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ON BAZILEVIC FUNCTIONS

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ABSTRACT. Let $B(\beta)$ be the class of Bazilevic functions of type $\beta(\beta>0)$. A function $f\varepsilon B(\beta)$ if it is analytic in the unit disc E and $\text{Re}\,\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}>0$, where g is a starlike function. We generalize the class $B(\beta)$ by taking g to be a function of radius rotation at most $k\pi(k\geq 2)$. Archlength, difference of coefficient, Hankel determinant and some other problems are solved for this generalized class. For k=2, we obtain some of these results for the class $B(\beta)$ of Bazilevic functions of type β.

KEY WORDS AND PHRASES. Bazilevic functions, functions of bounded boundary rotation; Hankel determinant, close-to-convex functions, radius of α-convexity.
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1. INTRODUCTION.

Bazilevic [1] introduced a class of analytic function f defined by the following relation. For $z\varepsilon E$: $E = \{z:|z|<1\}$ let

$$f(z) = \frac{\beta}{1+a^2}\left[\int_0^z (h(\xi)-ai)\,\xi^{\frac{-\beta ai}{2}-1}\, g^{\frac{\beta}{1+a^2}}(\xi)\, d\xi\right]^{\frac{1+ai}{\beta}} \tag{1.1}$$

where a is real, $\beta > 0$, $\text{Re}\, h(z)>0$ and g belongs to the class S^* of starlike functions. Such functions, he showed, are univalent [1]. With a=0 in (1.1), we have for $z\varepsilon E$

$$\text{Re}\,\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} > 0 \tag{1.2}$$

This class of Bazilevic functions of type β was considered in [2]. We denote this class of functions by $B(\beta)$. We notice that if $\beta=1$ in (1.2), we have the class K of close-to-convex functions. We need the following definations.

Definition 1.1

A function f analytic in E belongs to the vlass V_k of functions with bounded boundary rotation, if $f(0) = 0$, $f'(0) = 1$, $f'(z) \neq 0$, such that for $z = re^{i\theta}\varepsilon E$, $0\leq r<1$

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \leq k\pi , \qquad k \geq 2 \tag{1.3}$$

For k=2, we obtain the class C of convex functions. It is known [3] that for $2 \leq k \leq 4$, V_k consists entirely of univalent functions. The class V_k has been studied by many authors, see [3], [4], [5] etc.

Definition 1.2

Let f be analytic in E and f(0)=0, f'(0)=1. Then f is said to belong to the class R_k of functions with bounded radius rotation, if $z=re^{i\theta}\varepsilon E$, $0<r<1$

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{zf'(z)}{f(z)} \right| d\theta \leq k\pi , \qquad k \geq 2 \tag{1.4}$$

It is clear that $f \varepsilon V_k$ if and only if $zf' \varepsilon R_k$. We also note that $R_2 = S^*$.

We now give the following generalized form of the class $B(\beta)$.

Definition 1.3

Let f be analytic in E and f(0)=1, f'(0)=1. Then f belongs to the class $B_k(\beta)$, $\beta>0$ if there exists a $g \varepsilon R_k$; $k \geq 2$ such that

$$\operatorname{Re} \frac{zf'(z)}{f^{1-\beta}(z)\; g^{\beta}(z)} > 0, \qquad z\varepsilon E \tag{1.5}$$

We notice that, when $\beta=1$, $B_k(1) \equiv T_k$, a class of analytic functions introduced and discussed in [6]. Also $B_2(\beta) = B(\beta)$ and $B_2(1) = K$, the class of close-to-convex functions.

2. PRELIMINARIES

We shall give here the results needed to prove our main theorems in the preceeding section.

Lemma 2.1 [3].

