ABELIAN THEOREMS FOR WHITTAKER TRANSFORMS

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ABSTRACT. Initial and final value Abelian theorems for the Whittaker transform of functions and of distributions are obtained. The Abelian theorems are obtained as the complex variable of the transform approaches 0 or ∞ in absolute value inside a wedge region in the right half plane.

1.   INTRODUCTION.

Abelian theorems for the Whittaker transform of functions and of distributions are obtained as the complex variable of the transform approaches 0 or ∞ in absolute value inside a wedge region in the right half plane. The setting for these Abelian theorems is motivated by the initial and final value results of Doetsch [1, sections 33 and 34] for the Laplace transform of which the Whittaker transform is a generalization. Our results here generalize Abelian theorems previously obtained by Zemanian [2, sections 8.6 and 8.7], Akhaury [3-4], Moharir and Saxena [5], and Tiwari and Ko [6] in the generality of the transform variable being in a wedge and in the generality of the parameters.

2.   THE WHITTAKER FUNCTION.

Let $k = k_1 + ik_2$ and $m = m_1 + im_2$ be complex parameters which satisfy $(m_1 - k_1 + (1/2)) > 0$. Let s be a complex variable with $s \in \mathcal{C}_> = \{s = s_1 + is_2 \in \mathcal{C}: s_1 > 0\}$. For $s \in \mathcal{C}_>$ the Whittaker function $W_{k,m}(s)$ (Erdélyi et al [7, (2), p. 264]) is given by

$$W_{k,m}(s) = \frac{e^{-s/2} \, s^{m+(1/2)}}{\Gamma(m-k+(1/2))} \int_0^\infty e^{-su} \, u^{m-k-(1/2)} \, (1+u)^{m+k-(1/2)} \, du \; . \tag{2.1}$$

Let p be a positive real parameter. For $K \geq 0$ being a fixed real number put $P_K = \{s = s_1 + is_2 \in \mathcal{C}: s_1 > 0 \text{ and } |s_2| \leq Ks_1\}$; P_K is a wedge in the right half plane $\mathcal{C}_>$. Using (2.1) with the gamma function taken to the left of the equality and estimate

analysis on the exponentials and powers as in Carmichael and Hayashi [8, Lemma 2.2, p. 70] and Carmichael and Hayashi [8, proofs of Theorems 3.1 and 3.2], we have the following important estimate for this paper:

$$|W_{k,m}(pst)| \leq (1+K^2)^{(2m_1+1)/4} \exp(\pi|m_2|/2) \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|} W_{k_1,m_1}(ps_1 t) ,$$

$$s \in P_K , \ t > 0. \tag{2.2}$$

3.   THE WHITTAKER TRANSFORM FOR FUNCTIONS.

Let k,m, and r be complex parameters, and let p and q be real parameters. Let s be a complex variable. The function

$$F_{p,q,r}^{k,m}(s) = \int_0^\infty (st)^{r-(1/2)} \exp(-qst/2) W_{k,m}(pst) f(t) \, dt \tag{3.1}$$

is the Whittaker transform of the function $f(t)$ where $W_{k,m}(pst)$ is the Whittaker function. The general Whittaker transform defined in (3.1) was first considered by Srivastava [9] for certain values of the parameters and variable which depend upon the order of growth of $f(t)$.

In this section we prove initial and final value Abelian theorems for the Whittaker transform defined in (3.1). For this purpose we now place conditions on the parameters and variable noted above; these restrictions will hold throughout the remainder of this section. The complex parameters k,m, and r satisfy $\mathrm{Re}(m-k+(1/2)) > 0$ and $\mathrm{Re}(r) \geq \mathrm{Re}(m) \geq 0$ with $\mathrm{Re}(r) \geq (1/2)$. The real parameters p and q are positive. The complex variable s is in $\mathcal{C}_>$, that is $\mathrm{Re}(s) > 0$.

We now state and prove an initial value Abelian theorem.

THEOREM 3.1. Let $\eta = \eta_1 + i\eta_2$ be complex with $\mathrm{Re}(\eta) = \eta_1 > (-1)$. Let $f(t)$, $0 \leq t < \infty$, be a complex valued function such that there is a real number $c > 0$ for which $(e^{-ct} f(t))$ is absolutely integrable over $0 \leq t < \infty$ and such that $(f(t)/t^\eta)$ is bounded on $0 < t \leq y$ for all $y > 0$. Let the Whittaker transform $F_{p,q,r}^{k,m}(s)$ of $f(t)$ exist for $s \in \mathcal{C}_>$. If there is a complex number α for which

$$\lim_{t\to 0+} \frac{f(t)}{t^\eta} = \alpha \tag{3.2}$$

then for each fixed $K \geq 0$

$$\lim_{\substack{|s|\to\infty \\ s \in P_K}} \frac{s^{\eta+1} F_{p,q,r}^{k,m}(s)}{\Lambda(\eta,k,m,p,q,r)} = \alpha \tag{3.3}$$

where

$$\Lambda(\eta,k,m,p,q,r) = \frac{p^{m+(1/2)} \Gamma(\eta+m+r+1) \ \Gamma(\eta+r-m+1)}{((p+q)/2)^{\eta+m+r+1} \ \Gamma(\eta+r-k+(3/2))}$$

$$\times {}_2F_1\left[\eta+m+r+1, m-k+(1/2); \eta+r-k+(3/2); \frac{q-p}{q+p}\right].$$

