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λ-SIMILAR BASES

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ABSTRACT. Corresponding to an arbitrary sequence space λ, a sequence $\{x_n\}$ in a locally convex space (l.c.s.) (X,T) is said to be λ-similar to a sequence $\{y_n\}$ in another l.c.s. (Y,S) if for an arbitrary sequence $\{\alpha_n\}$ of scalars, $\{\alpha_n\, p(x_n)\} \in \lambda$, for all $p \in D_T \iff \{\alpha_n\, q(y_n)\} \in \lambda$, for all $q \in D_S$, where D_T and D_S are respectively the family of all T - and S - continuous seminorms generating T and S.

In this note we investigate conditions on λ and the spaces (X,T) and (Y,S) which ultimately help to characterize λ - similarity between two Schauder bases. We also determine relationship of this concept with λ-bases.

KEY WORDS AND PHRASES. (K)-property, normal topology, semi-λ-base, λ-base, fully λ-base, similar sequences, λ-similar sequences, normal and symmetric sequence spaces.
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1. MOTIVATION.

Similar bases in the literature have been studied with a view to find the comparison between an arbitrary base and a known base of apparently simple looking elements. However, for certain neutral and obvious reasons, the similarity of bases is defined in terms of the convergence of the underlying infinite series. Since there are different modes of convergence, one may talk of similarity of bases depending upon the particular kind of convergence or a particular type of a Schauder base, e.g. similar bases, absolutely similar bases etc. (see [1]). In the literature of Schauder bases, we have recently introduced different types of " λ-bases" which generalize several concepts of absolute bases in a locally convex space (cf. [2]) and hence it becomes imperative to exploit this notion and introduce a kind of similarity, that is, λ-similarity which should have in a natural way some relationship with λ-bases and at the same time should extend the results on absolute similarity. This is what we have done in this note and would discuss the same in subsequent pages.

2. TERMINOLOGY.

Unless specified otherwise, we write throughout (X,T) and (Y,S) for two arbitrary Hausdorff locally convex spaces (l.c. TVS) containing Schauder bases (S.b.)

$\{x_n\} \equiv \{x_n;f_n\}$ and $\{y_n\} \equiv \{y_n;g_n\}$ respectively (cf. [3] for details and unexplained terms on basis theory hereafter). We write D_T to mean the family of all T-continuous seminorms generating the topology T of (X,T) and attach a similar meaning for D_S. The symbol $(X,T) \simeq (Y,S)$ is used to mean the existence of a topological isomorphism R from X onto Y and conversely. Let

$$\delta_X = \{\{f_n(x)\}: \ x \in X\};$$

$$\delta_Y = \{\{g_n(y)\}: \ y \in Y\}; \text{ and}$$

$$\Delta_X = \bigcap_{p \in D_T} \frac{\lambda}{p^*}; \quad \Delta_Y = \bigcap_{p \in D_S} \frac{\lambda}{q^*},$$

where $p^* \equiv \{p(x_n)\}$; $q^* = \{q(y_n)\}$ and λ is an arbitrary sequence space (s.s) For a Köthe set P, we write $\Lambda(P)$ for its Köthe space. For discussion and unexplained terms on s.s., we refer to [4]; cf. also [5] whereas for the theory of l.c. TVS our references are [6] and [7].

An s.s. λ is said to satisfy the (K)-<u>property</u> if there exists β in λ^x such that

$$K \equiv K_\beta = \inf |\beta_n| > 0 \tag{2.1}$$

Without further notice, whenever we consider an s.s. λ as an l.c. TVS, it would be assumed that λ is equipped with its normal topology $\eta(\lambda,\lambda^x)$.

Finally, following [8], we recall

DEFINITION 2.2: An s.b. $\{x_n;f_n\}$ for an l.c. TVS (X,T) is called (i) a <u>semi</u>-λ-<u>base</u> if $\delta_X \subset \Delta_X$, (ii) a λ-<u>base</u> if $\delta_X = \Delta_X$' and (iii) a <u>fully</u> λ-<u>base</u> if $\delta_X \subset \Delta_X$ and for each p in D_T' the map $\Psi_p: (X,T) \to (\lambda,\eta(\lambda,\lambda^x))$, $\Psi_p(x) = \{f_n(x)p(x_n)\}$ is continuous.