Let $f \varepsilon V_k$. Then there exist two starlike functions S_1, S_2 such that for $z \varepsilon E$

$$f'(z) = \frac{(S_1(z)/z)^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4}-\frac{1}{2}}} , \quad k \geq 2 \tag{2.1}$$

Lemma 2.2

Let H be analytic in E, $|H(0)| \leq 1$ and be defined as

$$H(z) = (\tfrac{k}{2} + \tfrac{1}{2})h_1(z) - (\tfrac{k}{4} - \tfrac{1}{2})h_2(z), \ \operatorname{Re} h_i(z) > 0, \qquad i=1,2, k \geq 2.$$

Then, for $z = re^{i\theta}$,

(i) $$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta \leq \frac{1-(k^2-1)r^2}{1-r^2} \tag{2.2}$$

and

(ii) $$\frac{1}{2\pi}\int_0^{2\pi} |H'(z)|\,d\theta \leq \frac{k}{1-r^2} \tag{2.3}$$

This result is known [6] and, for k=2, we obtain Pommerenke's result [7] for functions of positive real parts.

Lemma 2.3

Let S_1 be univalent in E. Then:

(i) there exists a z_1 with $|z_1| = r$ such that for all z, $|z| = r$

$$|z-z_1||S_1(z)| \leq \frac{2r^2}{1-r^2}, \qquad \text{see [8]} \tag{2.4}$$

and

(ii) $$\frac{r}{(1+r)^2} \leq |S_1(z)| \leq \frac{r}{(1-r)^2}, \qquad \text{see [9]} \tag{2.5}$$

Definition 2.1.

Let f be analytic in E and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then the qth Hankel determinant of f is defined for $q \geq 1$, $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & \cdots & & a_{n+2q-2} \end{vmatrix} \tag{2.6}$$

Definition 2.2.

Let z_1 be a non-zero complex number. Then, with $\Delta_0(n,z_1,f) = a_n$; $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we define for $j \geq 1$,

$$\Delta_j(n,z_1,f) = \Delta_{j-1}(n,z_1,f) - \Delta_{j-1}(n+1,z_1,f) \tag{2.7}$$

Lemma 2.4

Let f be analytic in E and let the Hankel determinant of f be defined by (2.6). Then, writing $\Delta_j = \Delta_j(n,z_1,f)$, we have

$$H_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \Delta_{q-1}(n+q-1) \\ \Delta_{2q+3}(n+1) & \Delta_{2q-4}(n+2) & \Delta_{q-2}(n+q) \\ \vdots & & \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) \;\ldots & \Delta_0(n+2q-3) \end{vmatrix} \tag{2.8}$$

Lemma 2.5

With $z_1 = \frac{n}{n+1}y$, and $v \geq 0$ any integer,

$$\Delta_j(n+v,z_1,zf') = \sum_{k=0}^{j} \binom{j}{k} \frac{y^k(v-(k-1)n)}{(n+1)^k} \Delta_{j-k}(n+v+k,y,f)$$

Lemmas 2.4 and 2.5 are due to Noonan and Thomas [10].

Lemma 2.6 [11].

Let N and D be anlytic in E, N(0)=D(0) and D maps E onto many sheeted region which is starlike with respect to the origin. Then Re $\frac{N'(z)}{D')z)}>0$ implies Re $\frac{N(z)}{D(z)}>0$.

3. MAIN RESULTS.

THEOREM 3.1: Let $f\varepsilon B_k(\beta)$; $k\geq 2$, $0<\beta\leq 1$. Then

$$L_r(f)\leq C(k,\beta)\ M^{1-\beta}(r)\left(\frac{1}{1-r}\right)^{\beta(\frac{k}{2}+1)}, \quad \text{where}$$

$C(k,\beta)$ is a constant depending on k,β only. $L_r(f)$ denotes the length of the closed curve $f(|z|=r<1)$ and $M(r)=\max\limits_{|z|=r}|f(z)|$