PROOF. Using Erdélyi et al [10, (16), p. 216] we have

$$\int_0^\infty t^\eta (st)^{r-(1/2)} \exp(-qst/2) W_{k,m}(pst) \, dt = s^{-\eta-1} \Lambda(\eta,k,m,p,q,r). \tag{3.4}$$

From (3.1) and (3.4) we obtain

$$\left| s^{\eta+1} F^{k,m}_{p,q,r}(s) - \alpha \Lambda(\eta,k,m,p,q,r) \right| =$$

$$= \left| s^{\eta+1} \int_0^\infty (st)^{r-(1/2)} \exp(-qst/2) W_{k,m}(pst) (f(t)-\alpha t^\eta) dt \right| \tag{3.5}$$

$$\leq I_1 + I_2$$

where

$$I_1 = \left| s^{\eta+1} \int_0^y (st)^{r-(1/2)} \exp(-qst/2) W_{k,m}(pst) (f(t)-\alpha t^\eta) dt \right|$$

$$\tag{3.6}$$

$$I_2 = \left| s^{\eta+1} \int_y^\infty (st)^{r-(1/2)} \exp(-qst/2) W_{k,m}(pst) (f(t)-\alpha t^\eta) dt \right|$$

for $y > 0$ arbitrary for the moment. For $s \in P_K$ we use the boundedness hypothesis on $(f(t)/t^\eta)$, (2.2), and estimates as in obtaining (2.2) to obtain

$$I_1 = \left| s^{\eta+1} \int_0^y t^\eta (st)^{r-(1/2)} \exp(-qst/2) W_{k,m}(pst) ((f(t)/t^\eta)-\alpha) dt \right|$$

$$\leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2) \tag{3.7}$$

$$\left(\sup_{0 < t \leq y} |(f(t)/t^\eta)-\alpha| \right) s_1^{\eta_1+1} \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|}$$

$$\int_0^y t^{\eta_1} (s_1 t)^{r_1-(1/2)} \exp(-qs_1 t/2) W_{k_1,m_1}(ps_1 t) dt.$$

The integrand in the integral of this last estimate is positive valued for $t \geq 0$ for the variable $s_1 > 0$ and the parameters η_1, p, q, k_1, m_1, and r_1. We thus replace the integral in the last estimate in (3.7) by an integral over $0 < t < \infty$ and use (3.4) for the variable $s_1 > 0$ and the parameters η_1, p, q, k_1, m_1, and r_1; by so doing the estimate (3.7) can be continued as

$$I_1 \leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2)$$

$$\tag{3.8}$$

$$\Lambda(\eta_1,k_1,m_1,p,q,r_1) \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|} \sup_{0<t\leq y} |(f(t)/t^\eta)-\alpha|$$

and this bound is independent of $s \in P_K$ and holds for each fixed $y > 0$. We now consider I_2 given in (3.6). By hypothesis there is a real number $c > 0$ for which $(e^{-ct} f(t))$ is absolutely integrable over $0 \leq t < \infty$. Putting e^{-ct} into I_2 and using (2.2) and estimates for $s \in P_K$ we have

$$I_2 = |\ s^{\eta+1} \int_y^\infty (st)^{r-(1/2)} \exp(-(qs-2c)t/2)\ W_{k,m}(pst)(e^{-ct}(f(t)-\alpha t^\eta))\ dt|$$

$$\leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2)\ s_1^{\eta_1+1} \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|}$$

(3.9)

$$\int_y^\infty (s_1 t)^{r_1-(1/2)} \exp(-(qs_1-2c)t/2)\ W_{k_1,m_1}(ps_1 t)\ |\ e^{-ct}(f(t)-\alpha t^\eta)\ |\ dt.$$

In this theorem we desire to prove (3.3) as $|s| \to \infty$, $s \in P_K$. As $|s| \to \infty$, $s \in P_K$, then necessarily $s_1 = \mathrm{Re}(s) \to \infty$. Thus we now assume without loss of generality that $s_1 = \mathrm{Re}(s) > (2c/q)$ in the remainder of this proof for the fixed $c > 0$ and $q > 0$; and for such $s_1 = \mathrm{Re}(s)$ we know that

$$\exp(-(qs_1-2c)t/2) \leq 1,\ t \geq 0. \tag{3.10}$$

As $t \to \infty$ then $(ps_1 t) \to \infty$ here since $p > 0$ and $s_1 > 0$. From properties of $W_{k_1,m_1}(ps_1 t)$ (Whittaker and Watson [11, Chapter 16]) including the growth at ∞ (see also Tiwari and Ko [6, line 2+, p. 351]) we have that

$$\left[(ps_1 t)^{r_1-(1/2)}\ W_{k_1,m_1}(ps_1 t) \right] \text{ is positive, finite valued, and continuous as a}$$

function of t on $y \leq t < \infty$ and

$$\lim_{t\to+\infty} (ps_1 t)^{r_1-(1/2)}\ W_{k_1,m_1}(ps_1 t) = 0;$$

thus $\left[(ps_1 t)^{r_1-(1/2)}\ W_{k_1,m_1}(ps_1 t) \right]$ attains its maximum at some point $t = t_{s_1}$ depending on s_1 such that $y \leq t_{s_1} < \infty$. Using this fact and (3.10) we continue (3.9) as

$$I_2 \leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2)$$

$$\frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|}\ s_1^{\eta_1+1}\ p^{-r_1+(1/2)}\ (ps_1 t_{s_1})^{r_1-(1/2)}\ W_{k_1,m_1}(ps_1 t_{s_1})$$

$$\int_0^\infty |\ e^{-ct}(f(t)-\alpha t^\eta)\ |\ dt$$

(3.11)

$$\leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2)$$

$$\frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|} \; p^{-r_1+(1/2)} \quad (py)^{-\eta_1-1} \quad (ps_1t_{s_1})^{\eta_1+r_1+(1/2)}$$

$$W_{k_1,m_1}(ps_1t_{s_1}) \; \int_0^\infty |e^{-ct}(f(t) - \alpha t^\eta)| \; dt$$

where the last inequality is obtained from the fact that $y \le t_{s_1} < \infty$ and $\eta_1+1 > 0$. For

