Internat. J. Math. & Math. Sci.
Vol. 10 No. 2 (1987) 303-309

FINITE p'-NILPOTENT GROUPS. II

S. SRINIVASAN
Department of Mathematics and Computer Science
Austin Peay State University
Clarksville, Tennessee 37044 USA

(Received February 8, 1986 and in revised form September 15, 1986)

ABSTRACT: In this paper we continue the study of finite p'-nilpotent groups that was started in the first part of this paper. Here we give a complete characterization of all finite groups that are not p'-nilpotent but all of whose proper subgroups are p'-nilpotent.

KEY WORDS AND PHRASES. *Frattini subgroup, p'-nilpotent group, maximal subgroup, nilpotent group, solvable group*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 20 D 15, 20 D 05

1. INTRODUCTION.

We consider only finite groups. The concept of p'-nilpotency was introduced in [1]. Briefly, a p-closed group is p'-nilpotent if it has a nilpotent Sylow p-complement. In this paper we consider groups which possess a large number of p'-nilpotent groups where the prime p remains the same for the several subgroups or it differs from subgroup to subgroup. Here we rely heavily on the theorem of N.Ito in which he proves that a minimal non-p-nilpotent group is a minimal non-nilpotent group. K.Iwasawa separately.

We show that a group in which every two generator proper subgroup is p'-nilpotent is either p'-nilpotent or a p-nilpotent minimal non-nilpotent group. Then we study the case when the proper subgroups are either p'-nilpotent or q'-nilpotent and show that such groups are always solvable. The main theorem of this paper completely classifies all simple groups with every proper subgroup p'-nilpotent for some prime p. Notation and terminology are standard as in [2].

2. DEFINITIONS AND KNOWN RESULTS.

For the sake of completeness we give the following definition and result from [1].

DEFINITION 2.1 : G is a π-nilpotent group, π a set of primes, if $G_{\pi'} \lhd G$ and $G/G_{\pi'}$ a nilpotent π-group. Let $\underline{P}$ denote the set of all primes. When $\pi = \underline{P} - \{p\}$, we say that G is a p'-nilpotent group.

LEMMA 2.2 : G is p'-nilpotent if and only if G is q-nilpotent $\forall$ q $\neq$ p. (see Corollary 2.4 of [1])

THEOREM 2.3 : Let G be a group such that all proper subgroups are p-nilpotent but G is not p-nilpotent. Then

(i) every proper subgroup of G is nilpotent,

(ii) $|G| = p^a q^b$, $p \neq q$,

(iii) G has a normal Sylow p-subgroup; for p > 2 exp $(G_p) = p$ and for p = 2 the exponent is at most 4,

(iv) Sylow q-subgroups are cyclic. (see Satz 5.4 of [2])

Combining Lemma 2.2 and Theorem 2.3 we have the following theorem.

THEOREM 2.4 : Let G be a group with the property that all its proper subgroups are p'-nilpotent for the prime p. Then G is either p'-nilpotent or G is a p-nilpotent minimal non-nilpotent group.

3. MINIMAL NON-p'-NILPOTENT GROUPS.

In Theorem 2.4 we required that all proper subgroups be p'-nilpotent. We now weaken the hypothesis in Theorem 2.4 by requiring only that those proper subgroups that are generated by two elements be p'-nilpotent.

THEOREM 3.1 : Let G be a group with every proper subgroup generated by two elements p'-nilpotent for the prime p. Then G is either p'-nilpotent or G is a p'nilpotent SRI-group.

PROOF : Suppose G is not p'-nilpotent. Using 2.2 G is not q-nilpotent for some $q \neq p$. Using Theorem 14.4.7, p217 of [3], there exists an r-element x and a q-subgroup Q such that $x \in N_G(Q) - C_G(Q)$, $r \neq q$. Consider $H = Q\langle x\rangle$. Clearly $|H| = q^a r^b$.

CASE 1. r = p.

