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ON THE RECONSTRUCTION OF THE MATCHING POLYNOMIAL AND THE RECONSTRUCTION CONJECTURE

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ABSTRACT. Two results are proved. (i) It is shown that the matching polynomial is both node and edge reconstructable. Moreover a practical method of reconstruction is given. (ii) A technique is given for reconstructing a graph from its node-deleted and edge-deleted subgraphs. This settles one part of the Reconstruction Conjecture.

KEY WORDS AND PHRASES. Matching, perfect matching, matching polynomial, matching matrix, Reconstruction Conjecture, edge reconstruction, node reconstruction.
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1. INTRODUCTION.

The graphs considered here will be finite and will have no loops. Let G be such a graph. The *matching polynomial* of G has been defined (see Farrell [1]) as

$$m(G) = \sum_k a_k \, w_1^{p-2k} w_2^{k}, \tag{1.1}$$

where w_1 and w_2 are indeterminates (or weights) associated with each node and edge respectively in G, a_k is the number of matchings in G with k edges and the summation is taken over all the k-matchings in G.

The famous Reconstruction Conjecture (see Harary [2]) is the following. If G is a graph with $p > 2$ nodes and if the *deck* (i.e. the p subgraphs $G - v_i$) is given, then the entire graph G can be reconstructed, uniquely up to isomorphism, from these node-deleted subgraphs. For an interesting historical account of this conjecture, we refer the reader to [2]. An analogous form of this conjuecture, with "node" replaced by "edge" is called the Edge-Reconstruction Conjecture.

In this paper we will first concern ourselves with the reconstruction of m(G). We will therefore answer the following question. Suppose that the deck of G is given, can m(G) be found? Dually, we will also give an answer to the edge version of this question. If the answer is yes, we will say that m(G) is reconstructible (node-reconstructible). In the case of the edge version of the question, we say that m(G) is edge-reconstructible.

It is important to know whether or not m(G) is reconstructible. One reason is this. If m(G) is reconstructible, then any graph that is characterized by m(G) will be reconstructible. In general, the reconstruction of graph polynomials can shed some light on the Reconstruction Conjecture itself. For example, the reconstruction of the characteristic polynomial was investigated by several authors. In Particular, Gutman and Cvetkovic [3] investigated the reconstruction of the characteristic polynomial and its implications to Ulam's Conjecture. The existence of a reconstruction for the characteristic polynomial was eventually established by Tutte [4]. Tutte (in [4]) also established the reconstructibility of the rank polynomial and the chromatic polynomial.

Although the reconstruction of several graph polynomials has been established, no practical means of reconstruction exists for any of them. For example, given the deck of the graph G, there is no method available for finding either the characteristic polynomial or the chromatic polynomial of G. The reconstruction results are merely existence results. It is of interest therefore, to find *practical* methods of reconstructing the polynomials.

In this article, we will not only show that $m(G)$ is node and edge-reconstructible, but in doing so, we will give practical methods for reconstruction. We note that Godsil [5] has given a result (Theorem 4.1) which essentially establishes the node-reconstruction of a special form of $m(G)$. However his result does not provide a practical reconstruction technique, since he used Tutte's existence result to establish the reconstruction of the number of perfect matchings in G.

The problem of reconstructing a graph from a given deck has always been an interesting one, because of its connection with Ulam's Conjecture (see [3]). We will give a solution to this problem, using the matching polynomial of the graph. In fact, our technique gives all the graphs with the given deck. Thus we will have established the first part of the Reconstruction Conjecture i.e. given the deck of G, G itself can be reconstructed. However we are unable to say whether or not G is unique. This would have settled the Reconstruction Conjecture.