$\eta_1 = Re(\eta) > (-1)$ we have $(t^\eta e^{-ct})$ is absolutely integrable over $0 < t < \infty$; this com-

bined with the assumption in this theorem that $(e^{-ct}f(t))$ is absolutely integrable over

$0 < t < \infty$ yield that the integral in the estimate (3.11) is finite. From the hypothesis

(3.2) and the estimate (3.8), which is independent of $s \in P_K$, given $\boldsymbol{\epsilon}/2$ we can choose

$y > 0$ small enough so that $I_1 < \boldsymbol{\epsilon}/2$. We now fix this $y > 0$ and it depends only on

$\boldsymbol{\epsilon} > 0$. As noted above, as $|s| \to +\infty$, $s \in P_K$, then necessarily $s_1 = Re(s) \to +\infty$, and as

this happens $(ps_1t_{s_1}) \to +\infty$ since $p > 0$ is fixed and $y \le t_{s_1} < \infty$. Using the growth

property at ∞ of the Whittaker function W_{k_1,m_1}, (3.11) shows that I_2 can be made

smaller than $\boldsymbol{\epsilon}/2$ if $|s|$ is chosen large enough, $s \in P_K$. Combining these facts with

(3.5) yields the desired conclusion (3.3). The proof is complete.

As an example for which Theorem 3.1 is applicable, let $f(t) = t/(1+t^2)$, $\eta=1$, $c=1$,

and $\alpha=1$. Another example is obtained if we take $f(t) = t^2/(1+t^2)$, $\eta = 1+i$, $c=1$, and

$\alpha=0$.

We now obtain a final value Abelian theorem for the Whittaker transform of

functions.

THEOREM 3.2. Let $\eta = \eta_1 + i\eta_2$ with $\eta_1 > (-1)$. Let $f(t)$, $0 \le t < \infty$, be a complex

valued function such that there is a real number $c > 0$ for which $(e^{-ct} f(t))$ is abso-

lutely integrable over $0 \le t < \infty$ and such that $(f(t)/t^\eta)$ is bounded on $y \le t < \infty$ for all

$y > 0$. Let the Whittaker transform $F_{p,q,r}^{k,m}(s)$ of $f(t)$ exist for $s \in \mathcal{C}_>$. If there is a

complex number α for which

$$\lim_{t\to+\infty} \frac{f(t)}{t^\eta} = \alpha \tag{3.12}$$

then for each fixed $K \ge 0$

$$\lim_{\substack{|s|\to 0 \\ s \in P_K}} \frac{s^{\eta+1} F_{p,q,r}^{k,m}(s)}{\Lambda(\eta,k,m,p,q,r)} = \alpha. \tag{3.13}$$

PROOF. Using (3.4) and arguing as in (3.5) we have

$$|s^{\eta+1} F_{p,q,r}^{k,m}(s) - \alpha \Lambda(\eta,k,m,p,q,r)| \le I_1 + I_2 \tag{3.14}$$

where I_1 and I_2 are defined in (3.6) for $y > 0$ arbitrary. Let $K \ge 0$ be arbitrary but

fixed and $s \in P_K$. Using the boundedness hypothesis on $(f(t)/t^\eta)$, (2.2), and analysis

as in (3.7) and (3.8) we have

$$I_2 \leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2)$$

$$\Lambda(\eta_1,k_1,m_1,p,q,r_1) \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|} \left[\sup_{t \geq y} | (f(t)/t^\eta) - \alpha| \right] \tag{3.15}$$

and the right side of (3.15) is independent of $s \in P_K$. We now consider I_1 in (3.6). For the real number $c > 0$ in the hypothesis we argue as in (3.9) and obtain

$$I_1 \leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2)$$

$$p^{-r_1+(1/2)} s_1^{\eta_1+1} \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|} \tag{3.16}$$

$$\int_0^y (ps_1 t)^{r_1-(1/2)} \exp(-(qs_1-2c)t/2) W_{k_1,m_1}(ps_1 t) |e^{-ct}(f(t)-\alpha t^\eta)| dt$$

where we have put $p^{r_1-(1/2)}$ into the right side and hence have also put $p^{-r_1+(1/2)}$ there. The desired conclusion (3.13) in this theorem is to be obtained as $|s| \to 0$, $s \in P_K$; as $|s| \to 0$, $s \in P_K$, then necessarily $s_1 = \text{Re}(s) \to 0 +$; we thus may assume without loss of generality here that $0 < s_1 < (1/p)$ for the fixed parameter $p > 0$. As $t \to 0+$, $0 < ps_1 t < t \to 0+$ for all $s_1 < (1/p)$; by the growth condition at 0 in Tiwari and Ko [6, line 4↓, p. 351] or Whittaker and Watson [11, Chapter 16] we can choose constants M and T such that