If H < G, then $\forall\, y \in Q$, $\langle x, y\rangle$ is p'-nilpotent by hypothesis, i.e., $\langle x, y\rangle$ is p-closed. Since $|H_p| = |x|$, this means that $y \in N_G(\langle x\rangle)$ $\forall\, y \in Q$; i.e., $Q \leqq N_G(\langle x\rangle)$, i.e., $H = Q \times \langle x\rangle$, a nilpotent group, i.e., $x \in C_G(Q)$, a contradiction. Hence H = G with $H_q = Q = G_q \lhd G$ and $G_p = \langle x\rangle \not\lhd G$. Let K < G. Then $K = Q_1\langle x^i\rangle$ where $Q_1 \leqq Q$. $G_q \lhd G$ implies $K_q \lhd K$. If K is generated by two elements, then K is p'-nilpotent by hypothesis, so $K_p \lhd K$. Thus K is nilpotent. If K is not generated by two elements, then $\forall\, k \in K$, $\langle k, x^i\rangle$ is p'-nilpotent and hence $\langle k, x^i\rangle$ is nilpotent. Hence $x^i \in C_G(k)$. Thus x^i commutes with all q-elements in K and hence K is nilpotent. Thus all proper subgroups of G are nilpotent, so G is a p-nilpotent minimal non-nilpotent group.

CASE 2. $r \neq p$.

$|H| = q^a r^b$. Suppose H < G. $\forall\, y \in Q$, $\langle x, y\rangle \leqq H < G$. By hypothesis $\langle x, y\rangle$ is p'-nilpotent. $p \nmid |H|$ implies then that $\langle x, y\rangle$ is nilpotent. i.e., $xy = yx$ $\forall\, y \in Q$; i.e., $x \in C_G(Q)$, a contradiction. Hence H = G. As in Case 1 we can conclude again that G is a p-nilpotent minimal non-nilpotent group. Q.E.D.

Since p'-nilpotency is inherited by subgroups the condition of 2.4 follows if all maximal subgroups of G are p'-nilpotent. In 3.1 we required only the proper subgroups generated by two elements to be p'-nilpotent. In both cases G was solvable. We now show that if we require only the core-free maximal subgroups to be p'-nilpotent, then G is solvable under suitable conditions.

THEOREM 3.2 : Let G be a group with at least one core-free maximal subgroup. If G has the following properties:

(i) Sylow 2-subgroups of G have all their proper subgroups abelian,

(ii) all core-free maximal subgroups of G are p'-nilpotent for the prime p, then G is solvable.

PROOF : Suppose that all maximal subgroups of G are core-free. By hypothesis then all maximal subgroups of G are p'-nilpotent. Using 2.4 G is then solvable. So assume that G has at least one $M <\cdot G$ with $M_G \neq 1$. Thus G is not a simple group. We now assume that G is not solvable and arrive at a contradiction. First we show that all core-free maximal subgroups of G are conjugate; clearly we can assume that G has at least two core-free maximal subgroups M_1 and M_2. Let N be a minimal normal subgroup of G. Then $G = M_1N = M_2N$, so $[G : N] = [M_1 : M_1 \cap N]$ and $[G : N] = [M_2 : M_2 \cap N]$.

CASE 1. $p \mid [G : N]$.

Hence $p \mid |M_i|$, $i = 1,2$. M_i p'-nilpotent implies $M_i = N_G(Pi)$, where P_i is the Sylow p-subgroup of M_i. Hence P_i is a Sylow p-subgroup of G. Since P_1 and P_2 are conjugate, this means that M_1 and M_2 are conjugate.

CASE 2. $p \nmid [G : N]$.

Hence $p \nmid [M_i : M_i \cap N]$. If $p \nmid |M_i|$, then M_i are nilpotent. Just as in Case 1, M_1 will then be conjugate to M_2. Thus we assume that $p \mid |M_1|$ and $p \nmid |M_2|$. Hence $M_1 = N_G(P_1)$ and M_2 is nilpotent. Moreover, the argument of Case 1 shows that M_2 is a Hall subgroup of G. If M_2 is of odd order, then using Thompson's theorem on solvability of a group with a nilpotent maximal subgroup of odd order we see that G is solvable. Since we have assumed that G is not solvable, this means that M_2 is of even order. If M_2 is not a Sylow 2-subgroup of G, then using Satz 7.3, p.444 of [2] we see that $G = M_2N$ with $M_2 \cap N = 1$. Since $2 \nmid |N|$, N is solvable. Thus N and G/N are solvable implies G is solvable. Hence we have by choice of G that M_2 is a Sylow 2-subgroup G. Hence $G = M_2N$, $M_2 \cap N \neq 1$. Let T be a Sylow 2-subgroup of N. Since $N \lhd G$ and $[G : N] = 2^n$, N contains all Sylow p-subgroups of G for $p \neq 2$. Hence $M_2 \cap N < M_2$. By hypothesis (i) $M_2 \cap N$ is abelian. G/N is a 2-group. Now using

