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GAPS IN THE SEQUENCE $n^2 \vartheta$ (mod 1)

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ABSTRACT: Let ϑ be an irrational number and let $\{t\}$ denote the fractional part of t. For each N let $I_0, I_1, \ldots, I_N$ be the intervals resulting from the partition of [0,1] by the points $\{k^2\vartheta\}$, k = 1,2,...,N. Let T(N) be the number of distinct lengths these intervals can assume. It is shown that $T(N) \to \infty$. This is in contrast to the case of the sequence $\{n\vartheta\}$, where $T(N) \leq 3$.

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1. INTRODUCTION.

Let ϑ be an irrational number and let $\{t\}$ denote the fractional part of t ($\{t\}$ = t (mod 1) = t - [t], where [.] is the greatest integer function). For each fixed N the points $\{\vartheta\}, \{2\vartheta\}, \{3\vartheta\}, \ldots, \{N\vartheta\}$ partition on the interval [0,1] into N+1 subintervals. It is well known that the lengths of these intervals can assume only 3 values: α, β and $\alpha+\beta$. The values of α and β can be actually given explicitly in terms of N and the continued fraction expansion of ϑ. This is known as Steinhaus conjecture and it was first proved by Swierczkowski in [1]. For an excellent exposition of all this, see [2]. In this note we investigate the analogous problem for the sequence $\{n^2\vartheta\}$. It turns out that in this case the number of different lengths these subintervals can assume, is unbounded. More precisely we have the following results.

2. MAIN RESULTS.

<u>Theorem 1</u> Let ϑ be an irrational. For each integer N let $I_0, I_1, \ldots, I_N$ be the N+1 subintervals resulting from partition of [0,1] by the points $\{k^2\vartheta\}$, k = 1,2,...,N. Let T(N) be the number of distinct lengths these subintervals assume. Then for each $\varepsilon > 0$,

$$T(N) \geq N\exp\{-(1+\varepsilon)\ln 2^2 \frac{\ln N}{\ln \ln N}\} \qquad \text{for } N \geq N(\varepsilon) . \qquad (2.1)$$

In particular $T(N) \geq N^{1-\delta}$ for every $\delta > 0$ and $N \geq N(\delta)$.

In what follows $\vartheta > 0$ is some fixed irrational. We need the following four simple lemmas.

LEMMA 1. For any integers r,s

$$\{(r+s)\vartheta\} = \{r\vartheta\} + \{s\vartheta\} - E \tag{2.2}$$

where $E = 0$ or 1.

PROOF. We have

$$(r + s)\vartheta = \{(r+s)\vartheta\} + [(r+s)\vartheta]$$

$$= \{r\vartheta\} + \{s\vartheta\} + [r\vartheta] + [s\vartheta] = \{r\vartheta\} + \{s\vartheta\} + \text{integer}$$

Thus, if $0 < \{r\vartheta\} + \{s\vartheta\} < 1$ then (2) holds with $E = 0$,
and if $1 < \{r\vartheta\} + \{s\vartheta\} < 2$ then (2) holds with $E = 1$.

LEMMA 2. Suppose x,y are integers, $\{x\vartheta\} < \{y\vartheta\}$. Then

$$\{y\vartheta\} - \{x\vartheta\} = \begin{matrix} \{(y-x)\vartheta\} \\ 1 - \{(x-y)\vartheta\} \end{matrix} \quad \begin{Bmatrix} \text{if } x < y \\ \text{if } y < x \end{Bmatrix}. \tag{2.3}$$

PROOF. Suppose $x < y$ so that $y = x + k$. Then by Lemma 1

$$\{y\vartheta\} = \{x\vartheta\} + \{k\vartheta\} - E .$$

If $E = 1$ then $\{y\vartheta\} < \{x\vartheta\}$ contrary to hypothesis, so that $E = 0$ and (2.3) holds.
If $y < x$, let $x = y + k$, $k > 0$. Again, by Lemma 1

$$\{x\vartheta\} = \{y\vartheta\} + \{k\vartheta\} - E .$$

If $E = 0$ then $\{x\vartheta\} > \{y\vartheta\}$ so that $E = -1$ and (2.3) holds again.

LEMMA 3. For any two non-negative integers x,y, $\{x\vartheta\} \neq 1 - \{y\vartheta\}$.

PROOF. If $\{x\vartheta\} + \{y\vartheta\} = 1$ then by Lemma 1

$$\{(x+y)\vartheta\} = \{x\vartheta\} + \{y\vartheta\} - E = 1 - E = 0 \text{ or } 1$$

contradicting the fact that ϑ is irrational.