PROOF: We have

$$L_r(f)=\int_0^{2\pi}|zf'(z)|d\theta, \qquad z=re^{i\theta}$$

$$=\int_0^{2\pi}|f^{1-\beta}(z)g^{\beta}(z)h(z)|d\theta, \quad \text{using (1.5), where } g\varepsilon R_k \text{ and } \operatorname{Re} h(z)>0.$$

$$\leq M^{1-\beta}(r)\int_0^{2\pi}|g^{\beta}(z)\ h(z)|d\theta$$

$$\leq M^{1-\beta}(r)\int_0^{2\pi}\int_0^{r}|\beta g'(z)g^{\beta-1}(z)h(z)+g^{\beta}(z)h'(z)|drd\theta.$$

$$\leq M^{1-\beta}(r)\ \left\{\int_0^{2\pi}\int_0^{r}\frac{\beta}{r}\left|\frac{zg'(z)}{g(z)}\ g^{\beta}(z)h(z)\right|drd\theta + \int_0^{2\pi}\int_0^{r}|\ g^{\beta}(z)zh'(z)|drd\theta\right\}$$

$$=M^{1-\beta}(r)\left\{\int_0^{2\pi}\int_0^{r}\frac{\beta}{r}|H(z)g^{\beta}(z)h(z)|drd\theta+\int_0^{2\pi}\int_0^{r}\frac{1}{r}|g^{\beta}(z)zh'(z)|drd\theta\right\}$$

where $\frac{zg'(z)}{g(z)}=H(z)$ is defined as in Lemma 2.2

Using Lemma 2.1, Lemma 2.3 (ii), Schwarz inequality and then Lemma 2.2 for both general and special cases ($k\geq 2$, $k=2$), we have

$$L_r(f)\leq C(k,\beta)M^{1-\beta}(r)\left(\frac{1}{1-r}\right)^{\beta(\frac{k}{2}+1)},$$ $0<\beta\leq 1$, $C(k,\beta)$ is a constant depending on k,β only.

Corollary 3.1

For $k=2$, $f\varepsilon B(\beta)$ and $L_r(f)\leq C(\beta)M^{1-\beta}(r)\ \left(\frac{1}{1-r}\right)^{2\beta}$

THEOREM 3.2.

Let $f\varepsilon B_k(\beta)$, $0<\beta\leq 1$, $k\geq 2$ and $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$. Then for $n\geq 2$

$$|a_n|\leq C_1(\beta,k)M^{1-\beta}\left(1-\frac{1}{n}\right).n^{\beta(\frac{k}{2}+1)-1},$$

$C_1(\beta,k)$ is a constant depending only upon k and β .

PROOF: Since, with $z=re^{i\theta}$, Cauchy's theorem gives

$$n\, a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)e^{-in\theta}d\theta,$$

then

$$n|a_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi} |zf'(z)|d\theta = \frac{1}{2\pi r^n} L_r(f)$$

Using theorem 3.1 and putting $r=1-\frac{1}{n}$, we obtain the required result.

Corollary 3.2

When $\beta=1, f\varepsilon T_k$ and from theorem 3.2 we have

$|a_n| \le A(k)n^{k/2}$, $A(k)$ being a constant depending on k only.

This result was proved in [6].

Corollary 3.3

For k=2, $f\varepsilon B(\beta)$ and

$$|a_n| \le A_1(\beta)\ M^{1-\beta}(1-\tfrac{1}{n}).n^{2\beta-1}, \quad n\ge 2.$$

THEOREM 3.3.

Let f be as defined in theorem 3.2. Then, for $n\ge 2$, $k>\frac{5}{\beta}-2$,

$$||a_{n+1}|-|a_n|| = 0(1)M^{1-\beta}(1-\tfrac{1}{n}).n^{\beta(\frac{k}{2}+1)-2}$$

where 0(1) depends only on k and β.

PROOF: For $z_1\varepsilon E$, $n\ge 2$ and $z=re^{i\theta}\varepsilon E$, we have

$$|(n+1)z_1|a_{n+1}|-n|a_n|| \le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi}|z-z_1||zf'(z)|d\theta$$

$$= \frac{1}{2\pi r^{n+1}}\int_0^{2\pi}|z-z_1||f(z)^{1-\beta}g^{\beta}(z)h(z)|d\theta,$$ where we have used (1.5).

