } \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x) \succcurlyeq x,\\
&\text{then } \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x)) \succcurlyeq \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x)\\
&(\text{If } \Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(x) \preccurlyeq x,\\
&\quad \text{then } \Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(\Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(x)) \preccurlyeq \Pi_K \circ \wedge_X(\mathrm{id}_K - \lambda\Gamma)(x)).
\end{aligned} \tag{3.60}$$

The properties in (3.60) imply that, under the condition $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving (lower $\succcurlyeq$-preserving), if an element $x \in K$ is nonascending (nonascending) with respect to the mapping Γ, then $\Pi_K \circ \vee X(\mathrm{id}_K - \lambda\Gamma)(x)$ is nondescending (nonascending) with respect to the mapping Γ.

Definition 3.8. For every $y \in K$, we denote

$$\begin{aligned}
y^{\Gamma} &= \{x \in K : x \succcurlyeq y, \text{ and } x \text{ is nondescending with respect to the mapping } \Gamma\},\\
y_{\Gamma} &= \{x \in K : x \preccurlyeq y, \text{ and } x \text{ is nonascending with respect to the mapping } \Gamma\}.
\end{aligned} \tag{3.61}$$

A point $z \in K$ is said to be an *upper (lower) absorbing point* with respect to a set-valued mapping Γ, if there exists $y \in K$ with $y^{\Gamma} \neq \emptyset$ $(y_{\Gamma} \neq \emptyset)$ such that $z = \vee_X y^{\Gamma}$ $(z = \wedge_X y_{\Gamma})$.

The following proposition describes some existence and uniqueness properties of upper (lower) absorbing point with respect to a set-valued mapping Γ.

Theorem 3.9. *Let $(X; \succcurlyeq)$ be a Hilbert lattice and K a subcomplete $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Then, the set of upper (lower) absorbing point with respect to the mapping Γ is not empty. In addition, if $\mathrm{id}_K - \lambda\Gamma$ is upper (lower) $\succcurlyeq$-preserving, for some function $\lambda : X \to R_{++}$, then upper (lower) absorbing point with respect to the mapping Γ is unique.*

Proof. Since K is a subcomplete $\succcurlyeq$-sublattice of X, it contains minimum u_*(maximum u^*). It is clear that $u_* \in u_*^{\Gamma}$ $(u^* \in u^{*\Gamma})$, which implies $u_*^{\Gamma} \neq \emptyset$ $(u^{*\Gamma} \neq \emptyset)$. Let $u_*^* = \vee_X u_*^{\Gamma}$ $(u^{**} = \wedge_X u^{*\Gamma})$. Since K is a subcomplete $\succcurlyeq$-sublattice of X, so $u_*^* \in K$ $(u^{**} \in K)$. It implies that $u_*^*(u^{**})$ is an upper (lower) absorbing point with respect to the mapping Γ.

In addition, suppose that $\mathrm{id}_K - \lambda\Gamma$ is upper (lower) $\succcurlyeq$-preserving, for some function $\lambda : X \to R_{++}$, we prove that $u_*^*(u^{**})$ is the unique upper (lower) absorbing point with respect to the mapping Γ. Assume that $y_*^*(y^{**})$ is an upper (lower) absorbing point with respect to the mapping Γ, such that $y_*^* = \vee_X y_*^\Gamma (y^{**} = \wedge_X y^{*\Gamma})$, for some $y_*(y^*) \in K$. It is clear that $y_*^\Gamma \subseteq u_*^\Gamma (y^{*\Gamma} \subseteq u^{*\Gamma})$ which implies

$$y_*^* = \vee_X y_*^\Gamma \preccurlyeq \vee_X u_*^\Gamma = u_*^* \left(y^{**} = \wedge_X y^{*\Gamma} \succcurlyeq \wedge_X u^{*\Gamma} = u^{**} \right). \tag{3.62}$$