NOTE: If $\{x_n;f_n\}$ is a semi-λ-base, then for each p in D_T and γ in λ^x, the seminorm $Q_{p,\gamma}$ on X is well defined, where

$$Q_{p,\gamma}(x) = \sum_{n\geq 1} |f_n(x)||\gamma_n|p(x_n), \text{ for all } x \in X. \tag{2.2}$$

If λ satisfies (2.1) and $\{x_n;f_n\}$ is a fully λ-base, then T is also generated by the family $\{Q_{p,\gamma}: p \in D_T, \gamma \in \lambda^x\}$ of seminorms on x; in particular, for each p in D_T, there exists r in D_T so that for all x in X,

$$p(x) \leq \sum_{n\geq 1} |f_n(x)|p(x_n) \leq \frac{1}{K_\beta} Q_{p,\beta}(x) \leq r(x). \tag{2.3}$$

3. λ-SIMILAR BASES.

Let us recall from [1] the following:

DEFINITION 3.1. A sequence $\{x_n\} \subset X$ is said to be <u>similar</u> to a sequence $\{y_n\} \subset Y$, written as $\{x_n\} \sim \{y_n\}$, if for α in ω

$$\sum_{i=1}^{\infty} \alpha_i x_i \text{ converges in } X \Longleftrightarrow \sum_{i=1}^{\infty} \alpha_i y_i \text{ converges in } Y. \tag{3.2}$$

The concept of absolute similarity introduced in [1], is abstracted to λ-similarity as follows:

DEFINITION 3.3. For an arbitrary s.s. λ, a sequence $\{x_n\} \subset X$ is said to be λ-<u>similar</u> to a sequence $\{y_n\} \subset Y$, to be written as $\{x_n\} \overset{\lambda}{\sim} \{y_n\}$, provided for α in ω,

$$\{\alpha_n p(x_n)\} \in \lambda, \text{ for all } p \in D_T \iff \{\alpha_n q(y_n)\} \in \lambda, \text{ for all } q \in D_S. \tag{3.4}$$

The following are straightforward.

PROPOSITION 3.5. $\{x_n\} \sim \{y_n\} \iff \delta_X = \delta_Y$.

PROPOSITION 3.6. $\{x_n\} \overset{\lambda}{\sim} \{y_n\} \iff \Delta_X = \Delta_Y$.

PROPOSITION 3.7. If λ is normal, then (3.4) is equivalent to

$$\{p(\alpha_n x_n)\} \in \lambda, \text{ for all } p \in D_T \iff \{q(\alpha_n y_n)\} \in \lambda, \text{ for all } q \in D_S. \tag{3.8}$$

for α in ω.

NOTE. In particular, if $\lambda = \ell^1$, we have

$\{x_n\} \overset{\ell^1}{\sim} \{y_n\} \iff x_n$ is absolutely similar to $\{y_n\}$ as discussed in [1].

$\iff \Lambda(P) = \Lambda(Q)$, where $\Lambda(P)$ and $\Lambda(Q)$ are Köthe spaces corresponding to the Köthe sets $P = \{\{p(x_n)\}: p \in D_T\}$ and $Q = \{\{q(y_n)\}: q \in D_S\}$ respectively.

PROPOSITION 3.8. Let $\{x_n\}$ and $\{y_n\}$ be λ-bases. Then $\{x_n\} \sim \{y_n\} \iff \{x_n\} \overset{\lambda}{\sim} \{y_n\}$.

PROOF. It suffices to observe that $\delta_X = \Delta_X$, $\delta_Y = \Delta_Y$ and now apply Propositions 3.5 and 3.6.

4. FURTHER CONDITIONS.

In order to obtain more characterizations of λ-similarity of bases, let us bring-forth the following conditions for two arbitrary sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$.

$$\text{for all } p \in D_T, \exists\, q \in D_S \text{ such that } p(x_n) \le q(y_n), \text{ for all } n \ge 1. \tag{4.1}$$

$$\text{for all } q \in D_S, \exists\, p \in D_T \text{ such that } q(y_n) \le p(x_n), \text{ for all } n \ge 1. \tag{4.2}$$

PROPOSITION 4.3. If λ is normal and conditions (4.1) and (4.2) are satisfied, then $\{x_n\} \overset{\lambda}{\sim} \{y_n\}$.

For obtaining the converse of the preceding result, we restrict ourselves to metrizable l.c. TVS. If (X,T) is a metrizable l.c. TVS, then T is also generated by $D_X = \{p_i: i \ge 1\}$, where $p_1 \le p_2 \le \ldots$ Similarly, when (Y,S) is metrizable, S is generated by $D_Y = \{q_1 \le q_2 \le \ldots\}$. With this background, (4.1) and (4.2) take the following forms:

$$\text{for all } i \ge 1, \exists\, j \ge 1 \text{ and } M_i > 0 \text{ such that } p_i(x_n) \le M_i q_j(y_n), \text{ for all } n \ge 1. \tag{4.4}$$

$$\text{for all } j \ge 1, \exists\, i \ge 1 \text{ and } K_j > 0 \text{ such that } q_j(y_n) \le K_j p_i(x_n), \text{ for all } n \ge 1. \tag{4.5}$$

Hereafter, in all subsequent pages, we will let λ be a normal s.s., satisfying (2.1). Also, we consider another s.s. μ which will throughout be assumed to be normal and symmetric along with $\lambda \subset \mu \subsetneq \lambda^x$.