Taking $M(r)=\max\limits_{|x|=r} f(z)$, and using (2.1) (2.4) and (2.5), we have

$$|(n+1)z_1|a_{n+1}|-n|a_n|| \le \frac{M^{1-\beta}(r)}{2\pi r^{n+1}}\left(\frac{4}{r}\right)^{\beta(\frac{k}{4}-\frac{1}{2})}\cdot\frac{2r^2}{1-r^2}\int_0^{2\pi}|S_1(z)|^{\beta(\frac{k+2}{4}-1)}|h(z)|d\theta,$$

where S_1 is a starlike function.

Schwarz inequality, together with Lemma 2.2 (k=2) and subordination for starlike functions [12] yields

$$|(n+1)z_1|a_{n+1}|-n|a_n|| \le \frac{M^{1-\beta}(r)}{2\pi r^{n+1}}\left(\frac{4}{r}\right)^{\beta(\frac{k-2}{4})}\frac{2r^2}{1-r^2}\left(\int_0^{2\pi}\frac{r^{\beta(\frac{k}{2}+1)-2}}{|1-re^{i\theta}|^{\beta(k+2)-4}}d\theta\right)^{\frac{1}{2}}.\left(2\pi\,\frac{1+3r^2}{1-r^2}\right)^{\frac{1}{2}}$$

$$\le C(k,\beta)M^{1-\beta}(r)\ \left(\frac{1}{1-r}\right)^{\beta(\frac{k}{2}+1)-1},$$

where $C(k,\beta)$ is a constant depending only on k and β. Choosing $|z_1| = r = \frac{n}{n+1}$, we obtain the required result.

Corollary 3.4

If $\beta=1$, $f\in T_k$ and we obtain a known [6] result, for $k>3$,

$$||a_{n+1}|-|a_n|| = O(1).n^{\frac{k}{2}-1}$$

We now proceed to study the Hankel determinant problem for the class $B_k(\beta)$.

THEOREM 3.4.

Let $f\in B_k(\beta)$, $0<\beta\le 1$, $k\ge 2$ and let the Hankel determinant $H_q(n)$ of f be defined as in definition 2. Then

$$H_q(n)=O(1)M^{1-\beta}(r)\begin{cases} n^{\beta(\frac{k}{2}+1)-1}, & q=1 \\ n^{\beta(\frac{k}{2}+1)\,q-q^2}, & q\ge 2,\quad k\ge\frac{8q-8}{\beta}-2 \end{cases}$$

PROOF: Since $f\in B_k(\beta)$, we can write

$$zf'(z) = f^{1-\beta}(z)\, g^{\beta}(z)h(z),\ \text{Re } h(z)>0,\ g\in R_k.$$

Let $F(z)=zf'(z)$. Then for $j\ge 1$, z_1 any non-zero complex number and $z=re^{i\theta}$, consider $\Delta_j(n,z_1,F)$ as defined by (2.7). Then

$$|\Delta_j(n,z_1,F)| = \frac{1}{2\pi r^{n+j}}\left|\int_0^{2\pi}(z-z_1)^j F(z)e^{-i(n+j)\theta}d\theta\right|$$

$$\le \frac{1}{2\pi r^{n+j}}\int_0^{2\pi}|z-z_1|^j\ |f^{1-\beta}(z)g^{\beta}(z)\ h(z)|d\theta$$

$$\le \frac{M^{1-\beta}(r)}{2\pi r^{n+j}}\int_0^{2\pi}|z-z_1|^j|g^{\beta}(z)||h(z)|d\theta$$

Using (2.1), (2.4) and (2.5), we have

$$|\Delta_j(n,z_1,F)|\le\frac{M^{1-\beta}(r)}{2\pi r^{n+j}}\left(\frac{4}{r}\right)^{\beta(\frac{k-2}{4})}\left(\frac{2r^2}{1-r^2}\right)^j\int_0^{2\pi}|\ S_1(z)|^{\beta(\frac{k}{4}+\frac{1}{2})-j}|h(z)|d\theta$$

Schwarz inequality together with subordination for starlike functions [12] and (2.2) gives us, for $\beta(\frac{k}{4}+\frac{1}{2})-j\ge 0$,

$$|\Delta_j(n,z_1,F)|\le C(k,\beta,j)M^{1-\beta}(r)\left(\frac{1}{1-r}\right)^{\beta(\frac{k}{2}+1)-j},$$

where $C(k,\beta,j)$ is a constant which depends upon k, β and j only.