On the other hand, similar to the proof of (3.10), we have

$$\begin{aligned}
\Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)(y_*^* \vee_X u_*^*) &= \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)\left(\left(\vee_X y_*^\Gamma\right) \vee_X \left(\vee_X u_*^\Gamma\right)\right) \\
&\succcurlyeq \left(\vee_X \Pi_K \circ (\mathrm{id}_K - \lambda\Gamma)\left(y_*^\Gamma\right)\right) \vee_X \left(\vee_X \Pi_K \circ \vee_X (\mathrm{id}_K - \lambda\Gamma)\left(u_*^\Gamma\right)\right) \\
&\succcurlyeq \left(\vee_X y_*^\Gamma\right) \vee_X \left(\vee_X u_*^\Gamma\right) \\
&= y_*^* \vee_X u_*^*,
\end{aligned} \tag{3.63}$$

where the second $\succcurlyeq$-inequality in (3.63) follows from the definitions of y_*^Γ and u_*^Γ, that contain all nonascending with respect to the mapping Γ greater than y^*, u^*, respectively. The $\succcurlyeq$-inequalities (3.63) implies $y_*^* \vee_X u_*^*$ is nonascending with respect to the mapping Γ. It is clear that $y_*^* \vee_X u_*^* \succcurlyeq y_*$, and, therefore, $y_*^* \vee_X u_*^* \in y_*^\Gamma$. The definition $y_*^* = \vee_X y_*^\Gamma$ implies

$$y_*^* \succcurlyeq y_*^* \vee_X u_*^*. \tag{3.64}$$

Combining (3.62) and (3.64), we get $y_*^* = u_*^*$ (similar to (3.62) and (3.64), we can prove $y^{**} = u^{**}$). It shows the uniqueness of upper (lower) absorbing point with respect to the mapping Γ. The proposition is proved. □

Now, we apply the concepts of absorbing points with respect to a mapping to extend Theorem 3.1 to the following theorem with conditions that are weaker than those in Theorem 3.1.

Theorem 3.10. *Let $(X; \succcurlyeq)$ be a Hilbert lattice and K a nonempty, closed, bounded, and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Then, one has*

(1) *if $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving with upper bound $\succcurlyeq$-closed value at the unique upper absorbing point with respect to the mapping Γ, for some function $\lambda : X \to R_{++}$, then the problem $GVI(K; \Gamma)$ is solvable and the unique upper absorbing point with respect to the mapping Γ is a solution to $GVI(K; \Gamma)$,*

(2) *if $\mathrm{id}_K - \lambda\Gamma$ is lower $\succcurlyeq$-preserving with lower bound $\succcurlyeq$-closed value at the unique lower absorbing points with respect to the mapping Γ, for some function $\lambda : X \to R_{++}$, then the problem $GVI(K; \Gamma)$ is solvable and the unique lower absorbing point with respect to the mapping Γ is a solution to $GVI(K; \Gamma)$.*

Proof of Theorem 3.10. Part (1)

Since (as in the proof of Theorem 3.1) K is a subcomplete $\succcurlyeq$-sublattice of X. From Theorem 3.9, $u_*^* = \vee_X u_*^\Gamma$ is the unique upper absorbing point with respect to the mapping Γ, where u_* is the minimum of K. The assumptions of Part (1) imply

$$\vee_X(\mathrm{id}_K - \lambda\Gamma)(u_*^*) \in (\mathrm{id}_K - \lambda\Gamma)(u_*^*). \tag{3.65}$$

From the upper $\succcurlyeq$-preserving property of $\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)$, the equation $u_*^* = \vee_X u_*^\Gamma$, and the definition of u_*^Γ, we get

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(u_*^*) \succcurlyeq \Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x) \succcurlyeq x, \quad \forall x \in u_*^\Gamma, \tag{3.66}$$

which implies

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(u_*^*) \succcurlyeq u_*^*. \tag{3.67}$$

