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ON POLYNOMIAL EQUATIONS IN BANACH SPACE, PERTURBATION TECHNIQUES AND APPLICATIONS

IOANNIS K. ARGYROS

Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242

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ABSTRACT. We use perturbation techniques to solve the polynomial equation in Banach space. Our techniques provide more accurate information on the location of solutions and yield existence and uniqueness in cases not covered before. An example is given to justify our method.

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1. INTRODUCTION.

In this paper we use perturbation techniques to find solutions of the abstract polynomial equation of degree k,

$$x = P_k(x) = M_k x^k + M_{k-1} x^{k-1} + \cdots + M_1 x + M_0 \tag{1.1}$$

in a Banach space X over the field F of real or complex numbers.

Obviously (1.1) is a natural generalization of the scalar polynomial equation of the first kind to the more abstract setting of a Banach space.

The case $k = 2$ has been examined in [1], [7], [8]. Here we investigate the case $k \geq 2$. The principal new idea in this paper is the introduction of an equation similar to (1.1),

$$z = F_k(z) = N_k z^k + N_{k-1} z^{k-1} + \cdots + N_2 z^2 + N_1 z + M_0. \tag{1.2}$$

The results are then obtained under suitable choices of the N_p's, $p = 1,2,\cdots,k$.

Our method is a generalization of the one's discussed [8], [9] by L. B. Rall, namely, the method of successive substitutions and Newton's method. It always provides a more accurate information on the location of the solutions and it also yields existence and uniqueness results for (1.1) in the cases not covered before. For $z = 0$ and $A_2 = M_2 - I$ our results coincide with theorems in [8], [9], but even then we are able to provide more accurate information on the location of the solutions. In order to justify this, in Part 2 of our paper we compare our results with the results in [8], [9], [10] using as an example a special case of (1.1), namely the famous Chandrasekhar's equation

$$x(s) = 1 + \lambda x(s)\int_0^1 \frac{s}{s+t}\, x(t)dt \tag{1.3}$$

for $X = C[0,1]$ and $0 \le \lambda \le 0.5$. Stronger results for an even more general form of (1.3) have already been obtained in [3], [6] and elsewhere. In this paper we just use (1.3) as an example to justify our method.

2. BASIC CONCEPTS AND THEOREMS.

DEFINITION 1. Denote by $L(X,Y)$ the linear space over the field F of the linear operators from a linear space X into a linear space Y. For $k = 2,3,\cdots$ a linear operator from X into the space $L(X^{k-1},Y)$ of $(k-1)$-linear operators from X into Y is called <u>k-linear</u> operator from X into Y. For example, if a k-linear operator M_k from X into Y and k points $x_1,x_2,\cdots,x_k \in X$ are given, then

$$z = M_k x_1 x_2 \cdots x_k$$

will be a point of Y, *the convention being that* M_k *operates on* x_1, the $(k-1)$-linear operator $M_k x_1$ operates on x_2, and so on. The order of operation is important. Finally, denote $L(X,Y)$ by $L(X)$ if $X = Y$.

NOTATION 1. Given a k-linear operator M_k from X into Y and a permutation $i = (i_1,i_2,\cdots,i_k)$ of the integers $1,2,\cdots,k$, the notation $M_k(i)$ can be used for the k-linear operator from X into Y such that

$$M_k(i)x_1x_2\cdots x_k = M_k x_{i_1} x_{i_2} \cdots x_{i_k}$$

for all $x_1,x_2,\cdots,x_k \in X$.

Thus, there are $k!$ k-linear operators $M_k(i)$ associated with a given k-linear operator M_k.

DEFINITION 2. A k-linear operator M_k from X into Y is said to be <u>symmetric</u> if

$$M_k = M_k(i)$$

for all $i \in R_k$, where R_k denotes the set of all permutations of the integers $1,2,\cdots,k$. The symmetric k-linear operator

$$\bar{M}_k = \frac{1}{k!} \sum_{i \in R_k} M_k(i)$$

is called the <u>mean</u> of M_k.