LEMMA 4.6. Let (X,T) and (Y,S) be metrizable. Suppose that λ is also symmetric and let λ contain an element α with $\alpha >> 0$, that is, $\alpha_n > 0$, for each $n \ge 1$. Assume the truth of the following statement for an α in ω:

$$\text{for all } i \geq 1, \ \{p_i(\alpha_n x_n)\} \in \lambda \Longrightarrow \{q_j(\alpha_n y_n)\} \in \mu, \text{ for all } j \geq 1. \tag{4.7}$$

Then (4.5) holds.

PROOF. Since λ^x is also symmetric, we may find an element γ in $\lambda^x \setminus \mu$ with $\gamma >> 0$.

Let now (4.5) be not satisfied. Then there exists $j \geq 1$ so that for each $i \geq 1$, one finds a positive integer $n_i (n_i < n_{i+1}, i \geq 1)$ with

$$q_j(y_{n_i}) > \frac{\gamma_i}{\alpha_i} p_i(x_{n_i}), \text{ for all } i \geq 1.$$

Define

$$\delta_n = \begin{cases} \gamma_i / q_j(y_{n_i}); & n = n_i \\ \\ 0 & ; \text{otherwise}. \end{cases}$$

Now $q_j(\delta_{n_i} Y_{n_i}) = \gamma_i$ and 0 otherwise and so $\{q_j(\delta_n Y_n)\} \notin \mu$, for otherwise its close up being γ woulb be in μ, a contradiction. On the other hand, let $m \geq 1$; then one finds $i \geq m$ so that

$$p_m(\delta_{n_i} x_{n_i}) \leq p_i(\delta_{n_i} x_{n_i}) < \alpha_i, \text{ for all } i \geq m.$$

Hence $\{p_m(\delta_n x_n)\} \in \lambda$ for each $m \geq 1$. This contradicts (4.7) and the lemma is proved.

Symmetry considerations lead to

THEOREM 4.8. Let (X,T) and (Y,S) be metrizable, along with λ,μ satisfying the conditions laid down in Lemma 4.6. Then $\{x_n\} \overset{\lambda}{\sim} \{y_n\} \Longleftrightarrow$ (4.4) and (4.5) hold.

5. λ-BASES AND SIMILARITY.

In this section, we investigate conditions when two fully λ-bases are similar. First, we need

LEMMA 5.1. Let $\{x_n\}$ be a fully λ-base for (X,T) and $\{y_n\}$ an arbitrary sequence in (Y,S). Then (4.2) is equivalent to the following statement:

for all $q \in D_S$, $\exists$ $p \in D_T$ such that

$$q\left(\sum_{i=1}^{k} \alpha_i y_i\right) \leq p\left(\sum_{i=1}^{k} \alpha_i x_i\right) \tag{5.2}$$

for all finite sequences $\{\alpha_1,\ldots,\alpha_k\}$.

PROOF. Let (4.2) be satisfied. Then for some $r \in D_T$

$$q\left(\sum_{i=1}^{k} \alpha_i y_i\right) \leq \frac{1}{K_\beta} \sum_{i=1}^{k} |\alpha_i \beta_i| p(x_i) \leq \frac{1}{K_\beta} r\left(\sum_{i=1}^{k} \alpha_i x_i\right)$$

and this proves (5.2). The other part is obvious.

Symmetry considerations in Lemma 5.1 now easily lead to

THEOREM 5.3. Let $\{x_n\}$ and $\{y_n\}$ be fully λ-bases for ω-complete l.c. TVS's (X,T) and (Y,S) respectively. Then the truth of (4.1) and (4.2) yields that $\{x_n\} \sim \{y_n\}$.

Conversely, if (X,T) and (Y,S) are Fréchet spaces with λ, μ satisfying the conditions of Lemma 4.6, then $\{x_n\} \sim \{y_n\} \implies$ (4.4) and (4.5) hold.

PROOF. First part follows from Lemma 5.1 and its symmetrization. For the converse, observe that $\{x_n\}$ and $\{y_n\}$ are λ-bases by Propositions 3.5 and 3.6 of [8]. Now make use of Proposition 3.8 and Theorem 4.8.

REMARKS. Observe that Lemma 5.1 and the first part of Theorem 5.3 and valid for an arbitrary sequence space λ satisfying the K-property.