Since $u_*^* \succcurlyeq u_*$, the $\succcurlyeq$-inequality (3.67) implies that u_*^* is nonascending with respect to the mapping Γ, and, therefore,

$$u_*^* \in u_*^\Gamma. \tag{3.68}$$

Applying the property (3.60) that if $x \in K$ is nonascending with respect to a mapping Γ satisfying that $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving, for some function $\lambda : X \to R_{++}$, then so is $\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x)$, from (3.67) and (3.68), it yields

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(u_*^*) \in u_*^\Gamma. \tag{3.69}$$

The definition $u_*^* = \vee_X u_*^\Gamma$, the above relation, and (3.67) together imply

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(u_*^*) = u_*^*. \tag{3.70}$$

From (3.65) and the above equation, we obtain that the unique upper absorbing point with respect to the mapping Γu_*^* is a solution to $\mathrm{GVI}(K;\Gamma)$. Then, the solvability of the problem $\mathrm{GVI}(K;\Gamma)$ is proved. It completes the proof of part (1).

Similar to the proof of part (1), we can prove Part (2). □

Theorem 3.11. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and K a nonempty, closed, bounded, and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Then, one has*

(1) *if $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving with upper bound $\succcurlyeq$-closed value at the unique upper absorbing point with respect to the mapping Γ, for some function $\lambda : X \to R_{++}$, then the unique upper absorbing point is the $\succcurlyeq$-maximum solution to $GVI(K;\Gamma)$,*

(2) *if $\mathrm{id}_K - \lambda\Gamma$ is lower $\succcurlyeq$-preserving with lower bound $\succcurlyeq$-closed value at the unique lower absorbing point with respect to the mapping Γ, for some function $\lambda : X \to R_{++}$, then the unique lower absorbing point is the $\succcurlyeq$-minimum solution to $GVI(K;\Gamma)$,*

(3) *if* $id_K - \lambda\Gamma$ *is* $\succcurlyeq$*-preserving with both of upper and lower bounds* $\succcurlyeq$*-closed value at the unique upper absorbing point and the unique lower absorbing point with respect to the mapping* Γ *in* K*, for some function* $\lambda : X \to R_{++}$*, then the unique upper absorbing point and the unique lower absorbing point are the* $\succcurlyeq$*-maximum and* $\succcurlyeq$*-minimum solutions to* $GVI(K;\Gamma)$*, respectively.*

Proof of Theorem 3.11. Part (1)

Let $S(K;\Gamma)$ denote the set of solutions to the problem $\mathrm{GVI}(K;\Gamma)$. From Theorem 3.10, we have $S(K;\Gamma) \neq \emptyset$. Since K is a subcomplete $\succcurlyeq$-sublattice of X, $\vee_X S(K;\Gamma) \in K$. Denote

$$x^* = \vee_X S(K;\Gamma). \tag{3.71}$$

Then, similar to (3.10) in the proof of Theorem 3.1, we can show

$$\Pi_K \circ \vee_X(\mathrm{id}_K - \lambda\Gamma)(x^*) \succcurlyeq x^*. \tag{3.72}$$

It implies $x^* \in x^{*\Gamma}$. Let

$$x^{**} = \vee_X x^{*\Gamma} \succcurlyeq x^*. \tag{3.73}$$

Then, x^{**} is the unique upper absorbing point in K with respect to the mapping Γ. From the Part (1) of Theorem 3.10 that the unique upper absorbing point with respect to the mapping Γ is a solution to the problem $\mathrm{GVI}(K;\Gamma)$, we obtain that $x^{**} \in S(K;\Gamma)$. Combining the definition $x^* = \vee_X S(K;\Gamma)$ and the $\succcurlyeq$-inequality (3.73), we get $x^{**} = x^*$. Hence, $x^*(= x^{**})$ is a solution to $\mathrm{GVI}(K;\Gamma)$, and therefore the problem $\mathrm{GVI}(K;\Gamma)$ has a maximum solution. Part (1) of this theorem is proved. Part (2) can be similarly proved. Part (3) is an immediate consequence of Part (1) and Part (2). This theorem is proved. □