For the graph G, with p nodes and q edges, the node set will be $V(G) = \{v_1, v_2, \ldots, v_p\}$ and the edge set, $E(G) = \{e_1, e_2, \ldots, e_q\}$. We will denote by $G-v_i$, the graph obtained from G by removing node V. $G(-)e_j$ will denote the graph obtained from G by removing the nodes at the ends of the edge e_j. $G - e_j$ will be the graph obtained from G by deleting the edge e_j. Throughout the paper, we will assume that the general graph G has p nodes and q edges unless otherwise specified.

2. THE NODE-RECONSTRUCTION OF $m(G)$.

In order to establish our main result, we will need the following lemma.

LEMMA 1. Let G be a graph with p nodes and q edges. Then

$$\text{(i)}\ \frac{\partial}{\partial w_1}(m(G)) = \sum_{i=1}^{p} m(G-v_i)$$

$$\text{(ii)}\ \frac{\partial}{\partial w_2}(m(G)) = \sum_{j=1}^{q} m(G(-)e_j)$$

PROOF. We establish (i) by showing that the two polynomials $A = \frac{\partial}{\partial w_1} m(G)$ and $B = \sum_{i=1}^{p} m(G-v_i)$ have precisely the same terms with equal coefficients. The proof (ii) will be similar.

Let $w_1^j w_2^k$ be a term of A. Then m(G) has a term in $w_1^{j+1} w_2^k$. It follows that G has a matching S with (j+1) nodes and k edges. Let $v_r \in V(G)$. Then $G-v_r$ will contain the matching $S-v_r$. Hence B also contains a term in $w_1^j w_2^k$.

Conversely, if B contains a term in $w_1^j w_2^k$, then there exists a node v_r such that $G-v_r$ has a matching with j nodes and k edges. Therefore G has a matching with (j+1) nodes and k edges. It follows that m(G) has a term in $w_1^{j+1} w_2^k$. Hence A has a term in $w_1^j w_2^k$. We conclude that A and B have the same kinds of terms.

We will show that the coefficients of like terms are equal. Let $a_k w_1^j w_2^k$ be a term in m(G). Then the corresponding term in A will be $ja_k w_1^{j-1} w_2^k$. Hence the coefficient of $w_1^j w_2^k$ in A will be ja_k. Now, for each matching in G with j nodes, there will be exactly j corresponding matchings in the graphs $G-v_r$ with j-1 nodes. Since G contains a_k such matchings, then B will contain the term $w_1^{j-1} w_2^k$ with coefficient ja_k. Therefore the coefficients of like terms are equal. Hence the result follows. The proof of (ii) is similar. □

We define a *perfect matching* in G to be a matching which contains edges only. we will denote the number of perfect matchings in G by $\gamma(G)$. Clearly $\gamma(G)$ is the coefficient of the term independent of w_1 in m(G).

THEOREM 1.

$$m(G) = \sum_{i=1}^{p} \int m(G-v_i)\,dw_1 + \gamma(G)$$

PROOF. This is straightforward from the lemma, by integrating with respect to w_1. □

Since we are given the deck of G (i.e. the $G-v_i$'s) we can find $m(G-v_i)$ for $i = 1,2, \ldots, p$. Hence the summation on the RHS of Theorem 1 can be found. The problem which confronts us now is that of finding $\gamma(G)$. In order to give a technique for finding $\gamma(G)$, we will use the concept of the matching matrix A(G) and the d-function of A(G) (see Farrell and Wahid [6]).

Let G be a graph with p nodes, we define the *matching matrix* A(G) of G as follows.

$$A(G) = (a_{ij}), \text{ where } a_{ij} = \begin{cases} \sqrt{w_2} & \text{if } i < j. \\ -\sqrt{w_2} & \text{if } i > j, \\ w_1 & \text{if } i = j. \end{cases}$$

The *d-function* is defined recursively as follows. If A(G) is 3 x 3 or smaller, $d(A(G)) = |A(G)|$. Otherwise,

$$d(A(G)) = w_1 d(A(G-v_i)) + w_2 \sum_{v_i v_j \in E(G)} d(A(G-v_i-v_j)).$$

The following lemma was established in [6].