NOTATION 2. The notation

$$M_k x^p = M_k \overbrace{xx\cdots x}^{p},$$

$p \le k$, $M_k \in L(X^k,Y)$, for the result of applying M_k to $x \in X$ p-times will be used. If $p < k$, then $M_k x^p$ will represent a $(k-p)$-linear operator from X into Y. For $p = k$, note that

$$M_k x^k = \bar{M}_k x^k = M_k(i)x^k \tag{2.1}$$

for all $i \in R_k$, $x \in X$. It follows from (2.1) that the multilinear operators $M_2,\cdots,M_k$ in (1.1) may be assumed to be symmetric without loss of generality, since each M_i in (1.1) may be replaced by $\bar{M}_i$, $i = 2,3,\cdots,k$, without changing the value of $P_k(x)$. Unless the contrary is explicitly stated, the multilinear operators M_i, $i = 2,3,\cdots,k$ will be assumed to be symmetric.

Assume from now on that X, Y are Banach spaces.

DEFINITION 3. A linear operator L from X into Y is said to be bounded if

$$\|L\| = \sup_{\|x\|=1} \|Lx\| \tag{2.2}$$

is finite. The quantity $\|L\|$ is called the bound (or norm) of L.

DEFINITION 4. For $k \geq 2$, a k-linear operator M_k from X into Y is said to be <u>bounded</u> if it is a bounded linear operator from X into $L(X^{k-1},Y)$, the Banach space of bounded (k-1)-linear operators from X into Y. The <u>bound</u> (or norm) $\|M_k\|$ of M_k is defined by (2.2), with M_k being considered to be an element of $L(X,L(X^{k-1},Y))$.

NOTATION 3. The space of bounded k-linear operators from X into Y will be denoted henceforth by $L(X^k,Y)$. Note that by Definitions (3) and (4) if $M_k \in L(X^k,Y)$ and $p \leq k$ then

$$\|M_k x^p\| \leq \|M_k\| \; \|x\|^p.$$

DEFINITION 5. An abstract polynomial operator P_k from X into Y of degree k defined by

$$P_k(x) = M_k x^k + M_{k-1} x^{k-1} + \cdots + M_2 x^2 + M_1 x + M_0,$$

is said to be <u>bounded</u> if its coefficients M_i, $i = 1,2,\cdots,k$ are bounded multilinear operators from X into Y. From now on we assume P_k is bounded.

DEFINITION 6. Let z be fixed in X and define the polynomial q_k of degree k on $\mathbb{R}^+$ by

$$q_k(r) = \|P_k(z)-z\| + \|M_1\| \left[\frac{(r+\|z\|)-\|z\|}{(r+\|z\|)-\|z\|}\right] r + \cdots + \|M_k\| \left[\frac{(r+\|z\|)^k-\|z\|^k}{(r+\|z\|)-\|z\|}\right] r.$$

Note that by Descartes rule of signs [5] the equation $q(r) = q_k(r)-r = 0$ has two positive solutions $s_1 \leq s_2$ or none.

THEOREM 1. Assume that $q(r)$ has two positive solutions $s_1 < s_2$ such that $q'(r) < 1$, $r \in (s_1,s_2)$. Then P_k has a unique fixed point in the ball $\bar{U}(z,r) = \{x \in X \mid \|x-z\| \leq r\}$, where $r \in (s_1', s_2') \subset (s_1,s_2)$.

PROOF. Claim 1. P_k maps $\bar{U}(z,r)$ into $\bar{U}(z,r)$.

$$\begin{aligned}\|P_k(x)-z\| &= \|P_k(x)-P_k(z)+P_k(z)-z\| \leq \|P_k(x)-P_k(z)\| + \|P_k(z)-z\| \\ &\leq \|M_1(x-z)+\cdots+M_k(x-z)x^{k-1}+M_k x^{k-2}(x-z)z+\cdots+M_k z^{k-1}(x-z)\| \\ &\leq [\,\|M_1\|+\|M_2\|(r+\|z\|)+\cdots+\|M_k\|((r+\|z\|)^{k-1}+(r+\|z\|)^{k-2}\|z\|+\cdots+\|z\|^{k-1}]r \\ &\qquad + \|P_k(z)-z\| \leq r\end{aligned}$$

or

$$q(r) \leq 0 \text{ which is true by hypothesis.}$$

(Note that claim 1 is true even if $s_1 = s_2$ and $r \in [s_1,s_2]$).