Notice that in Example 3.7, the conditions of Theorem 3.1 are not satisfied, that is, $\mathrm{id}_K - \lambda\Gamma$ does not have upper bound $\succcurlyeq$-closed value at every point (except at point $(0,0)$). But there exists a $\succcurlyeq$-maximum solution to $\mathrm{GVI}(K;\Gamma)$ in Example 3.7. On the other hand, in Example 3.7, there is a unique upper absorbing point for the mapping Γ, which is $(0,0)$. It satisfies

$$(\mathrm{id}_K - \lambda\Gamma)(0,0) = (0,0), \tag{3.74}$$

that is,

$$\vee_X(\mathrm{id}_K - \lambda\Gamma)(0,0) = (0,0) \in (\mathrm{id}_K - \lambda\Gamma)(0,0). \tag{3.75}$$

Hence, Example 3.7 satisfies the conditions of Theorem 3.10 (Part (1)), and therefore there exists a $\succcurlyeq$-maximum solution to $\mathrm{GVI}(K;\Gamma)$, which coincides with the result of Example 3.7.

Remark 3.12. Theorem 3.1 is a special case of Theorem 3.10.

4. The Existence of Maximum and Minimum Solutions to Some General Variational Inequalities Defined on Unbounded Subsets in Hilbert Lattices

The difficulty to extend the results in bounded subsets in Hilbert lattices to unbounded subsets in Hilbert lattices is that the subcomplete property of unbounded closed convex $\succcurlyeq$-sublattice of a Hilbert lattice X does not hold. All the proofs of Theorems 3.1–3.10 in last section are based on the property that any bounded closed convex $\succcurlyeq$-sublattice in Hilbert lattices is a subcomplete $\succcurlyeq$-sublattice. So, the techniques in those proofs are not applicable in the unbounded case. Hence, except new techniques are developed in unbounded case, we have to apply the results in last section to investigate the solvability of some GVI$(C;\Gamma)$ problems. The following results are similar to Theorem 3.3 in [7] and Smithson Theorem 1.1 in [15].

Theorem 4.1. *Let $(X;\succcurlyeq)$ be a Hilbert lattice and C a closed convex $\succcurlyeq$-sublattice of X. Let $\Gamma : C \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Suppose that $id_C - \lambda\Gamma$ is upper $\succcurlyeq$-preserving with upper bound $\succcurlyeq$-closed values at all points in C, for some function $\lambda : X \to R_{++}$. In addition, assume*

- (i) *$id_C - \lambda\Gamma$ is $\succcurlyeq$-preserving and there exist $x^\circ, x_\circ \in C$ with $x^\circ \succcurlyeq x_\circ$ and $\Gamma(x^\circ) \succcurlyeq 0 \succcurlyeq \Gamma(x_\circ)$, or*
- (ii) *C has a $\succcurlyeq$-minimum and there exists $x^\circ \in C$ with $\Gamma(x^\circ) \succcurlyeq 0$.*

Then, $GVI(C;\Gamma)$ is solvable.

Proof of Theorem 4.1. *Part (i)*

Set

$$K = \{x \in C : x^\circ \succcurlyeq x \succcurlyeq x_\circ\}. \tag{4.1}$$

Then, K is $\succcurlyeq$-bounded closed convex $\succcurlyeq$-sublattice of X. Then, from [7, Lemma 2.2]), it is a subcomplete $\succcurlyeq$-sublattice of X. Similar to the proof of Theorem 3.3 in [7] or the proof of Theorem 1.1 in [15], we can show that under the conditions of part (i), the following $\succcurlyeq$-inequalities hold:

$$x^\circ \succcurlyeq \Pi_C \circ (id_C - \lambda\Gamma)(x^\circ), \qquad x_\circ \preccurlyeq \Pi_{C^\circ}(id_C - \lambda\Gamma)(x_\circ), \tag{4.2}$$

which imply

$$\Pi_C \circ (id_C - \lambda\Gamma)(K) \subseteq K. \tag{4.3}$$

It is clear that $\Gamma|_K$ satisfies all conditions of Theorem 3.1 and K is a subcomplete $\succcurlyeq$-sublattice of X. Notice that the only application of the nonempty closed bounded condition in Theorem 3.1 in the proof of Theorem 3.1 is to guarantee that the subset is a subcomplete $\succcurlyeq$-sublattice of X. Here, the subset K has been showed to be a subcomplete $\succcurlyeq$-sublattice of X.