LEMMA 2. $$d(A(G)) = m(G).$$

Let $G-v_i$ be any element of the given deck. Then we can write

$$A(G) = \left[\begin{array}{c|c} A(G-v_i) & A_p \\ \hline -A_p & w_1 \end{array}\right], \tag{2.1}$$

where A_p is the first $p-1$ elements in the (unknown) pth column of $A(G)$. A will contain $p-1$ boolean unknowns $x_{1p}, x_{2p}, \ldots, x_{p-1,p}$. By using Lemma 2, we can obtain $[p/2]$-*linear* equations involving these unknowns. These equations can be solved to obtain the x_{ip}'s. The coefficient of $w_2^{p/2}$ in the expression for $d(A(G))$ could then be found. This coefficient will be $\gamma(G)$.

From Theorem 1 and the above analysis, we obtain the following result which establishes the practical reconstruction of $m(G)$.

THEOREM 2. THe matching polynomial is node-reconstructible.

By integrating with respect to w_2, the expressions given in part (ii) of Lemma 1, we get

$$m(G) = \sum_{j=1}^{q} \int m(G(-)e_j)\, dw_2 + C(w_1), \tag{2.2}$$

where $C(w_1)$ is a function of w_1. The only term independent of w_2 in $m(G)$ is w_1^p. (N.B. We can also put $w_2 = 0$ in Equation (2.2)). Therefore $C(w_1) = w_1^p$. Hence we have the following result.

THEOREM 3. $$m(G) = \sum_{j=1}^{q} \int m(G(-)e_j)dw_2 + w_1^p.$$

Suppose that we are given the q subgraphs $G(-)e_j$ of G. Then both terms on the RHS of Theorem 3 can be found. Hence $m(G)$ could be found. We therefore have the following theorem.

THEOREM 4. The matching polynomial is reconstructible from the set of subgraphs obtained by removing the pairs of nodes defined by the edges in the graph.

Again, we have given a practical means by which the reconstruction of $m(G)$ can be carried out. This is explicit from Theorem 3.

3. THE EDGE-RECONSTRUCTION OF m(G).

The following lemma is the basic tool in the practical edge-reconstruction of $m(G)$.

LEMMA 3. $$qm(G) = w_2 \frac{\partial}{\partial w_2}(m(G)) + \sum_{j=1}^{q} m(G-e_j).$$

PROOF. Let e_j be an edge in G. We can partition the matchings in G into two classes (i) those in which e_j is used and (ii) those in which e_j is not used. The matchings in class (i) are all matchings in the graph $G(-)e_j$. The matchings in class (ii) are matchings in the graph $G-e_j$. Hence we get

$$m(G) = w_2\, m(G(-)e_j) + m(G-e_j).$$

By summing over the q edges in G, we get

$$\sum_{j=1}^{q} m(G) = w_2 \sum_{j=1}^{q} m(G(-)e_j) + \sum_{j=1}^{q} m(G-e_j).$$

The result follows by using (ii) of Lemma 1. □

Suppose that the q edge-deleted subgraphs $G-e_j$ of G are given. Then the summation term $\sum_{j=1}^{q} m(G-e_j)$ on the RHS of the equation can be found. By using the general expression for $m(G)$ given in Equation (1.1), we get

$$q \sum_k a_k w_1^{p-2k} w_2^k = w_2 \sum_k ka_k w_1^{p-2k} w_2^{k-1} + \sum_{j=1} m(G-e_j).$$

Hence we have,

$$\sum_{k=0}^{[p/2]} a_k(q-k)\, w_1^{p-2k} w_2^k = \sum_{j=1} m(G-e_j).$$

By comparing coefficients, a_k could be found for all values of k. Hence $m(G)$ could be found. Therefore Theorem 5 establishes the practical edge-reconstruction of $m(G)$.

THEOREM 6. The matching polynomial is edge-reconstructible.