Claim 2. P_k is a contraction operator on $\bar{U}(z,r)$. IF $x_1, x_2 \in \bar{U}(z,r)$, then as in claim 1,

$$\|P_k(x_1)-P_k(x_2)\| \le \tilde{q}_k(r)\|x_1-x_2\|$$

but

$$\tilde{q}_k(r) < 1 \text{ by hypothesis.}$$

The result now follows from the contraction mapping principle.

DEFINITION 7. Define the polynomial $\tilde{q}_k(r)$ of degree k on $\mathbb{R}^+$ by

$$\tilde{q}_k(r) = \|M_1-N_1\|\|z\| + \|M_2-N_2\|\|z\|^2 + \cdots + \|M_k-N_k\|\cdot\|z\|^k + q_k(r) + \|P_k(z)-z\|.$$

Note that

$$\begin{aligned}\|P_k(z)-z\| &= \|M_0 + M_1 z + M_2 z^2 + \cdots + M_k z^k - z\| \\ &= \|M_0-M_0 + M_1 z - N_1 z + \cdots + M_k z^k - N_k z^k + (F_k(z)-z)\| \\ &\le \|M_1-N_1\|\cdot\|z\| + \|M_2-N_2\|\cdot\|z\|^2 + \cdots + \|M_k-N_k\|\cdot\|z\|^k\end{aligned}$$

if z is a fixed point of (1.2).

The proof of the following theorem follows from Theorem 1 and the above observation.

THEOREM 2. Suppose that there exists a solution z satisfying (1.2) and that $\tilde{q}(r) = \tilde{q}_k(r)-r$ has two positive solutions $s_1 < s_2$. Then P_k has a unique fixed point in $\bar{U}(z,r)$, where $r \in (s_1', s_2') \subset (s_1, s_2)$.

3. APPLICATIONS.

From now on we assume that $k = 2$ and $M_1 = 0$. Then Theorem 2 becomes

THEOREM 3. Consider the equation

$$z = M_0 + N_2 z^2. \tag{3.1}$$

Suppose that there exists a solution z satisfying (3.1) and

$$\|z\| < [2\sqrt{\|M_2\|}\,(\sqrt{\|M_2-N_2\|} + \sqrt{\|M_2\|})]^{-1}.$$

(I) Then the equation

$$x = M_0 + M_2 x^2 \tag{3.2}$$

has a unique solution $x \in U(z,a)$, where

$$a = \frac{1}{2\|M_2\|} - \|z\|.$$

(II) Moreover, $x \in \bar{U}(z,b)$ where

$$b = \{1-2\|M_2\|\cdot\|z\| - [(2\|M_2\|\cdot\|z\|-1)^2 - 4\|M_2-N_2\|\|M_2\|\cdot\|z\|^2]^{1/2}\}(2\|M_2\|)^{-1}.$$

In practice, an exact solution of the auxiliary equation (3.1) can seldom be obtained. The following theorem, whose proof is similar to that of Theorem 2, guarantees that the original equation (3.2) has a solution even when we can only find an approximate solution of (3.1).

THEOREM 4. Let z be fixed in X and set

$$a = \frac{1}{2\|M_2\|} - \|z\|$$

$$\epsilon = \|N_2 z^2 + M_0 - z\|\cdot\|z\|^2$$

$$b = a - \left[a^2 - \frac{\|M_2-N_2\|\|z\|^2 + \epsilon\|z\|^2}{\|M_2\|}\right]^{1/2}.$$

Assume that

$$\|z\| < [2\sqrt{\|M_2\|}(\sqrt{\|M_2\|} + \sqrt{\|M_2-N_2\| + \epsilon})]^{-1},$$

then

(I) equation (3.2) has a unique solution in $U(z,a)$;

(II) this solution actually lies in $\bar{U}(z,b)$.

