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WEAKLY α-CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper, we introduce the notion of weakly α-continuous functions in topological spaces. Weak α-continuity and subweak continuity due to Rose [1] are independent of each other and are implied by weak continuity due to Levine [2]. It is shown that weakly α-continuous surjections preserve connected spaces and that weakly α-continuous functions into regular spaces are continuous. Corollary 1 of [3] and Corollary 2 of [4] are improved as follows: If $f_1 : X \to Y$ is a semi continuous function into a Hausdorff space Y, $f_2 : X \to Y$ is either weakly α-continuous or subweakly continuous, and $f_1 = f_2$ on a dense subset of X, then $f_1 = f_2$ on X.

KEY WORDS AND PHRASES. *weakly continuous, subweakly continuous, weakly α-continuous, semi continuous, almost continuous.*
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1. INTRODUCTION.

Weak continuity due to Levine [2] is one of the most important weak forms of continuity in topological spaces. Recently, Rose [1] has introduced the notion of subweakly continuous functions and investigated the relationship between subweak continuity and weak continuity. In [3], Baker has obtained further properties of subweakly continuous functions. In this paper, we introduce a new class of functions called weakly α-continuous. Subweak continuity and weak α-continuity are independent of each other and are implied by weak continuity. §3 deals with fundamental properties of weakly α-continuous functions. In §4, we investigate similarities and dissimilarities between subweak continuity and weak α-continuity. It is shown that connectedness is preserved under weakly α-continuous surjections. Baker's result [3, Corollary 1] and Janković's one [4, Corollary 2] will be improved. In the last section, we investigate the interrelation among weak α-continuity, almost continuity [5], semi continuity [6], weak quasi continuity [7] and almost weak continuity [8].

2. PRELIMINARIES.

Throughout the present paper, spaces mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let S be a subset of a space (X, τ). The closure of S and the interior of S in (X, τ) are denoted by $Cl_\tau(S)$ and $Int_\tau(S)$, respectively. When there is no possibility of confusion, we will simply denote them by $Cl(S)$ and $Int(S)$, respectively. A subset S of (X, τ) is said to be *α-open* [9] (resp. *semi-open* [6], *pre-open* [10]) if $S \subset Int(Cl(Int(S)))$ (resp. $S \subset Cl(Int(S))$, $S \subset Int(Cl(S))$). The complement of an α-open set is called *α-closed*. We denote the family of α-open (resp. semi-open, pre-open) sets of (X, τ) by τ^α (resp. $SO(X, \tau)$, $PO(X, \tau)$). It is shown in [9] that $\tau \subset \tau^\alpha \subset SO(X, \tau)$ and τ^α is a topology for X. It is shown in [11, Lemma 3.1] that $\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$.

DEFINITION 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly α-continuous* (resp. *weakly continuous* [2]) if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau^\alpha$ (resp. $U \in \tau$) containing x such that $f(U) \subset Cl(V)$. "weakly α-continuous" will be denoted by "w.α.c.".

Every weakly continuous function is w.α.c. but the converse is not true by Example 5.4 (below). Let (X, τ) be a space, S a subset of X and x a point of X. We say that x is in the *θ-closure* of S [12] if $S \cap Cl(U) \neq \emptyset$ for every $U \in \tau$ containing x. The θ-closure of S is denoted by $[S]_\theta$.

LEMMA 2.2. For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(a) f is w.α.c.

(b) $f : (X, \tau^\alpha) \to (Y, \sigma)$ is weakly continuous.

(c) $f^{-1}(V) \subset Int_{\tau^\alpha}(f^{-1}(Cl_\sigma(V)))$ for every $V \in \sigma$.

(d) $Cl_{\tau^\alpha}(f^{-1}(V)) \subset f^{-1}(Cl_\sigma(V))$ for every $V \in \sigma$.

(e) $f(Cl_{\tau^\alpha}(A)) \subset [f(A)]_\theta$ for every subset A of X.

(f) $Cl_{\tau^\alpha}(f^{-1}(B)) \subset f^{-1}([B]_\theta)$ for every subset B of Y.