4. CONNECTIONS WITH THE RECONSTRUCTION CONJECTURE.

Lemma 2 provides a technique for constructing a graph with a given matching polynomial. Suppose that $m(G)$ is known. Then from Lemma 2 we will obtain a system of equations in the boolean unknowns x_{ij} associated with off-diagonal elements of $A(G)$. These equations can then be solved to obtain values for the x_{ij}'s. Hence $A(G)$ could be found. $A(G)$ defines the graph G. Hence G itself can be found. This method of constructing G from $m(G)$ is given in [6].

Given the deck of G we can node-reconstruct $m(G)$ according to Theorem 2. From $m(G)$, G itself can be found. Hence G can be reconstructed from its deck. We state the result formally in the following theorem.

THEOREM 7. If the deck of G is given, then the entire graph G can be reconstructed (though perhaps not uniquely).

The edge analogue of this result follows by a similar argument using Theorem 6.

THEOREM 8. If the edge-deleted subgraphs of G is given, then the entire graph G can be reconstructed (though perhaps not uniquely).

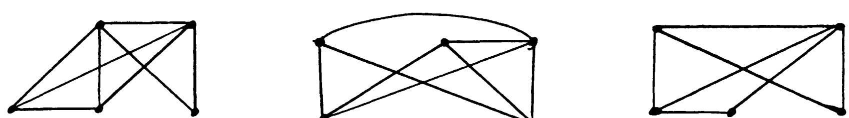
The reconstruction of the graph G is carried out from the mathcing matrix $A(G)$. $A(G)$ in turn is defined by the solutions for the boolean unknowns x_{ij} ($i = 1,2, \ldots, p$, and $j = 1,2, \ldots, p$ $(i \neq j)$). In practice, we have found that although there might be different solutions for the x_{ij}'s, the graphs defined by the resulting matching matrices are always isomorphic.

5. ILLUSTRATIONS OF THE MAIN RESULTS.

In this section we will give two examples which illustrate Theorems 2, 6, 7 and 8.

<u>Example 1</u>

Let the following graphs be the node-deleted subgraphs of a graph G.



H_1 H_2 H_3

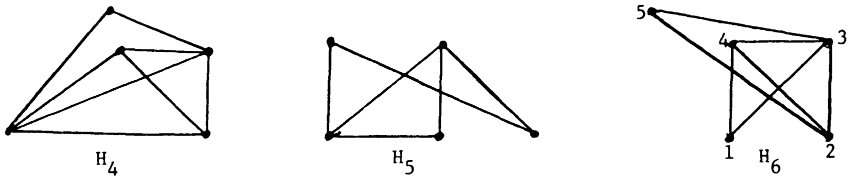


Figure 1

The following matching polynomials can be easily obtained.

$$m(H_1) = w_1^5 + 8w_1^3w_2 + 9w_1w_2^2; \quad m(H_2) = w_1^5 + 8w_1^3w_2 + 10w_1w_2^2$$

$$m(H_3) = w_1^5 + 7w_1^3w_2 + 7w_1w_2^2; \quad m(H_4) = w_1^5 + 8w_1^3w_2 + 9w_1w_2^2$$

$$m(H_5) = w_1^5 + 6w_1^3w_2 + 6w_1w_2^2; \quad m(H_6) = w_1^5 + 7w_1^3w_2 + 7w_1w_2^2.$$