$$| W_{k_1,m_1}(ps_1 t) | = W_{k_1,m_1}(ps_1 t) \leq M (ps_1 t)^{(1/2)-m_1},$$

$$0 < ps_1 t < t < T < y, \quad s_1 < (1/p); \tag{3.17}$$

here M is independent of $s_1 < (1/p)$ and of $t < T$ and of $ps_1 t < t < T$ and T is independent of $s_1 < (1/p)$. Returning to (3.16) and using (3.17) and the fact that $\exp(-qs_1 t/2) \leq 1$, $t > 0$, we have

$$I_1 \leq (1+K^2)^{(\eta_1+m_1+r_1+1)/2} \exp(\pi(|\eta_2|+|m_2|+|r_2|)/2) p^{-r_1+(1/2)} e^{cy}$$

$$s_1^{\eta_1+1} \frac{\Gamma(m_1-k_1+(1/2))}{|\Gamma(m-k+(1/2))|} \left[M (ps_1 y)^{r_1-m_1} \int_0^\infty |e^{-ct} (f(t) - \alpha t^\eta) | dt \right. \tag{3.18}$$

$$\left. + \int_T^y (ps_1 t)^{r_1-(1/2)} W_{k_1,m_1}(ps_1 t) | e^{-ct} (f(t) - \alpha t^\eta) | dt \right].$$

In (3.18), as in (3.11), we have

$$\int_0^\infty \mid e^{-ct} (f(t) - \alpha t^\eta) \mid dt < \infty;$$

$(e^{-ct} f(t))$ is absolutely integrable over $0 \le t < \infty$ by assumption and $(e^{-ct} t^\eta)$ is absolutely integrable over $0 \le t < \infty$ since $c > 0$ and $Re(\eta) > (-1)$. Now

$$\left[(ps_1 t)^{r_1 - (1/2)} W_{k_1, m_1} (ps_1 t) \right] \text{ is a continuous function of t on the closed bounded}$$

interval $T \le t \le y$ for each fixed s_1, $0 < s_1 < (1/p)$; hence this product attains its maximum on $T \le t \le y$ at a point t_{s_1} depending on $s_1 < (1/p)$. Continuing (3.18) we then have

$$I_1 \le (1+K^2)^{(\eta_1 + m_1 + r_1 + 1)/2} \exp(\pi(|\eta_2| + |m_2| + |r_2|)/2) \; p^{-r_1 + (1/2)}$$

$$(3.19)$$

$$e^{cy} s_1^{\eta_1 + 1} \frac{\Gamma(m_1 - k_1 + (1/2))}{|\Gamma(m - k + (1/2))|} \left[M (ps_1 y)^{r_1 - m_1} + (ps_1 t_{s_1})^{r_1 - (1/2)} W_{k_1, m_1} (ps_1 t_{s_1}) \right]$$

$$\int_0^\infty |e^{-ct}(f(t) - \alpha t^\eta) | \; dt.$$

The estimate (3.19) holds for all s_1, $0 < s_1 < (1/p)$. The constants c, K, and M are independent of s_1, $0 < s_1 < (1/p)$, as are all of the parameters p, $k = k_1 + ik_2$, $m = m_1 + im_2$, $r = r_1 + ir_2$ and $\eta = \eta_1 + i\eta_2$ and the yet to be chosen constant $y > 0$. Again as $|s| \to 0$, $s \in P_K$, then $s_1 = Re(s) \to 0+$. Recall that $T \le t_{s_1} \le y$ for $0 < s_1 < (1/p)$ in (3.19) and T is independent of s_1, $0 < s_1 < (1/p)$, but dependent on $y > 0$. (Recall (3.17).) As $s_1 \to 0+$ then $0 < ps_1 t_{s_1} \le ps_1 y \to 0+$. Thus as

$|s| \to 0$, $s \in P_K$, the growth condition Tiwari and Ko [6, line 4↓, p. 351] and Whittaker and Watson [11, Chapter 16] applied to $W_{k_1, m_1} (ps_1 t_{s_1})$ yields constants M' and T' which are independent of s_1 such that

$$\mid W_{k_1, m_1} (ps_1 t_{s_1}) \mid = W_{k_1, m_1} (ps_1 t_{s_1}) \le M' (ps_1 t_{s_1})^{(1/2) - m_1},$$

$$(3.20)$$

$$0 < ps_1 t_{s_1} \le ps_1 y < T',$$

for $0 < s_1 < (1/p)$. Using (3.20) in (3.19) and combining the resulting estimate on I_1 with the estimate (3.15) on I_2, the proof can be completed similarly as in the proof of Theorem 3.1.

Examples of the applicability of Theorem 3.2 similar to those after the proof of Theorem 3.1 for Theorem 3.1 can be constructed by the reader.

4.   THE WHITTAKER TRANSFORM FOR DISTRIBUTIONS.

We construct a Whittaker transform for distributions as in Pathak [12]. We define the differential operators D_α, $\alpha = 0,1,2, \ldots$, as in Pathak [12, (3), p. 4]; the seminorms γ_α, $\alpha = 0,1,2, \ldots$, and the test functions $V_a(0,\infty)$ are given in Pathak [12, p. 6].

LEMMA 4.1.   Let $s \neq 0$ and $Re(s) > (a/p)$. Let $Re(m) \geq 0$. We have $((st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst)) \in V_a(0,\infty)$ as a function of $t \in (0,\infty)$.

PROOF.   The proof is by Pathak [12, Lemma 1, p. 6]. Nothing different is introduced by having $p > 0$ in the exponential term and in $W_{k,m}(pst)$ here.