Satz 7.4, p.445 of [2] we get $M_2 \cap N = 1$. This is contrary to $M_2 \cap N \neq 1$. This impossible situation shows that it can not happen that $p \mid |M_1|$, $p \nmid |M_2|$. Thus, using previous arguments we see that M_1 and M_2 are conjugate. Suppose G has another miniman normal subgroup $N_1 \neq N$. Then $G = M_1N = M_1N_1$. By hypothesis M_1 is p'-nilpotent, so M_1 is solvable. Hence $G \simeq G/(N \cap N_1) \hookrightarrow (G/N) \times (G/N_1)$ shows that G is solvable. By choice of G this means that G has a unique minimal normal subgroup of G. Since all core-free maximal subgroups of G are conjugate they all have the same index in G. Now using Lemma 3, p.121 of [4] N is solvable and hence G is solvable. This final contradiction completes the proof. Q.E.D.

COROLLARY 3.3 : Let G be a group with the property that all of its nonnormal maximal subgroups are p'-nilpotent. If Sylow 2-subgroups of G have all their proper subgroups abelian, then G is solvable.

PROOF : Suppose that all maximal subgroups of G are normal in G. Then G is nilpotent and hence G is solvable. On the other hand if G has no normal maximal subgroups, then by hypothesis all maximal subgroups are p'-nilpotent and hence G is solvable using 2.4. Assume now that G has at least two nonnormal maximal subgroups M, M_1. By hypothesis M, M_1 are p'-nilpotent, hence solvable. Suppose that $M_G \neq 1$. If $M_G \nleq M_1$, then $G = M_GM_1$. M_G and G/M_G are solvable implies that G is solvable. Assume that $M_G \leqq M_1$. Hence $M_G \leqq (M_1)_G$. Using a similar argument with $(M_1)_G$ we have $(M_1)_G \leqq M_G$. Hence $M_G = (M_1)_G$; i.e., all nonnormal maximal subgroups having nontrivial core have the same core. If all nonnormal maximal subgroups have nontrivial core, then by the above argument they have the same core, say N. Consider G/N. Using 3.2 G/N is solvable and since N is solvable we have G solvable. Finally, if all the nonnormal maximal subgroups are core-free, then using 3.2 G is solvable. Q.E.D.

So far we considered the condition that many subgroups of G are p'-nilpotent for the same prime p. In the next theorem we consider the situation that the proper subgroups are either p'-nilpotent or q'-nilpotent.

THEOREM 3.4 : Let G be a group with the property that all its proper subgroups are either p'-nilpotent or q'-nilpotent, $p \neq q$ are primes that are fixed. Then G is solvable.

PROOF : If G is p'-nilpotent or q'-nilpotent, then G is solvable. Assume that G is neither p'-nilpotent nor q'-nilpotent. If $|G|$ is divisible by p and q alone, then using Burnside's theorem on solvability of groups of order p^aq^b, G is solvable. Assume that $|G|$ has at least 3 distinct primes, say p,q,r. By hypothesis all proper subgroups of G are r-nilpotent using Lemma 2.2. Using Theorem 2.3 we see that G is r-nilpotent; i.e. $G^r \lhd G$ and $G = G_rG^r$ where G^r is the Sylow r-complement of G. G^r is solvable by hypothesis and $G/G^r \simeq G_r$ is solvable. Hence G is solvable. Q.E.D.

EXAMPLE 3.5 : Let $G = A_5$. Every proper subgroup of G is either 2'-nilpotent, 3'-nilpotent or 5'-nilpotent. G is not solvable.

This example shows that in Theorem 3.4 we can not, in general, replace 2 primes by 3 primes.

4. MAIN THEOREM.

Example 3.5 shows that when we vary the prime p in the requirement that all proper subgroups be p'-nilpotent, then the group need not be solvable. In this section we completely classify all finite simple groups with this property. First we prove the following lemma.

LEMMA 4.1 : Let G be nonnilpotent dihedral group of order 2m. If G is p'-nilpotent, then $m = 2^a p^b$.