Since λ-bases and fully λ-bases are the same in a Fréchet space (cf. [8], p.82), we can rephrase Theorem 5.3 for Fréchet spaces as follows:

THEOREM 5.4. Let (X,T) and (Y,S) be Fréchet spaces and $\{x_n\}$, $\{y_n\}$ be λ-bases along with λ, μ satisfying the conditions of Lemma 4.6. Then $\{x_n\} \sim \{y_n\} \iff$ (4.4) and (4.5) hold.

NOTE. The proof of the above theorem is also immediate from Proposition 3.8 and Theorem 4.8.

6. THE ISOMORPHISM THEOREM.

In order to obtain the main result of this section, let us introduce the following two conditions:

$$\{p(\alpha_n x_n)\} \in \lambda, \text{ for all } p \in D_T \implies \{q(\alpha_n y_n)\} \in \lambda, \text{ for all } q \in D_S; \tag{6.1}$$

$$\{p(\alpha_n x_n)\} \in \lambda, \text{ for all } p \in D_T \implies \{q(\alpha_n y_n)\} \in \mu, \text{ for all } q \in D_S, \tag{6.2}$$

where $\alpha \in \omega$, $\{x_n\}$, $\{y_n\}$ are arbitrary sequences in X, Y respectively and λ, μ are the s.s. as specified in Lemma 4.6.

NOTE. When (X,T) and (Y,S) are metrizable, the conditions (6.1) and (6.2) will be considered with D_T and D_S being replaced by D_X and D_Y (cf. remarks following Proposition 4.3). Thus we have

PROPOSITION 6.3. If the hypothesis of Lemma 4.6 holds, then (4.5) $\implies$ (6.1) $\implies$ (6.2) $\implies$ (4.5).

Next, we have

PROPOSITION 6.4. Let (X,T), (Y,S) be a metrizable l.c. TVS and $R:X \to Y$ a continuous linear map. For a sequence $\{x_n\} \subset X$, let $y_n = Rx_n$. Then either of the equivalent conditions (4.5), (6.1) and (6.2) holds.

PROOF. For each $j > i$ there exist $i \geq 1$ such that $q_j(Rx) \leq K_j p_i(x)$ for all x in X and so (4.5) holds. The result now follows from the preceding proposition.

Conversely, we have

PROPOSITION 6.5. Let (X,T) be a metrizable l.c. TVS containing a fully λ-base $\{x_n\}$, (Y,S) a Fréchet space containing a sequence $\{y_n\}$. If either of the equivalent conditions (4.5), (6.1) or (6.2) holds, then there exists a continuous linear map $R:(X,T) \to (Y,S)$ with $Rx_n = y_n$, $n \geq 1$.

PROOF. Define

$$R\left(\sum_{m=1}^{k} \alpha_m x_m\right) = \sum_{m=1}^{k} \alpha_m y_m,$$

then by (4.5) and (2.3), $R:\text{sp}\,\{x_n\} \to Y$ is continuous and so its unique extension is the required continuous linear map. This completes the proof.

PROPOSITION 6.6. Let (X,T) and (Y,S) be ω-complete such that $(X,T) \simeq (Y,S)$. Let $\{x_n\}$ be a semi λ-base for (X,T) and $y_n = Rx_n$, $n \geq 1$. Then $\{y_n\}$ is a semi λ-base for (Y,S) and $\{x_n\} \overset{\lambda}{\sim} \{y_n\}$.

PROOF. $\{y_n\}$ is clearly an S.b. and let $\{g_n\}$ be the s.a.c.f. corresponding to $\{y_n\}$; indeed, $g_n = f_n o R^{-1}$, $n \geq 1$. Then, for each y in Y, ther exists a unique x in X with $g_n(y) = f_n(x)$ for all $n \geq 1$. Since (4.2) holds and λ is normal, $\{y_n;g_n\}$ is a semi λ-base for (Y,S). Consequently, both of these bases are λ-bases, cf. Proposition 3.5 of [8]. Thus $\delta_X = \Delta_X$ and $\delta_Y = \Delta_Y$. But $\delta_X = \delta_Y \Longrightarrow \Delta_X = \Delta_Y$ and now apply Proposition 3.6.

Finally, we have

PROPOSITION 6.7. Let (X,T) and (Y,S) be ω-complete barrelled spaces having semi λ-bases $\{x_n\}$ and $\{y_n\}$ respectively such that $\{x_n\} \overset{\lambda}{\sim} \{y_n\}$. Then $(X,T) \simeq (Y,S)$ with $Rx_n = y_n$, $n \geq 1$.

PROOF. By Proposition 3.4, $\{x_n\} \sim \{y_n\}$ and so by a theorem of [9], $(X,T) \simeq (Y,S)$ with $Rx_n = y_n$, $n \geq 1$.

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