LEMMA 4. Suppose x_1, y_1, x_2, y_2 are non-negative integers and let

$$A = \{y_1\vartheta\} - \{x_1\vartheta\}>0 , \quad B = \{y_2\vartheta\} - \{x_2\vartheta\}>0 .$$

If $A = B$ then $y_1 - x_1 = y_2 - x_2$.

PROOF. We will use Lemmas 2 and 3 and consider 4 cases

I: $x_1 < y_1$, $x_2 < y_2$;

II: $x_1 < y_1$, $x_2 > y_2$;

III: $x_1 > y_1$, $x_2 < y_2$;

IV: $x_1 > y_1$, $x_2 > y_2$.

In case I we get from Lemma 2

$$A = \{(y_1 - x_1)\vartheta\} , \quad B = \{(y_2 - x_2)\vartheta\}$$

so $A = B$ implies $y_1 - x_1 = y_2 - x_2$.

In case II, by Lemma 2 we get

$$A = \{(y_1 - x_1)\vartheta\} , \quad B = 1 - \{(x_2 - y_2)\vartheta\}$$

so A = B cannot hold by Lemma 3.
Similarly, A = B cannot hold in case III, and A = B implies $y_1 - x_1 = y_2 - x_2$ in case IV.

We are now ready to prove the Theorem 1. Let N be fixed and consider the partition of [0,1] by the points $\{0^2\vartheta\} = 0$, $\{1^2\vartheta\}$, $\{2^2\vartheta\}$, $\{3^2\vartheta\}$, ..., $\{N^2\vartheta\}$. If we exclude the right most interval (i.e. the interval $[\{x^2\vartheta\},1]$ for some x), we are left with a collection A(N) of N intervals. If two of these intervals $[\{x_1^2\vartheta\}, \{y_1^2\vartheta\}]$ and $[\{x_2^2\vartheta\}, \{y_2^2\vartheta\}]$ are of equal length then

$$y_1^2 - x_1^2 = y_2^2 - x_2^2 \tag{2.4}$$

by Lemma 4. Let T(N) be the number of distinct lengths these intervals from A(N) can assume. The collection A(N) is then divided into T(N) subsets, any two intervals from one subset are of equal length. One of these subsets must contain N/T(N) intervals. Thus, by (2.4), there exists an integer k, $1 \le k \le N^2$ such that the equation

$$k = y^2 - x^2 = (y-x)(y+x) \tag{2.5}$$

has N/T(N) solutions in integers x,y, $1 \le x < y \le N^2$. Each such solution produces 2 distinct divisors of k. If $y_1^2 - x_1^2 = y_2^2 - x_2^2$, $1 \le x_i < y_i \le N^2$ for i = 1,2 and $(x_1,y_1) \neq (x_2,y_2)$, then $y_1 - x_1 \neq y_2 - x_2$ and $y_1 + x_1 \neq y_2 + x_2$. Thus

$$N/T(N) \le \tfrac{1}{2}d(k) \tag{2.6}$$

where d(z) is the number of divisors of d. It is well known that for each $\varepsilon > 0$

$$d(z) < \exp\{(1+\varepsilon) \ln 2 \frac{\ln z}{\ln \ln z}\} = \varphi(\varepsilon,z) \quad \text{for} \quad z \ge z(\varepsilon)$$

This was first proved by Wigert in [3], see also [4], Satz 5.2. Since $k < N^2$ we get from (2.6)

$$\begin{aligned} 2N/T(N) &\le \varphi(\varepsilon,k) < \varphi(\varepsilon,N^2) \\ &= \exp\{(1+\varepsilon) \ln 2 \frac{2 \ln N}{\ln 2 + \ln \ln N}\} \\ &\le \exp\{(1+\varepsilon) \ln 2^2 \frac{\ln N}{\ln \ln N}\} \quad \text{for } N > N_1(\varepsilon) \end{aligned} \tag{2.7}$$

Solving this inequality for T(N) gives (2.1).

The argument carries over almost without any change to the sequence $\{n^p\vartheta\}$ for any integer $p > 1$. The corresponding estimate is then as follows.

THEOREM 2. Let ϑ be an irrational and $p > 1$ an integer. For each integer N let $I_0, I_1, \ldots, I_N$ be the N+1 subintervals resulting from partition of [0,1] by the points $\{k^2\vartheta\}$, k = 1,2,...,N. Let $T_p(N)$ be the number of distinct lengths these intervals can assume. Then for each $\varepsilon > 0$

$$T_p(N) \ge N \exp\{-(1+\varepsilon) \ln 2^p \frac{\ln N}{\ln \ln N}\} \quad \text{for} \quad N \ge N(\varepsilon) .$$

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