So, applying Theorem 3.1, the problem $GVI(K, \Gamma|_K)$ is solvable and it has a maximum solution. Let x^* be a solution to GVI$(K, \Gamma|_K)$. Then,

$$x^* \in \Pi_K \circ (\mathrm{id}_K - \lambda\, \Gamma|_K)(x^*) \subseteq K. \tag{4.4}$$

Since $K \subseteq C$, we have (piecewise)

$$\begin{aligned} &\|(\mathrm{id}_C - \lambda\Gamma)(x^*) - \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(x^*)\| \\ &\quad = \|(\mathrm{id}_K - \lambda\,\Gamma|_K)(x^*) - \Pi_C \circ (\mathrm{id}_K - \lambda\,\Gamma|_K)(x^*)\| \\ &\quad \leq \|(\mathrm{id}_K - \lambda\,\Gamma|_K)(x^*) - \Pi_K \circ (\mathrm{id}_K - \lambda\,\Gamma|_K)(x^*)\|. \end{aligned} \tag{4.5}$$

From (4.4), there exists $y^* \in (\mathrm{id}_K - \lambda\Gamma|_K)(x^*)$ such that $x^* = \Pi_K(y^*)$. Since $(\mathrm{id}_K - \lambda\Gamma|_K)(x^*) = (\mathrm{id}_C - \lambda\Gamma)(x^*)$, so $y^* \in (\mathrm{id}_C - \lambda\Gamma)(x^*)$. Using (4.5), from $x^* = \Pi_K(y^*)$, we get

$$\|y^* - \Pi_C(y^*)\| \leq \|y^* - \Pi_K(y^*)\| = \|y^* - x^*\|. \tag{4.6}$$

Since $\Pi_C(y^*) \in K$ (from (4.3)), the above inequality implies $\Pi_C(y^*) = \Pi_K(y^*) = x^*$, that is, $x^* \in \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(x^*)$. Hence, x^* is a solution to GVI$(C; \Gamma)$. Part (i) is proved.

Part (ii)

Take $x_\circ$ to be the minimum of C. The inequality $x^\circ \succcurlyeq \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(x^\circ)$ can be proved by the condition $\Gamma(x^\circ) \succcurlyeq 0$ in part (ii). It is obvious that the inequality $x_\circ \preccurlyeq \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(x_\circ)$ holds, because $x_\circ$ is the minimum of C. Then, (4.3) can be proved for part (ii) and the rest of the proof will be the same as that in the proof of part (i). It completes the proof of this theorem. □

Theorem 4.2. *Let $(X; \succcurlyeq)$ be a Hilbert lattice and C a closed convex $\succcurlyeq$-sublattice of X. Let $\Gamma : C \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Suppose that $id_C - \lambda\Gamma$ is upper (lower) $\succcurlyeq$-preserving with upper (lower) bound $\succcurlyeq$-closed values at all points in C, for some function $\lambda : X \to R_{++}$. In addition, assume that $(id_C - \lambda\Gamma)(C)$ is a $\succcurlyeq$-bounded closed $\succcurlyeq$-sublattice of X. Then, $GVI(C; \Gamma)$ is solvable and it has a maximum (minimum) solution.*

Proof. Let $y^* = \vee_X(\mathrm{id}_C - \lambda\Gamma)(C)$ and $y* = \wedge_X(\mathrm{id}_C - \lambda\Gamma)(C)$. Then, the $\succcurlyeq$-preserving property of Π_C implies

$$\Pi_C(y^*) \succcurlyeq \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(C) \succcurlyeq \Pi_C(y^*). \tag{4.7}$$