Let $V_a'(0,\infty)$ denote the space of generalized functions defined on $V_a(0,\infty)$ (Pathak [12, p. 6]). Let $U \in V_a'(0,\infty)$. Let σ_U have the meaning given in Pathak [12, p. 7]; that is $U \in V_a'(0,\infty)$ for each $a > \sigma_U$ and $U \notin V_a'(0,\infty)$ for any $a < \sigma_U$. Because of Lemma 4.1 we can form

$$WT_p^{k,m}[U;s] = <U_t, (st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst)>, \quad s \in \Omega_U, \qquad (4.1)$$

where $\Omega_U = \{s : s \neq 0, Re(s) > \sigma_U, -\pi < Arg(s) < \pi\}$ and $Re(m) \geq 0$. We call $WT_p^{k,m}[U;s]$ the distributional Whittaker transform of $U \in V_a'(0,\infty)$; by Pathak [12, Theorem 1, p. 8] $WT_p^{k,m}[U;s]$ is an analytic function of s in Ω_U.

The set of seminorms $\{\gamma_\alpha\}$, $\alpha = 0,1,2, \ldots$, which defines and generates the topology of $V_a(0,\infty)$ is a multinorm in the sense of Zemanian [13, p. 8] as noted in Pathak [12, p. 6]; here γ_0 is a norm. Thus the hypotheses of Zemanian [13, Theorem 1.8-1, p. 18] are satisfied; by this result, given $U \in V_a'(0,\infty)$ there exist a positive constant C and a nonnegative integer N, which depend only on U, such that

$$| <U, \phi> | \leq C \max_{\alpha=0,1, \ldots, N} \gamma_\alpha(\phi), \quad \phi \in V_a(0,\infty). \qquad (4.2)$$

We shall call the number N here the order of the generalized function $U \in V_a'(0,\infty)$.

We obtain Abelian theorems for the distributional Whittaker transform; in so doing we use the results of section 3. Thus in the remainder of this section we assume that the complex parameters k and m satisfy $Re(m-k + (1/2)) > 0$. By comparing the forms (3.1) and (4.1) we see that the complex parameter r and the real parameter q of (3.1) and section 3 are now taken to be $r = m$ and $q = p$ in this section. We thus must make the restriction $Re(m) \geq (1/2)$ in the remainder of this section because $Re(r)$ was necessarily so restricted in section 3.

To prove our initial value theorem for $WT_p^{k,m}[U;s]$ we need the following lemma.

LEMMA 4.2.   Let $U \in V_a'(0,\infty)$ for $a > \sigma_U$ and let the support of U be in $T \leq t < \infty$, $T > 0$. Let $s \in P_K$ with $(p \, Re(s)) > \max \{1, p \, \sigma_U, 4a\}$. There are constants $B > 0$ and $y_0 > 0$ which are independent of s such that

$$| WT_p^{k,m}[U;s] | \leq B (1 + s_1)^{m_1+k_1+N-(1/2)} \exp(-ps_1 y_0/4), \quad s_1 = Re(s), \qquad (4.3)$$

where N is the order of U.

PROOF. The proof of this lemma is in the spirit of that of Zemanian [2, Lemma 1, p. 246]. Choose a function $\lambda(t) \in C^\infty$ such that for any nonnegative integer α we have

$$\left| \frac{d^\alpha(\lambda(t))}{dt^\alpha} \right| \le M_\alpha, \quad -\infty < t < \infty,$$

where M_α is a constant which depends only on α; $0 \le \lambda(t) \le 1$; $\lambda(t) = 1$ for $T \le t < \infty$; and the support of $\lambda(t)$, denoted $supp(\lambda)$, is contained in $y_0 \le t < \infty$, $0 < y_0 < T$. Then by the usual proof we have $<U,\phi> = <U, \lambda\phi>$, $\phi \in V_a(0,\infty)$, since $supp(U) \subseteq [T,\infty)$; and the value of $<U,\phi> = <U, \lambda\phi>$ is independent of the choice of $\lambda(t)$ satisfying the above properties. Let $Re(s) > \sigma_U$. With N being the order of $U \in V'_a(0,\infty)$ we apply (4.2) and the definition Pathak [12, (12), p. 6] of the seminorms to obtain

$$\left| WT^{k,m}_p[U;s] \right| = \left| <U_t, \lambda(t) (st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst) > \right| \le$$
(4.4)

$$\le C \max_{\alpha = 0,1,\dots,N} \sup_{0 < t < \infty} \left| e^{at} D_\alpha(\lambda(t) (st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst)) \right|$$

for $Re(s) > \sigma_U$. From Pathak [12, (8), p. 5] we have

$$D_\alpha(f(t)) = \sum_{\beta=0}^\alpha A_\beta t^\beta f^{(\beta)}(t), \quad \alpha = 0,1,2,\dots,$$
(4.5)

where the A_β are complex constants. Using (4.5), the Leibnitz rule, the boundedness of the derivatives of $\lambda(t)$ ($|d^{\beta-\delta}(\lambda(t))/dt^{\beta-\delta}| \le M_{\beta,\delta}$ in (4.6) below), and Slater [14, (2.4.17), p. 25] we continue (4.4) as

$$\left| WT^{k,m}_p[U;s] \right| \le$$

$$\le C \max_{\alpha=0,1,\dots,N} \sup_{0 < t < \infty} \left| e^{at} \sum_{\beta=0}^\alpha A_\beta t^\beta \right.$$

$$\left. \frac{d^\beta}{dt^\beta} \left[\lambda(t) (st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst) \right] \right|$$

$$= C \max_{\alpha=0,1,\dots,N} \sup_{0 < t < \infty} \left| e^{at} \sum_{\beta=0}^\alpha A_\beta t^\beta \right.$$