Next we state and prove the main theorem. In the proof of this theorem we will need Thompson's classification of minimal simple groups and Dickson's list of all subgroups of $PSL(2, p^n)$. Also, we need details of the Suzuki group which are given in [5].

MAIN THEOREM : Let G be a nonsolvable simple group with the property that all its proper subgroups are q'-nilpotent for some arbitrary prime q. Then G is one of the following types:

(a) $PSL(2, p)$, with $p^2 - 1 \not\equiv 0 \pmod 5$, $p^2 - 1 \not\equiv 0 \pmod{16}$, $p > 3$, $p - 1 = 2^2 r^i$ and $p + 1 = 2s^j$ or $p - 1 = 2r^i$ and $p + 1 = 2^2 s^j$ where r,s are odd primes, $i,j \geqq 0$.

(b) $PSL(2, 2^n)$, n is a prime, $2^n - 1 = r^i$, $2^n + 1 = s^j$, r,s,i,j as in (a),

(c) $PSL(2, 3^n)$, n is an odd prime, $3^n - 1 = 2^2 r^i$ and $3^n + 1 = 2s^j$ or $3^n - 1 = 2r^i$ and $3^n + 1 = 2^2 s^j$, r,s,i,j as in (a).

Conversely, if G is one of the groups listed above in (a), (b) or (c), then G is a simple group with all its proper subgroups q'-nilpotent for some prime q.

PROOF : Since a q'-nilpotent group is always solvable, all proper subgroups of G are solvable. Hence using Thompson's list of minimal simple groups (see [6]), we conclude that G is one of the following types:

(i) $PSL(2, p)$ where $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$,

(ii) $PSL(2, 2^r)$, r is a prime,

(iii) $PSL(2, 3^r)$, r is an odd prime,

(iv) $PSL(3, 3)$,

(v) the Suzuki group $Sz(2^r)$ where r is an odd prime.

Now we use the subgroups of $PSL(2, p^f)$ listed in Hauptsatz 8.27, pp.213-214 of [2]. For easy reference we give this list below and refer to it as Dickson's list. Dickson's list of subgroups of $PSL(2, p^f)$:

(i) elementary abelian p-groups,

(ii) cyclic groups of order z with $z | (p^f \pm 1)/k$, where $k = (p^f - 1, 2)$,

(iii) dihedral groups of order 2z where z is as in (ii),

(iv) alternating group A_4 for $p \neq 2$ or $p = 2$ and $f \equiv 0 \pmod 2$,

(v) symmetric group S_4 for $p^{2f} - 1 \equiv 0 \pmod{16}$,

(vi) alternating group A_5 for $p = 5$ or $p^{2f} - 1 \equiv 0 \pmod 5$,

(vii) semidirect product of elementary abelian group of order p^m with cyclic group of order t with $t \mid (p^m - 1)$ and $t \mid (p^f - 1)$,

(viii) groups $PSL(2, p^m)$ for $m \mid f$.

In Dickson's list, the subgroups in (i), (ii), (iv) and (vii) are q'-nilpotent for some prime q. Using the possible choices for G listed above, Dickson's list (viii) can not be a subgroup of G. S_4 is not q'-nilpotent for any prime q. Hence using Dickson's list (v) we have $p^{2f} - 1 \not\equiv 0 \pmod{16}$. Also, A_5 being a simple group can not be a proper subgroup of G. Thus, from Dickson's list (vi) we have $p^{2f} - 1 \not\equiv 0 \pmod 5$. Using Lemma 4.1, $z = 2^a v^b$ where v is a prime. Using these observations and Lemma 4.1 it is a matter of routine verification that the Thompson's list of groups (i) - (iii) given earlier would be a choice for G.

(i) PSL(3 , 3).

Considering K = PSL(3 , 3) as a doubly transitive group on 13 letters, the stabilizer of a point will be a maximal subgroup M with $|M| = 3^3 \cdot 2^4$. $M \cong GL(2,3) \cdot (Z_3 \times Z_3)$ shows that M is not p'-nilpotent for any prime p. So PSL(3 , 3) can not be a choice for G.

(ii) $Sz(2^q)$, p an odd prime.

Using the notation and results used in Suzuki [5], we will now verify that $Sz(2^q)$ has a subgroup, namely $N_L(A_1)$, which is not s'-nilpotent for any prime s, and thus $Sz(2^q)$ can not be a choice for G.