Define

$$K = \{x \in C : \Pi_C(y^*) \succcurlyeq x \succcurlyeq \Pi_C(y^*)\}, \tag{4.8}$$

that is,

$$\Pi_C(y^*) \succcurlyeq K \succcurlyeq \Pi_C(y^*). \tag{4.9}$$

It is easy to see that K is a $\succcurlyeq$-bounded $\succcurlyeq$-sublattice of C. Then, from Lemma 2.2 in [7], K is a subcomplete $\succcurlyeq$-sublattice of X containing the set $\Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(C)$. It is clear that $\Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(K) \subseteq \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(C) \subseteq K$.We see that $\Gamma|_K$ satisfies all conditions of Theorem 3.1. Similar to the proof of Theorem 4.1, from Theorem 3.1, the problem $\mathrm{GVI}(K, \Gamma|_K)$ is solvable. Let x^* be a solution to $\mathrm{GVI}(K, \Gamma|_K)$. Then, the proof that x^* is a solution to $\mathrm{GVI}(K, \Gamma)$ is exactly the same as that in Theorem 4.1. Moreover, the maximum solution to the problem $\mathrm{GVI}(K, \Gamma|_K)$ is also a solution to the problem $\mathrm{GVI}(C; \Gamma)$.

Since $\Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)(C) \subseteq K$, from the variational characterization of the metric projection, it yields that all solutions to the problem $\mathrm{GVI}(C; \Gamma)$ must be contained in K. Hence, the maximum solution to the problem $\mathrm{GVI}(K, \Gamma|_K)$ in K is the maximum solution to the problem $\mathrm{GVI}(C; \Gamma)$ in C. This theorem is proved. □

Next, we consider a special type of mappings which has been used by number of authors in the fields of variational inequality theory and complementarity theory (see [4–6]). Let C be a closed convex $\succcurlyeq$-sublattice of a Hilbert lattice $(X; \succcurlyeq)$. A set-valued correspondence $f : C \to 2^X/\{\emptyset\}$ is said to be a $\succcurlyeq$-*completely continuous mapping* if $f(C)$ is a $\succcurlyeq$-bounded and closed $\succcurlyeq$-sublattice of X. A set-valued correspondence $\Gamma : C \to 2^X/\{\emptyset\}$ is said to be a $\succcurlyeq$-*completely continuous field* if Γ has the representation: $\Gamma = \mathrm{id}_C - f$, for some $\succcurlyeq$-completely continuous mapping $f : C \to 2^X/\{\emptyset\}$. With these concepts, we provide an immediate consequence of Theorem 4.2 below.

Corollary 4.3. *Let $(X; \succcurlyeq)$ be a Hilbert lattice and C a closed convex $\succcurlyeq$-sublattice of X. Let $\Gamma : C \to 2^X/\{\emptyset\}$ be a set-valued $\succcurlyeq$-completely continuous mapping with the representation $\Gamma = id_C - f$, for some $\succcurlyeq$-completely continuous mapping $f : C \to 2^X/\{\emptyset\}$. In addition, if f is upper (lower) $\succcurlyeq$-preserving with upper (lower) bound $\succcurlyeq$-closed values at all points in C, then GVI$(C; \Gamma)$ is solvable and it has a maximum (minimum) solution.*

Proof. Taking $\lambda \equiv 1$ in Theorem 4.2, we get

$$\mathrm{id}_C - \Gamma = \mathrm{id}_C - \left(\mathrm{id}_C - f\right) = f. \tag{4.10}$$

From the condition of $\succcurlyeq$-completely continuous mapping, it implies that $(\mathrm{id}_C - \Gamma)(C)a$ $\succcurlyeq$-bounded closed $\succcurlyeq$-sublattice of X. So, Γ satisfies all conditions of Theorem 4.2. Then, this corollary follows immediately. □

The solvability of a general variational inequality in Theorem 4.2 can be extended as below. But, the existence of maximum or minimum solution will be failed.