$$\sum_{\delta=0}^\beta \frac{\beta!}{\delta!(\beta-\delta)!} \frac{d^{\beta-\delta}(\lambda(t))}{dt^{\beta-\delta}} \frac{d^\delta}{dt^\delta} \left[(st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst) \right] \left. \right|$$

$$\le C \max_{\alpha=0,1,\dots,N} \sup_{0 < t < \infty} e^{at} \sum_{\beta=0}^\alpha |A_\beta| \sum_{\delta=0}^\beta \frac{\beta!}{\delta!(\beta-\delta)!} M_{\beta,\delta}$$
(4.6)

$$\left| t^\beta p^{-m+(1/2)} (ps)^\delta (-1)^\delta (pst)^{m-(1/2)-(\delta/2)} \exp(-pst/2) \right.$$

$W_{k+(\delta/2),m-(\delta/2)}(pst)\ |$

$$= C \max_{\alpha=0,\ 1,\ \ldots\ ,N}\ \sup_{0\ <\ t\ <\ \infty}\ \sum_{\beta=0}^{\alpha}\ |\ A_{\beta}\ |\ \sum_{\delta=0}^{\beta}\ \frac{\beta!}{\delta!(\beta-\delta)!}\ M_{\beta,\delta}$$

$$|\ p^{-m+(1/2)+\delta-\beta}\ |\ \ |s|^{\delta-\beta}\ |e^{at}\ (pst)^{m-(1/2)-(\delta/2)+\beta}\ \exp(-pst/2)$$

$$W_{k+(\delta/2),m-(\delta/2)}(pst)\ |\ .$$

Now recall m_1 = Re(m) $\geq$ (1/2), (p Re(s)) > max {1, $p\sigma_U$, 4a}, and s ϵ P_K in the hypothesis of this lemma; under these restrictions we use the growth properties of the Whittaker function (Whittaker and Watson [11, Chapter 16] and Tiwari and Ko [6, p. 351]) and analysis as in the proof of Pathak [12, Lemma 2, pp. 7-8] to obtain a constant C_{β} such that

$$|e^{at}\ (pst)^{m-(1/2)-(\delta/2)+\beta}\ \exp(-pst/2)\ W_{k+(\delta/2),m-(\delta/2)}(pst)|\ \leq$$

$$\leq\ C_{\beta}\ (1+s_1)^{m_1+k_1+N-(1/2)}\ \exp(-ps_1 t/4)$$

(4.7)

holds for t > 0 and all relevant values of the parameters. Now in (4.6) we notice that

$|p^{-m+(1/2)+\delta-\beta}| = p^{-m_1+(1/2)+\delta-\beta}$. Since (p Re(s)) > max {1, $p\sigma_U$,4a} and s ϵ P_K, the term $|s|^{\delta-\beta}$, δ = 0,1, ... , β, β = 0, 1, ... ,α, in (4.6) is bounded by a constant which depends on δ and β. Now recall that supp(λ) $\subseteq$ [y_0,∞) with 0 < y_0 < T so that the sup in (4.4) and (4.6) is actually taken over $y_0 \leq$ t < ∞. Hence for the values of t actually being considered in (4.4) and (4.6), the estimate in (4.7) holds for $y_0 \leq$ t < ∞ with the term $\exp(-ps_1 t/4)$ replaced by $\exp(-ps_1 y_0/4)$. Combining these facts with (4.7) and (4.6) we obtain a constant B which depends on the generalized function U and on its order N such that the desired conclusion (4.3) holds; the constants B and y_0 are independent of s. The proof is complete.

We now prove an initial value Abelian theorem for the distributional Whittaker transform where the element U ϵ $V_a'(0,\infty)$ is assumed to have support in [0,∞).

THEOREM 4.1. Let U ϵ V_a' (0,∞) for a > σ_U such that over some right neighborhood (0, t_0) of zero U is a regular distribution corresponding to a complex valued function f(t) such that there is a real number c > 0 for which (e^{-ct} f(t)) is absolutely integrable over 0 $\leq$ t $\leq$ t_0. For η = $\eta_1+i\eta_2$, η_1 > -1, let (f(t)/t^{η}) be bounded on 0 < t $\leq$ y' for all y' $\leq$ t_0. Let the Whittaker transform $F_{p,p,m}^{k,m}$(s), s ϵ $\mathcal{C}_>$, exist for the function which is f(t) on 0 < t < y' and which is zero on y' $\leq$ t < ∞ for all y' $\leq$ t_0. If there is a complex number α for which

$$\lim_{t\to 0+}\ \frac{f(t)}{t^{\eta}} = \alpha$$

(4.8)

then for each fixed $K \geq 0$

$$\lim_{\substack{|s| \to \infty \\ s \in P_K}} \frac{s^{\eta+1} \, WT_p^{k,m}[U;s]}{\Lambda(\eta,k,m,p,p,m)} = \alpha. \tag{4.9}$$