Therefore

$$\sum_{i=1} \int m(G-v_i)dw_1 = \int (6w_1^5 + 44w_1w_2 + 48w_1w_2)dw_1,$$

$$= w_1^6 + 11w_1^4 + 24w_1^2w_2^2 + \gamma(G)w_2^3.$$

Using the labelled subgraph H_6, the matching matrix $A(G)$ defined in Equation (2.1) is

$$A(G) = \begin{bmatrix} w_1 & 0 & \sqrt{w_2} & \sqrt{w_2} & 0 & \sqrt{x_{16}w_2} \\ 0 & w_1 & \sqrt{w_2} & \sqrt{w_2} & \sqrt{w_2} & \sqrt{x_{26}w_2} \\ -\sqrt{w_2} & -\sqrt{w_2} & w_1 & \sqrt{w_2} & \sqrt{w_2} & \sqrt{x_{36}w_2} \\ -\sqrt{w_2} & -\sqrt{w_2} & -\sqrt{w_2} & w_1 & 0 & \sqrt{x_{46}w_2} \\ 0 & -\sqrt{w_2} & -\sqrt{w_2} & 0 & w_1 & \sqrt{x_{56}w_2} \\ -\sqrt{x_{16}w_2} & -\sqrt{x_{26}w_2} & -\sqrt{x_{36}w_2} & -\sqrt{x_{46}w_2} & -\sqrt{x_{56}w_2} & w_1 \end{bmatrix}$$

It can be confirmed that

$$d(A(G)) = w_1^6 + (x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + 7)w_1^4w_2$$
$$+ (5x_{16} + 4x_{26} + 3x_{36} + 4x_{46} + 5x_{56} + 7)w_1^2w_2^2$$
$$+(2x_{16} + x_{26} + x_{36} + x_{46} + 2x_{56})w_2^3.$$

From Lemma 2, we get, by comparing coefficients

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} = 4.$$

$$5x_{16} + 4x_{26} + 3x_{36} + 4x_{46} + 5x_{56} = 17.$$

It is clear that the only solutions to these equations are

$$(x_{16}, x_{26}, x_{36}, x_{46}, x_{56})^T = (1, 0, 1, 1, 1)^T$$

and $$(x_{16}, x_{26}, x_{36}, x_{46}, x_{56})^T = (1, 1, 1, 0, 1)^T.$$

By substituting into $d(A(G))$ we get in both cases $\gamma(G) = 6$. Hence we obtain

$$\underline{m(G) = w_1^{\,6} + 11w_1^{\,4}w_2 + 24w_1^{\,2}w_2 + 6w_2^{\,3}.}$$

We can now use $A(G)$ in order to reconstruct the graph G. THe following labelled graphs are constructed, corresponding to the two solutions obtained. G_1 is obtained by using the first solution.

Figure 2

It can be easily confirmed that the mapping defined by $\phi : G_1 \to G_2$ such that $\phi(1) = 5$; $\phi(2) = 4$; $\phi(3) = 3$; $\phi(4) = 2$; $\phi(5) = 1$ and $\phi(6) = 6$ is an isomorphism. Hence $G_1 \cong G_2$.

Example 2

Let the following graphs be the edge-deleted subgraphs of a graph G.

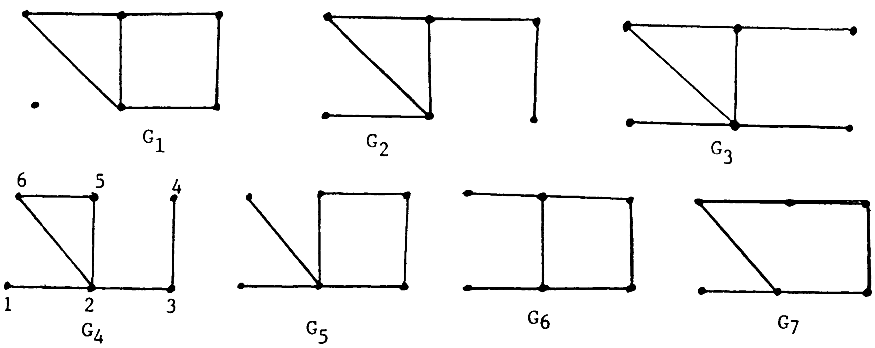


Figure 3

The following matching polynomials can be easily obtained.