Applying lemma 2.5 and putting $z_1=\left(\frac{n}{n+1}\right)e^{i\theta_n}$, $(n\to\infty)$, $r=1-\frac{1}{n}$, we have for $k\ge\frac{4j}{\beta}-2$

$$\Delta_j(n,e^{i\theta_n},f)=O(1)M^{1-\beta}(r)\ n^{\beta(\frac{k}{2}+1)-j-1},$$

where $O(1)$ depends on k,β and j only.

We now estimate the rate of growth of $H_q(n)$

For $q=1$, $H_q(n) = a_n = \Delta_0(n)$ and

$$H_q(n) = 0(1)M^{1-\beta}\, n^{\beta(\frac{k}{2}+1)-1}, \quad q=1$$

For $q\geq 2$, we use the Remark due to Noonan and Thomas in [10] and we have

$$H_q(n) = 0(1)M^{1-\beta}(r)\, n^{\beta(\frac{k}{2}+1)q-q^2}, \quad q\geq 2, \quad 0<\beta\leq 1,$$

and $k \geq \frac{8(q-1)}{\beta} - 2$. This completes the proof.

Corollary 3.5

When $\beta=1$, $f\varepsilon T_k$ and

$$H_q(n)=0(1)\begin{cases} n^{k/2}, & q=1 \\ n^{(\frac{k}{2}+1)q-q^2}, & q\geq 2, \quad k\geq 8q-10. \end{cases}$$

This result is known [13].

Definition 3.1

A function f is called α-convex if, for $\alpha\geq 0$,

$$\mathrm{Re}\,\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\} > 0, \quad z\varepsilon E.$$

We now prove the following

THEOREM 3.5.

Let $f\varepsilon B_k(\beta)$, $k\geq 2$ and $\beta\geq 1$. Then f is $\frac{1}{\beta}$ - convex for $|z|<r_0$, where

$$r_0 = \frac{1}{2\beta}\,[(k\beta+2)-\sqrt{(k\beta+2)^2-4\beta^2}] \qquad (3.1)$$

PROOF: Since $f\varepsilon B_k(\beta)$, we have

$zf'(z) = f^{1-\beta}(z)g^{\beta}(z)\, h(z)$; $\mathrm{Re}h(z)>0$, $g\varepsilon R_k$, from which it follows that

$$\frac{1}{\beta}\frac{(zf'(z))'}{f'(z)} + (1-\frac{1}{\beta})\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{1}{\beta}\frac{zh'(z)}{h(z)}, \quad (\beta\geq 1)$$

Thus

$$\mathrm{Re}\,[\frac{1}{\beta}\frac{(zf'(z))'}{f'(z)} + (1-\frac{1}{\beta})\frac{zf'(z)}{f(z)}] \geq \mathrm{Re}\,\frac{zg'(z)}{g(z)} - \frac{1}{\beta}\left|\frac{zh'(z)}{h(z)}\right|$$

Since $zG'=g\varepsilon R_k$ implies $G\varepsilon V_k$, we have from a known result [14],

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} = \mathrm{Re}\,\frac{(zG'(z))'}{G'(z)} \geq \frac{1-kr+r^2}{1-r^2}, \qquad (3.2)$$

and for functions h of positive real part, it is known [15] that

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1-r^2} \qquad (3.3)$$

Using (3.2) and (3.3), we have

$$\mathrm{Re}\,\,[\frac{1}{\beta}\frac{(zf'(z))'}{f'(z)} + (1-\frac{1}{\beta})\frac{zf'(z)}{f(z)}] \geq \frac{\beta(1-kr+r^2)-2r}{\beta(1-r^2)},$$

and this gives us the requaired result.