PROOF. Let us define the operator on X by

$$T(x) = M_0 + M_2x^2.$$

Claim 1. T maps $\bar{U}(z,r)$ into $\bar{U}(z,r)$ for $r \in [b,a)$. If $x \in \bar{U}(z,r)$ then

$$\begin{aligned} T(x) - z &= M_2x^2 + M_0 - z \\ &= (M_2-N_2)(x-z+z)^2 + N_2(x-z)^2 + 2N_2z(x-z) + (N_2z^2+M_0-z). \end{aligned}$$

Now, $\|T(x)-z\| \le r$ if

$$(\|M_2-N_2\|+\|N_2\|)r^2 + (2\|z\|\|M_2-N_2\|+2\|N_2\|\cdot\|z\|-1)r + \|M_2-N_2\|\cdot\|z\|^2 + \epsilon\|z\|^{-2} \le 0$$

which is true for $r \in [a,b)$.

Claim 2. T is a contraction operator on $\bar{U}(z,r)$. If $w,v \in \bar{U}(z,r)$ then

$$\begin{aligned} \|T(w)-T(v)\| &= \|M_2w^2 - M_2v^2\| \\ &= \|M_2(w-z+v-z+2z)(w-v)\| \\ &\le 2(r+\|z\|)\|w-v\|\|M_2\|. \end{aligned}$$

So T is a contraction on $\bar{U}(z,r)$ for $0 < r < a$.

Because Theorem 4 relies on the contraction mapping principle, it actually provides an iteration procedure for solving (3.2), namely, set

$$x_0 = z \quad \text{and}$$

$$x_{n+1} = M_0 + M_2x_n^2, \qquad n = 1,2,\cdots.$$

REMARK 1. The iteration

$$x_{n+1} = M_0 + M_2x_n^2, \qquad n = 1,2,\cdots$$

converges for any $x_0 \in \bar{U}(z,b)$ to the solution x of (3.2) at the rate of a geometric progression with quotient

$$q = 1 - [(2\|M_2\|\cdot\|z\|-1)^2 - 4\|M_2-N_2\|\|M_2\|\cdot\|z\|^2]^{1/2}.$$

PROOF. By Theorem 3 we have

$$\begin{aligned} q &= 2(b+\|z\|)\|M_2\| \\ &= 1 - [(2\|M_2\|\cdot\|z\|-1)^2 - 4\|M_2\|\cdot\|M_2-N_2\|\cdot\|z\|^2]^{1/2}. \end{aligned}$$

COROLLARY 1. Under the hypotheses of Theorem 3, the solution x obtained in Theorem 3 satisfies

$$\|x\| < \frac{1}{2\|M\|}.$$

PROOF. By Theorem 3,

$$\|x-z\| < a,$$

so that

$$\|x\| \le \|z\| + a,$$

i.e.,

$$\|x\| < \frac{1}{2\|M_2\|}.$$

COROLLARY 2. For any $M_0 \in X$ such that $4\|M_2\|\cdot\|M_0\| < 1$,

(I) equation (3.2) has a unique solution $x \in U(M_0,a)$, where

$$a = \frac{1-2\|M_2\|\|M_0\|}{2\|M_2\|};$$

(II) moreover, $x \in \bar{U}(M_0,b)$ where

$$b = \frac{1-2\|M_2\|\cdot\|M_0\|-\sqrt{1-4\|M_2\|\cdot\|M_0\|}}{2\|M_2\|}.$$

PROOF. Apply Theorem 3 with $M_2 = 0$ and $z = M_0$.

We now state Rall's theorem for comparison. The proof can be found in [8], [9].

THEOREM 5. If $4\|M_2\|\cdot\|M_0\| < 1$ then

(I) equation (3.2) has a solution $x \in X$ satisfying

$$\|x\| \le \frac{1-\sqrt{1-4\|M_2\|\|M_0\|}}{2\|M_2\|};$$

(II) moreover, x is unique in $U(x,R)$, where

$$R = \frac{\sqrt{1-4\|M_2\|\|M_0\|}}{2\|M_2\|}.$$

PROPOSITION 1. Assume:

(I) the hypotheses of Theorems 3, 5 are satisfied;

(II) $(\|M_2\|-\|M_2-N_2\|)\|z\|^2-\|z\|+\|M_0\| > 0$.

Then Theorem 3 provides a sharper estimate on $\|x\|$ than Theorem 5.