PROOF. We decompose U into $U_1 + U_2$ where $\mathrm{supp}(U_1) \subseteq [0,T]$ and $\mathrm{supp}(U_2) \subseteq [T,\infty)$, $0 < T < t_0$. We now have

$$WT_p^{k,m}[U;s] = WT_p^{k,m}[U_1;s] + WT_p^{k,m}[U_2;s]. \tag{4.10}$$

Lemma 4.2 is applicable to $WT_p^{k,m}[U_2;s]$ since $\mathrm{supp}(U_2) \subseteq [T,\infty)$, $T > 0$. Recall that as $|s| \to \infty$, $s \in P_K$, then necessarily $s_1 = \mathrm{Re}(s) \to \infty$. Thus by the conclusion (4.3) of Lemma 4.2 and the estimate

$$|s^{\eta+1}| \leq (1+K^2)^{(\eta_1+1)/2} \, (s_1)^{\eta_1+1} \exp(\pi|\eta_2|/2) \, , \, s \in P_K,$$

we have

$$\lim_{\substack{|s| \to \infty \\ s \in P_K}} \frac{s^{\eta+1} \, WT_p^{k,m}[U_2;s]}{\Lambda(\eta,k,m,p,p,m)} = 0. \tag{4.11}$$

Extend the function $f(t)$ in the hypothesis, which is known on $0 < t < t_0$, to $0 < t < \infty$ by

$$\bar{f}(t) = \begin{cases} f(t), & 0 < t < T , \\ 0, & T \leq t < \infty. \end{cases}$$

Then $\bar{f}(t)$ satisfies the hypotheses of Theorem 3.1 and $WT_p^{k,m}[U_1;s] = WT_p^{k,m}[\bar{f}(t);s]$ with this latter Whittaker transform equaling the function Whittaker transform of $\bar{f}(t)$ defined in (3.1). Thus by Theorem 3.1

$$\lim_{\substack{|s| \to \infty \\ s \in P_K}} \frac{s^{\eta+1} \, WT_p^{k,m}[U_1;s]}{\Lambda(\eta,k,m,p,p,m)} = \alpha. \tag{4.12}$$

Combining (4.10), (4.11), and (4.12) we have the desired result (4.9). The proof is complete.

COROLLARY 4.1. Let $U \in V'_a(0,\infty)$ for $a > \sigma_U$ such that over some right neighborhood $(0,t_0)$ of zero U is a regular distribution corresponding to a function $f(t)$ which is Lebesgue integrable over $0 \leq t < \infty$. Let $\eta = \eta_1 + i\eta_2$, $\eta_1 > (-1)$. Let $(f(t)/t^\eta)$ be bounded on $0 < t \leq y$ for all $y > 0$ and let (4.8) be satisfied for some complex number α. Then for each fixed $K \geq 0$ (4.9) holds.

PROOF. Since $f(t)$ is Lebesgue integrable over $0 \leq t < \infty$ then $(e^{-ct} f(t))$ is absolutely integrable over $0 \leq t < \infty$ for any fixed $c > 0$. The result thus follows immediately by Theorem 4.1.

We now proceed to obtain a final value Abelian theorem for the distributional Whittaker transform. First we need to make some comments concerning this transform. of a distribution of the form assumed in the final value theorem below. Let $U \in V'_a(0,\infty)$, $a > \sigma_U$, with $\mathrm{supp}(U) \subseteq [t_0,\infty)$, $t_0 > 0$. Assume over some interval $y < t < \infty$ with $0 < t_0 < y$ that U is a regular distribution corresponding to a complex valued function $f(t)$. Then U can be decomposed as $U = U_1 + U_2$ with $\mathrm{supp}(U_1) \subseteq [t_0,T]$ and $\mathrm{supp}(U_2) \subseteq [T,\infty)$, $T > y$. We then have

$$WT_p^{k,m} [U;s] = WT_p^{k,m} [U_1;s] + WT_p^{k,m} [U_2;s]. \tag{4.13}$$

From the definition (4.1) $WT_p^{k,m} [U;s]$ is defined for $s \in \Omega_U$ in which $\mathrm{Re}(s) > \sigma_U$. Because of the form of the $U \in V'_a (0,\infty)$ assumed above and the assumptions on $f(t)$ in the final value Theorem 4.2 below, where we use the above decomposition, we can take $\sigma_U = 0$ here, and (4.13) will be well defined in Theorem 4.2 for $s_1 = \mathrm{Re}(s) > 0$. Because of this we may let $|s| \to 0$, $s \in P_K$, in the final value theorem below as desired.

We now obtain a needed lemma for the final value result.

LEMMA 4.3. Let $U \in \mathcal{E}'$ with $\mathrm{supp}(U) \subseteq [t_0,T]$, $0 < t_0 < T < \infty$. Let $\eta = \eta_1 + i\eta_2$ with $\eta_1 > -1$. For each fixed $K \geq 0$

$$\lim_{\substack{|s| \to 0 \\ s \in P_K}} s^{\eta+1} WT_p^{k,m} [U;s] = 0. \tag{4.14}$$

PROOF. The distributional Whittaker transform $WT_p^{k,m} [U;s]$ of $U \in \mathcal{E}'$ exists for $\mathrm{Re}(s) > 0$. By Schwartz [15, Théorème, p. 91]

$$U = \sum_{\alpha=0}^{N} \frac{d^{\alpha}(g_{\alpha}(t))}{dt^{\alpha}}$$

for some nonnegative integer N, the order of U, where the $g_{\alpha}(t)$, $\alpha = 0,1, \ldots ,N$, are continuous functions with support in an arbitrary neighborhood $[t_0 - \epsilon, T + \epsilon]$, $\epsilon > 0$, of $[t_0, T]$. Since $\epsilon > 0$ is arbitrary we assume here that $t_0 - \epsilon > 0$. Using distributional differentation and the calculation Slater [14, (2.4.17), p. 25] we have

$$WT_p^{k,m} [U;s] =$$

$$= \sum_{\alpha=0}^{N} (-1)^{\alpha} \int_{t_0-\epsilon}^{T+\epsilon} g_{\alpha}(t) \frac{d^{\alpha}}{dt^{\alpha}} \left[(st)^{m-(1/2)} \exp(-pst/2) W_{k,m}(pst) \right] dt$$