Corollary 3.6

(i) For $k=2$, $\beta\geq 1$, $f\varepsilon B(\beta)$ is $\frac{1}{\beta}$ - convex for $|z|<r_1 = \frac{(\beta+1) - \sqrt{2\beta+1}}{\beta}$

(ii) $\beta=1$ implies $f\in T_k$ and it is convex for $|z|<r_2 = \frac{1}{2}[(k+2)-\sqrt{k^2+4k}]$.
This result is known, see [6].

(iii) When $k=2$, and $\beta=1$, $f\in K$ and it is convex for $|z|<2-\sqrt{3}$.

THEOREM 3.6.

Let $G\in R_k$, $k\geq 2$. Let, for β any positve integer, $\frac{1}{\alpha} = 2,3,\ldots,n$ be defined as

$$h(z) = \int_0^z t^{\frac{1}{\alpha}-2} G^{\beta}(t)\,dt$$

Then h is $(\frac{1}{\alpha} - 1 + \beta)$ - valently starlike for $|z|<r_0$, where

$$r_0 = \frac{1}{2}(k-\sqrt{k^2-4}) \qquad (3.4)$$

PROOF: $\frac{(zh'(z))'}{h'(z)} = (\frac{1}{\alpha} - 1) + \beta\frac{zG'(z)}{G(z)}$

Since $G\in R_k$, it is known [16] that $\text{Re}\,\frac{zG'}{G} > 0$ for $|z|<r_0 = \frac{1}{2}(k-\sqrt{k^2-4})$.
Hence h is convex and thus starlike in $|z|<r_0$. The $(\frac{1}{\alpha} - 1+\beta)$ valency follows from the argument principle.

THEOREM 3.7.

Let $G\in R_k$, $k\geq 2$ and

$$g^{\beta}(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2} G^{\beta}(t)\,dt,$$ where α and β are defined as in theorem 3.6. Then g is starlike for $|z|< r_0$, where r_0 is given by (3.4).

PROOF:

$$\beta\frac{zg'(z)}{g(z)} = \frac{(1-\frac{1}{\alpha})\int_0^z t^{\frac{1}{\alpha}-2} G^{\beta}(t)\,dt + z^{\frac{1}{\alpha}-1}G^{\beta}(z)}{\int_0^z t^{\frac{1}{\alpha}-2}G^{\beta}(t)\,dt} = \frac{N(z)}{D(z)}$$

where $D(z) = \int_0^z t^{\frac{1}{\alpha}-2} G^{\beta}(t)\,dt$, which is $(\beta+\frac{1}{\alpha}-1)$ - valently starlike for $|z|<r_0$, r_0 given by (3.4) from theorem 3.6.

Now

$$\frac{N'(z)}{D'(z)} = \beta\,\frac{zG'(z)}{G(z)},$$

and , for $|z|<r_0$, r_0 given by (3.4), we have

$$\text{Re}\,\frac{N'(z)}{D'(z)} = \beta\,\text{Re}\,\frac{zG'(z)}{G(z)} > 0,$$ since $G\in R_k$.

Thus, using lemma 2.6, for $|z|< r_0$, it follows that

$$\beta\,\text{Re}\frac{zg'(z)}{g(z)} = \text{Re}\,\frac{N(z)}{D(z)} > 0,$$ and this completes the proof.

<u>Corollary 3.7</u>

(i) For $k=2$, $\beta=1$, we obtain Bernardi's result [17] for starlike functions.

(ii)Also, for $k=2$, $\alpha=1/2$, we obtain a result proved in [18].

THEOREM 3.8.

Let $F\varepsilon B_k(\beta)$, $z\varepsilon E$ and let

$f^{\beta}(z) = \frac{1}{\alpha} z^{1-1/\alpha}\int_0^z t^{(1/\alpha)-2}F^{\beta}(t)\,dt$, α and β defined (3.5) as in theorem 3.6. Then $f\varepsilon B(\beta)$ for $|z|<r_0, r_0$ is given by (3.4).