PROOF. By Theorem 3,

$$\|x-z\| \le b \quad \text{so} \quad \|x\| \le b+\|z\|.$$

By Theorem 5,

$$x \le \frac{1-\sqrt{1-4\|M_2\|\|M_0\|}}{2\|M_2\|}$$

so it is enough to show

$$\frac{1-[(2\|M_2\|\cdot\|M_0\|-1)^2-4\|M_2\|\|M_2-N_2\|\|z\|^2]^{1/2}}{2\|M_2\|} < \frac{1-\sqrt{1-4\|M_2\|\|M_0\|}}{2\|M_2\|}$$

or

$$(\|M_2\|-\|M_2-N_2\|)\|z\|^2-\|z\|+\|y\| > 0$$

and the result follows from (II).

REMARK 2. If the evaluation of $\|M_2-N_2\|$ in Theorem 3 is difficult, then

(a) we can look for a z such that:

$$\|z\| < [2\sqrt{\|M_2\|}\,(\sqrt{\|M_2\|+\|N_2\|} + \sqrt{\|M_2\|})]^{-1}$$

$$\le [2\sqrt{\|M_2\|}\,(\sqrt{\|M_2-N_2\|} + \sqrt{\|M_2\|})]^{-1}$$

and start the iteration with $x_0 = z$;

(b) we can apply the theorem in the ball $\bar{U}(z,a')$, where $b \le a' < a$ and

$$a' = \frac{1-2\|M_2\|\cdot\|z\| - [(2\|M_2\|\cdot\|z\|-1)^2 - 4\|M_2\|+\|N_1\|)\|z\|^2]^{1/2}}{2\|M_2\|},$$

provided that the quantity under the radical is nonnegative. Also note that sice $b \le a' < a$, we have

$$\bar{U}(z,b) \subset \bar{U}(z,a') \subset U(z,a).$$

EXAMPLE 1. For the equation of Chandrasekhar,

$$x(s) = 1 + \lambda x(s)\int_0^1 \frac{s}{s+t}\,x(t)dt, \tag{3.3}$$

we have $X = C[0,1]$ with the sup-norm. The operator $Q : X \longrightarrow X$ defined by

$$Q(x) = x(s)\int_0^1 \frac{s}{s+t}\,x(t)dt$$

is quadratic since the symmetric bilinear operator $M_2 : X \times X \longrightarrow X$ defined by

$$M_2(x,y) = \frac{1}{2}\left[x(s)\int_0^1 \frac{s}{s+t}\,y(t)dt + y(s)\int_0^1 \frac{s}{s+t}\,x(t)dt\right]$$

satisfies

$$M_2(x,x) = Q(x) \quad \text{for all} \quad x \in X.$$

We will prove that the norm $\|M_2\| = \ln 2$. Now

$$\|Q\| = \max_s \int_0^1 \left|\frac{s}{s+t}\right| dt = \ln 2$$

and since always

$$\|Q\| \le \|M_2\|$$

we obtain

$$\ln 2 \le \|M_2\|.$$

The proof will be completed if we prove that

$$\|M_2\| \le \ln 2.$$

But by the definition of M_2,

$$\|M_2\| \le \frac{1}{2} \max_s 2\int_0^1 \left|\frac{s}{s+t}\right| dt = \ln 2$$

so

$$\|M_2\| = \ln 2.$$

We now apply Theorem 5 and Corollary 2 to (3.3) with $B = \lambda M_2$. According to Theorem 5, equation (3.3) has a unique solution in $U(x,R)$, where

$$R = \frac{\sqrt{1-4\lambda \ln 2}}{2\lambda \ln 2},$$

provided that $1 - 4\lambda \ln 2 > 0$, i.e., $\lambda < .36067\cdots$. According to Corollary 2,

(I) equation (3.3) has a unique solution in $U(x,R)$, where

$$R = \frac{1-2\lambda \ln 2}{2\lambda \ln 2};$$

(II) moreover, $x \in \bar{U}(1,a)$, where

$$a = \frac{1-2\lambda \ln 2 - \sqrt{1-4\lambda \ln 2}}{2\lambda \ln 2},$$

provided that $1-4\lambda \ln 2 > 0$, i.e., $\lambda < .36067\cdots$.

One can now see by comparing the above that Corollary 2 under the same condition on λ gives a better information on the location of the solution than Theorem 5 [8], [9].

Our next goal is to use Theorem 4 to obtain solutions of (3.3) for a wider range of λ. It is not necessary to assume B has any connection with Chandrasekhar's equation in Proposition 2 or 3.