CASE 1 : s = 2.

Using Proposition 15, p.121 of [5], $N_L(A_1)/A_1$ is cyclic. If $N_L(A_1)$ is 2'-nilpotent, since $|N_L(A_1)/A_1| = 4$ and $|A_1|$ is an odd number, we will have $N_L(A_1)$ to be nilpotent. Hence every element of odd order commutes with every 2-element. This is contrary to Lemma 11, p.135 of [5]. Hence $N_L(A_1)$ can not be 2'-nilpotent.

CASE 2 : $s \neq 2$.

In this case $N_L(A_1)$ has an abelian subgroup which is a complement of a Sylow s-subgroup of $N_L(A_1)$. Again, using Lemma 11, p.135 of [5], such a subgroup does not exist. Thus $N_L(A_1)$ is not s'-nilpotent for any prime s. Thus $Sz(2^q)$ can not be a choice for G.

Conversely, suppose that G is one of the groups listed in the statement. Clearly all the groups are simple. First consider G = PSL(2 , p) as in (a). From the list of subgroups of PSL(2 , p) given in Dickson's list, the subgroups in (i), (ii), (iv) and (vii) are q'-nilpotent for some prime q. (v) and (vi) can not be subgroups of G because $p^2 - 1 \not\equiv 0 \pmod 5$ and $p^2 - 1 \not\equiv 0 \pmod{16}$.

Suppose G has a subgroup H as in (iii). $|G| = p(p^2-1)/2$. $|H| = 2z$ with $z \mid (p \pm 1)/2$. Suppose $z \mid (p - 1)/2$. $(p - 1)/2 = 2^2r^i/2 = 2r^i$. $z \mid 2r^i$. $|H| = 2z$. Hence H has a cyclic normal subgroup of order z, say K. If $|K| = r^l$ where $l \leq i$, then $|H| = 2r^l$ and hence H is r'-nilpotent. If $|K| = 2r^l$, then K_r char K H implies K_r H. Also, $K_r = H_r$ since $|H| = 2^2r^l$. Thus H is r'-nilpotent in this case as well.

Suppose $z \mid (p + 1)/2$. If $p + 1 = 2^2s^j$, then as in the above argument we get H to be q'-nilpotent for some prime q, so assume that $p + 1 = 2s^j$. $z \mid (p + 1)/2 = 2s^j/2 = s^j$. Thus $z = s^{l_1}$ where $l_1 \leq j$. Clearly H is s'-nilpotent in this case as noted in the previous argument. Thus all proper subgroups of G are q'-nilpotent for some prime q when G is as in (a).

Next consider $G = PSL(2\ ,\ 2^n)$ as in (b). In this case $z \mid (2^n \pm 1)$ and $2^n - 1 = r^i$, $2^n + 1 = s^j$ where r,s are odd primes. Thus if H is a subgroup of G of order 2z, then clearly H is q'-nilpotent for some prime q. Thus all proper subgroups of $G = PSL(2\ ,\ 2^n)$ as in (b), are q'-nilpotent for some prime q. Finally consider $G = PSL(2\ ,\ 3^n)$ as in (c). In this case $z \mid (3^n \pm 1)/2$. The argument given earlier for the case $G = PSL(2\ ,\ p)$ applies here as well. Thus we complete the proof of the main theorem. Q.E.D.

ACKNOWLEDGMENT: The author wishes to thank Professor W.E. Deskins and Professor E. Arrington-Idowu for their comments and suggestions when these results were proved.

REFERENCES

1. S. SRINIVASAN, "Finite p'-nilpotent groups. I", Journal of International Mathematics and Mathematical Sciences, 10 (1987) 135-146.

2. B. HUPPERT, "Endlishe Gruppen. I", Springer Verlag, New York, 1967.

3. M. HALL, "Group Theory", Chelsea, New York, 1976.

4. R. BAER, "Classes of finite groups and their properties", Illinois Jl. of Math., 1(1957), 115 - 187.

5. M. SUZUKI, "On a class of doubly transitive groups", Annals of Math., 75(1962), 105 - 145.

6. J.G. THOMPSON, "Nonsolvable finite groups all of whose local subgroups are solvable", Bull. Amer. Math. Soc., 74(1968), 383 - 437.