Theorem 4.4. *Let $(X; \succcurlyeq)$ be a Hilbert lattice and C a closed convex $\succcurlyeq$-sublattice of X. Let $\Gamma : C \to 2^X/\{\emptyset\}$ be a set-valued correspondence. Suppose that $id_C - \lambda\Gamma$ is upper (lower) $\succcurlyeq$-preserving with upper (lower) bound $\succcurlyeq$-closed values at all points in C, for some function $\lambda : X \to R_{++}$. In addition, if there exists a nonempty, closed, bounded, and convex $\succcurlyeq$-sublatticeKsuch that $\Pi_C \circ (id_C - \lambda\Gamma)(K)$ is a nonempty closed bounded and convex $\succcurlyeq$-sublattice inK, then GVI$(C; \Gamma)$ is solvable.*

Proof. As restricting to the mapping $\Gamma|_K$, the proof of this theorem is very similar to the proof of the solvability in Theorem 4.2. It is omitted. □

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References

[1] R. W. Cottle and J. S. Pang, "A least-element theory of solving linear complementarity problems as linear programs," *Mathematics of Operations Research*, vol. 3, no. 2, pp. 155–170, 1978.

[2] C. W. Cryer and M. A. H. Dempster, "Equivalence of linear complementarity problems and linear programs in vector lattice Hilbert spaces," *SIAM Journal on Control and Optimization*, vol. 18, no. 1, pp. 76–90, 1980.

[3] C. W. Cryer and M. A. H. Dempster, "Equivalence of linear complementarity problems and linear programs in vector lattice Hilbert spaces," *SIAM Journal on Control and Optimization*, vol. 18, no. 1, pp. 76–90, 1980.

[4] G. Isac, *Complementarity Problems*, vol. 1528 of *Lecture Notes in Mathematics*, Springer, Berlin, UK, 1992.

[5] G. Isac and J. Li, "Complementarity problems, Karamardian's condition and a generalization of Harker-Pang condition," *Nonlinear Analysis Forum*, vol. 6, no. 2, pp. 383–390, 2001.

[6] G. Isac, V. M. Sehgal, and S. P. Singh, "An alternate version of a variational inequality," *Indian Journal of Mathematics*, vol. 41, no. 1, pp. 25–31, 1999.

[7] H. Nishimura and E. A. Ok, "Solvability of variational inequalities on Hilbert lattices," to appear.

[8] S. Park, "Generalized equilibrium problems and generalized complementarity problems," *Journal of Optimization Theory and Applications*, vol. 95, no. 2, pp. 409–417, 1997.

[9] D. M. Topkis, *Supermodularity and Complementarity*, Frontiers of Economic Research, Princeton University Press, Princeton, NJ, USA, 1998.

[10] J. C. Yao and J.-S. Guo, "Variational and generalized variational inequalities with discontinuous mappings," *Journal of Mathematical Analysis and Applications*, vol. 182, no. 2, pp. 371–392, 1994.

[11] C. J. Zhang, "Generalized variational inequalities and generalized quasi-variational inequalities," *Applied Mathematics and Mechanics*, vol. 14, no. 4, pp. 315–325, 1993.

[12] C.-J. Zhang, "Existence of solutions of two abstract variational inequalities," in *Fixed Point Theory and Applications. Vol. 2*, pp. 153–161, Nova Science, Huntington, NY, USA, 2001.

[13] J. M. Borwein and M. A. H. Dempster, "The linear order complementarity problem," *Mathematics of Operations Research*, vol. 14, no. 3, pp. 534–558, 1989.

[14] J. M. Borwein and D. T. Yost, "Absolute norms on vector lattices," *Proceedings of the Edinburgh Mathematical Society II*, vol. 27, no. 2, pp. 215–222, 1984.

[15] R. E. Smithson, "Fixed points of order preserving multifunctions," *Proceedings of the American Mathematical Society*, vol. 28, no. 1, pp. 304–310, 1971.