PROPOSITION 2. If $z \in X$ is a solution of the equation

$$z = M + \lambda M_2(z,z),$$

satisfying

$$2\lambda\|M_2\| \cdot \|z\| < 1,$$

then for

$$\lambda \le \lambda < C_1,$$

where

$$C_1 = [4\|M_2\| \; \|z\|(1-\lambda\|M_2\| \cdot \|z\|)]^{-1}$$

the conclusions of Theorem 3 for the equation

$$x = M_0 + \lambda_1 M_2(x,x)$$

hold.

PROOF. To apply Thoerem 3 we need

$$\|z\| < [2\sqrt{\lambda_1\|M_2\|}\;(\sqrt{|\lambda-\lambda_1|\,\|M_2\|} + \sqrt{\lambda_1\|M_2\|})]^{-1}$$

since

$$\lambda_1 < C_1 = [4\|M_2\| \cdot \|z\| \;(1-\lambda\|M_2\| \cdot \|z\|)]^{-1},$$

we have

$$\lambda_1^2-\lambda_1\lambda < \left(\frac{1}{2\|M_2\| \cdot \|z\|}\right)^2 + \lambda_1^2 - \frac{\lambda_1}{\|M_2\| \cdot \|z\|}$$

or by taking the square root of both sides of the above inequality and using

$$\lambda_1 < (2\|M_2\| \cdot \|z\|)^{-1}$$

we get

$$\sqrt{\lambda_1(\lambda_1-\lambda)} < \frac{1}{2\|M_2\| \cdot \|z\|} - \lambda_1.$$

The result now follows by solving the last inequality for $\|z\|$.

If z is not an exact solution of the quadratic equation

$$z = M_0 + N_2(z,z),$$

then we can use the following generalization of Proposition 2.

PROPOSITION 3. Let M_0, z be fixed in X and $\lambda > 0$. Set

$$\varepsilon = \|\lambda M_2(z,z)+M_0-z\| \cdot \|M_2\| \cdot \|z\|^{-2}$$

and

$$C_1 = \{4\|z\|M_2\|[1-(\lambda-\varepsilon)\|M_2\| \cdot \|z\|]\}^{-1}.$$

Then for any λ_1 satisfying $\lambda \le \lambda_1 < C_1$ the equation

$$x = M_0 + \lambda_1 M_2(x,x)$$

has a unique solution in $U(z,a)$, and in fact this solution lies in $\bar{U}(z,b)$. Here

$$a = \frac{1}{2\lambda_1\|M_2\|} - \|z\|$$

$$b = a - \left[a^2 - \left(1-\frac{\lambda}{\lambda_1}\right)\|z\|^2 - \frac{\varepsilon}{\lambda_1}\|z\|^2\right]^{1/2}.$$

PROOF. Similar to Proposition 2.

REMARK 3. According to Corollary 2 or Theorem 5 and the discussion following Example 1, Chandrasekhar's equation

$$z(s) = 1+\lambda M_2(z(s),z(s)) = 1+\lambda s z(s)\int_0^1 \frac{z(t)}{s+t}\,dt \tag{3.4}$$

has a solution z provided that $\lambda < .36067376\cdots$. But now using Proposition 3 and the iteration suggested in Remark 1 for a suitable $x_0 = Z_N(\lambda)$, we can extend the range of λ until $.424059379\cdots$. Here are some characteristic values for λ the norm of the corresponding approximate solution $Z_N(\lambda)$ and $C_1(\lambda)$.

λ	$\|Z_N(\lambda)\|$	$C_1(\lambda)$
.35	1.44474532	.384363732
⋮	⋮	⋮
.38	1.534201867	.394512252
.39	1.558263525	.399942101
⋮	⋮	⋮
.4	1.59821923	.405244331
⋮	⋮	⋮
.42	1.68363661	.420163281
⋮	⋮	⋮
.423	1.69644924	.423011429
⋮	⋮	⋮
.424	1.70085561	.424070047
⋮	⋮	⋮
.424059378	1.700973716	.424059379
.424059379	1.700973721	.424059379

Note that the above results coincide at least at six decimal places with the ones obtained in [2], [3] and [10].

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