$m(G_1) = w_1^{\,6} + 6w_1^{\,4}w_2 + 6w_1^{\,2}w_2^{\,2}$; $m(G_2) = w_1^{\,6} + 6w_1^{\,4}w_2 + 7w_1^{\,2}w_2^{\,2} + w_2^{\,3}$

$m(G_3) = w_1^{\,6} + 6w_1^{\,4}w_2 + 5w_1^{\,2}w_2^{\,2}$; $m(G_4) = w_1^{\,6} + 6w_1^{\,4}w_2 + 6w_1^{\,2}w_2^{\,2} + w_2^{\,3}$

$m(G_5) = w_1^{\,6} + 6w_1^{\,4}w_2 + 6w_1^{\,2}w_2^{\,2}$; $m(G_6) = w_1^{\,6} + 6w_1^{\,4}w_2 + 7w_1^{\,2}w_2^{\,2} + w_2^{\,3}$

$m(G_7) = w_1^{\,6} + 6w_1^{\,4}w_2 + 8w_1^{\,2}w_2^{\,2} + w_2^{\,3}$.

$$\sum_{i=1}^{7} m(G_i) = 7w_1^{\,6} + 42w_1^{\,4}w_2 + 45w_1^{\,2}w_2^{\,2} + 4w_2^{\,3} = C(w_1, w_2).$$

From Theorem 5, we get

$$\sum_{k=0}^{3} a_k(7-k)\, w_1^{\,6-2k}w_2^{\,k} = 7w_1^{\,6} + 42w_1^{\,4}w_2 + 45w_1^{\,2}w_2^{\,2} + 4w_2^{\,3}.$$

By comparing coefficients in this equation, we get

$$a_0 = 1, \quad a_1 = 7, \quad a_2 = 9 \text{ and } a_3 = 1.$$

Hence the matching polynomial of G is

$$m(G) = w_1^6 + 7w_1^4 w_2 + 9w_1^2 w_2^2 + w_2^3.$$

We can extend the reconstruction to G itself by using the labelled subgraph G_4 to form the following matching matrix of G.

$$A(G) = \begin{bmatrix} w_1 & \sqrt{w_2} & \sqrt{x_{13}w_2} & \sqrt{x_{14}w_2} & \sqrt{x_{15}w_2} & \sqrt{x_{16}w_2} \\ -\sqrt{w_2} & w_1 & \sqrt{w_2} & \sqrt{x_{24}w_2} & \sqrt{w_2} & \sqrt{w_2} \\ -\sqrt{x_{13}w_2} & -\sqrt{w_2} & w_1 & \sqrt{w_2} & \sqrt{x_{35}w_2} & \sqrt{x_{36}w_2} \\ -\sqrt{x_{14}w_2} & -\sqrt{x_{24}w_2} & -\sqrt{w_2} & w_1 & \sqrt{x_{45}w_2} & \sqrt{x_{46}w_2} \\ -\sqrt{x_{15}w_2} & -\sqrt{w_2} & -\sqrt{x_{35}w_2} & -\sqrt{x_{45}w_2} & w_1 & \sqrt{w_2} \\ -\sqrt{x_{16}w_2} & -\sqrt{w_2} & -\sqrt{x_{36}w_2} & -\sqrt{x_{46}w_2} & -\sqrt{w_2} & w_1 \end{bmatrix}$$

By using Lemma 2, all the boolean unknowns in $A(G)$ could be found. Hence $A(G)$ and consequently G can be found. The calculations involved in finding $d(A(G))$ are a bit tedious. We will not reproduce them here. One solution is

$$(x_{13}, x_{14}, x_{15}, x_{16}, x_{24}, x_{35}, x_{36}, x_{45}, x_{46})^T = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0)^T.$$

The graph G defined by this solution is the following.

G:

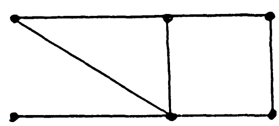


Figure 4

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