$$(4.15)$$

$$= \sum_{\alpha=0}^{N} (-1)^{\alpha} \int_{t_0-\epsilon}^{T+\epsilon} g_{\alpha}(t) \, p^{-m+(1/2)} \, (ps)^{\alpha} \, (-1)^{\alpha}(pst)^{m-(1/2)-(\alpha/2)}$$

$$\exp(-pst/2) \, W_{k+(\alpha/2),m-(\alpha/2)}(pst) \, dt.$$

Now each $g_{\alpha}(t)$ is continuous on $t_0 - \epsilon \le t \le T + \epsilon$; hence

$$|g_{\alpha}(t)| \le M_{\alpha}, \ t_0 - \epsilon \ \le t \le T + \epsilon \ , \ \alpha = 0,1, \ \ldots \ ,N, \qquad (4.16)$$

for constants M_{α}, $\alpha = 0,1, \ \ldots \ ,N$. For each $\alpha = 0,1, \ \ldots \ ,N$ we have

$$|s^{\eta+1+\alpha}| \ \le \ (1+K^2)^{(\eta_1+1+\alpha)/2} \ \exp(\pi|\eta_2|/2) \, s_1^{\eta_1+1+\alpha} \, , \ s \ \epsilon \ P_K, \ \eta_1 = Re(\eta) > -1. \qquad (4.17)$$

Using analysis like that in obtaining the estimate (4.7) in the proof of Lemma 4.2 we have

$$\left| \, (pst)^{m-(1/2)-(\alpha/2)} \, \exp(-pst/2) \, W_{k+(\alpha/2),m-(\alpha/2)}(pst) \right| \ \le$$

$$(4.18)$$

$$\le \ C_{\alpha} \ (1+s_1)^{m_1+k_1-(1/2)} \ \exp(-ps_1 t/4)$$

for constants C_{α} with $\alpha = 0,1, \ \ldots \ ,N$ and $s \ \epsilon \ P_K$. Again recall that as $|s| \to 0$, $s \ \epsilon \ P_K$, then necessarily $s_1 = Re(s) \to 0 \ +$. Thus by combining (4.15), (4.16), (4.17), and (4.18) we obtain (4.14) and the proof is complete.

We now obtain a final value Abelian theorem for the distributional Whittaker transform.

THEOREM 4.2. Let $U \ \epsilon \ V_a'(0,\infty)$, $a > \sigma_U$, with supp$(U) \subseteq [t_0,\infty)$, $t_0 > 0$. Over some interval $y < t < \infty$, $0 < t_0 < y$, let U be a regular distribution corresponding to a complex valued function $f(t)$ for which there is a real number $c > 0$ such that $(e^{-ct} f(t))$ is absolutely integrable over $y \le \ t < \infty$. For $\eta = \eta_1 + i\eta_2$, $\eta_1 > (-1)$, let $f(t)/t^{\eta}$ be bounded on $y' \le t < \infty$ for all $y' \ge y$; and assume that $F_{p,p,m}^{k,m}(s)$, $s \ \epsilon \ \mathcal{C}_{>}$, exists for the function which is $f(t)$ on $y' < t < \infty$ and which is zero on $0 \le t \le y'$ for all $y' \ge y$. If there is a complex number α for which

$$\lim_{t \to +\infty} \frac{f(t)}{t^{\eta}} = \alpha \qquad (4.19)$$

then for each fixed $K \ge 0$

$$\lim_{\substack{|s| \to 0 \\ s \ \epsilon \ P_K}} \frac{s^{\eta+1} \, WT_p^{k,m} \, [U;s]}{\Lambda(\eta,k,m,p,p,m)} = \alpha. \qquad (4.20)$$

PROOF. Decompose U into $U = U_1 + U_2$ with $supp(U_1) \subseteq [t_0,T]$ and
$supp(U_2) \subseteq [t,\infty)$, $T > y$, as in the paragraph above which contains equation (4.13);
and by the discussion in that paragraph, (4.13) holds for $s_1 = Re(s) > 0$. Let the
function $f(t)$ in the hypothesis, which is known on $y < t < \infty$, be extended to
$0 \le t < \infty$ by

$$\bar{f}(t) = \begin{cases} 0 & , \ 0 \le t \le T, \\ f(t), & T < t < \infty. \end{cases}$$

Then $\bar{f}(t)$ satisfies the hypotheses of Theorem 3.2. Now $WT_p^{k,m}[U_2;s] = WT_p^{k,m}[\bar{f}(t);s]$
with this latter Whittaker transform equaling the function Whittaker transform of $\bar{f}(t)$
defined in (3.1). Thus the conclusion (4.20) follows from (4.13), Lemma 4.3, and
Theorem 3.2. The proof is complete.

In future work we hope to extend the Abelian theorems of Joshi and Saxena [16]
and Malgonde and Saxena [17] for the H-transform to the general setting that the
complex variable of the transform approaches 0 or ∞ inside a wedge region in the right
half plane and for more general parameters, To do so we will need to use the proper-
ties of the H-function as in Srivastava et al [18]. Analysis which is associated with
that in this paper and with the corresponding H-transform problem is contained in
Sinha [19] and Joshi and Saxena [20]. The Meijer transform is studied in Pathak [21]
which we note here because of the similarity of the Meijer and Whittaker transforms.

5.   ACKNOWLEDGEMENTS.

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