PROOF: Let $G\varepsilon R_k$ and let g be defined as in theorem 3.7. Then g is starlike for $|z|<r_0$, where r_0 is given by (3.4).

Now, from (3.5), we obtain

$$\beta \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} = \frac{(1-\frac{1}{\alpha})\int_0^z t^{\frac{1}{\alpha}-2} F^{\beta}(t)dt + z^{\frac{1}{\alpha}-1} F^{\beta}(z)}{\int_0^z [t^{\frac{1}{\alpha}-2} G^{\beta}(t)]dt} = \frac{N(z)}{D(z)},$$

where $D(z) = \int_0^z t^{\frac{1}{\alpha}-2} G^{\beta}(t)dt$ is $(\beta+\frac{1}{\alpha}-1)$ valently starlike for $|z|<r_0$. Also

$$\frac{N'(z)}{D'(z)} = \beta\frac{zF'(z)}{F^{1-\beta}(z)G^{\beta}(z)} > 0, \text{ since } F\varepsilon B_k(\beta).$$

Thus , using lemma 2.6, we obtain the desired result that $f\varepsilon B(\beta)$ for $|z|<r_0$, where r_0 is given by (3.4).

Corollary 3.8.

(i) For k=2, $F\varepsilon B(\beta)$, $z\varepsilon E$ and it follows from theorem 3.8 that $f\varepsilon B(\beta)$ for $|z|<1$.

(ii) For k=2, $\beta=1$ implies $F\varepsilon K$ and from theorem 3.8 it follows that f also belongs to K for $|z|<1$.

(iii) Let $\beta=1$, then $F\varepsilon T_k$, and it follows from theorem 3.8 that f is close-to-convex for $|z|< r_0$, given by (3.4). This is a generalization of a result proved in [13] for $\alpha=\frac{1}{2}$.

THEOREM 3.9

Let $f\varepsilon B_k(\beta)$. Then for $z=re^{i\theta}$, $0<\theta_1<\theta_2\leq 2\pi$, $f(z)\neq 0$, $f'(z)\neq 0$ in E and $0<r<1$, we have

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{\frac{(zf'(z))'}{f'(z)} + (\beta-1)\frac{zf'(z)}{f(z)}\right\} d\theta > -\frac{1}{2}\beta k\pi.$$

PROOF: Sine $f\varepsilon B_k(\beta)$ we can write

$$\begin{aligned} zf'(z) &= f^{1-\beta}(z)\, g^{\beta}(z)h(z), \quad \text{Re } h(z)>0, \quad g\varepsilon R_k. \\ &= f^{1-\beta}(z)g^{\beta}(z)h_1(z), \quad \text{where } h(z)=h_1^{\beta}(z), \quad \text{Re } h_1(z)>0 \\ &= f^{1-\beta}(z)(zT'(z))^{\beta}, \quad \text{where } T\varepsilon T_k. \end{aligned}$$

Therefore,

$$\frac{(zf'(z))'}{f'(z)} + (\beta-1)\frac{zf'(z)}{f(z)} = \beta\frac{(zT'(z))'}{T'(z)} . \qquad (3.5)$$

Using a known result [6] for the class T_k, we have by integrating both sides of (3.5) between θ_1, θ_2, $0<\theta_1<\theta_2\leq 2\pi$

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\{\frac{(zf'(z))'}{f'(z)} + (\beta-1)\frac{zf'(z)}{f(z)}\}\, d\theta \geq -\frac{\beta}{2}k\pi .$$

The following theorem shows the relationship between the classes $B_k(\beta)$ and $B(\beta)$. More precisely it gives the necessary condition for $f\in B_k(\beta)$ to be univalent.

THEOREM 3.10

Let $f\in B_k(\beta)$. Then f is univalent if $k\leq\frac{2}{\beta}$, where $0<\beta\leq 1$.

PROOF:The proof follows immediately from Theorem 3.9 and the result of Sheil-Small [19].

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