PROOF. It follows from Definition 2.1 that (a) and (b) are equivalent. Therefore, the others follow immediately from [2, Theorem 1], [13, Theorem 1], [1, Theorem 7] and [14, Theorem 2].

LEMMA 2.3 (Andrijević [15]). Let A be a subset of a space (X, τ). Then the following hold:

(1) $Cl_{\tau^\alpha}(A) = A \cup Cl_\tau(Int_\tau(Cl_\tau(A)))$; (2) $Int_{\tau^\alpha}(A) = A \cap Int_\tau(Cl_\tau(Int_\tau(A)))$.

The following theorem is very useful in the sequel.

THEOREM 2.4. For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(a) f is w.α.c.

(b) $f^{-1}(V) \subset Int(Cl(Int(f^{-1}(Cl(V)))))$ for every $V \in \sigma$.

(c) $Cl(Int(Cl(f^{-1}(V)))) \subset f^{-1}(Cl(V))$ for every $V \in \sigma$.

(d) $f(Cl(Int(Cl(A)))) \subset [f(A)]_\theta$ for every subset A of X.

(e) $Cl(Int(Cl(f^{-1}(B)))) \subset f^{-1}([B]_\theta)$ for every subset B of Y.

PROOF. This follows from Lemmas 2.2 and 2.3.

3. FUNDAMENTAL PROPERTIES OF WEAK α-CONTINUITY.

In this section, we obtain several fundamental properties of w.α.c. functions.

THEOREM 3.1. If $f : (X, \tau) \to (Y, \sigma)$ is w.α.c. and $g : (Y, \sigma) \to (Z, \theta)$ is continuous, then the composition $g \circ f : (X, \tau) \to (Z, \theta)$ is w.α.c.

PROOF. Since f is w.α.c., by Lemma 2.2 $f : (X, \tau^\alpha) \to (Y, \sigma)$ is weakly continuous and hence $g \circ f : (X, \tau^\alpha) \to (Z, \theta)$ is weakly continuous [16, Lemma 1]. Therefore, $g \circ f : (X, \tau) \to (Z, \theta)$ is w.α.c. by Lemma 2.2.

The composition $g \circ f : X \to Z$ of a continuous function $f : X \to Y$ and a w.α.c. function $g : Y \to Z$ is not necessarily w.α.c. as the following example shows.

EXAMPLE 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{c\}\}$ and $\theta = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \theta)$ be the identity functions. Then f is continuous and g is w.α.c. by Example 5.4 (below). However, by Theorem 2.4 $g \circ f$ is not w.α.c. since there exists $\{a\} \in \theta$ such that $\{a\} = (g \circ f)^{-1}(\{a\}) \not\subset \text{Int}_\tau(\text{Cl}_\tau(\text{Int}_\tau((g \circ f)^{-1}(\text{Cl}_\theta(\{a\}))))) = \{c\}$.

THEOREM 3.3. Let $f : (X, \tau) \to (Y, \sigma)$ be an open continuous surjection. Then a function $g : (Y, \sigma) \to (Z, \theta)$ is w.α.c. if and only if $g \circ f : (X, \tau) \to (Z, \theta)$ is w.α.c.

PROOF. *Necessity*. Suppose that g is w.α.c. Let W be any open set of (Z, θ). By Theorem 2.4, $g^{-1}(W) \subset \text{Int}(\text{Cl}(\text{Int}(g^{-1}(\text{Cl}(W)))))$. Since f is open and continuous, we have $f^{-1}(\text{Int}(\text{Cl}(\text{Int}(B)))) \subset \text{Int}(\text{Cl}(\text{Int}(f^{-1}(B))))$ for every subset B of Y. Therefore, we obtain $(g \circ f)^{-1}(W) \subset \text{Int}(\text{Cl}(\text{Int}((g \circ f)^{-1}(\text{Cl}(W)))))$. It follows from Theorem 2.4 that $g \circ f$ is w.α.c.

Sufficiency. Suppose that $g \circ f$ is w.α.c. Let W be any open set of (Z, θ). By Theorem 2.4, $(g \circ f)^{-1}(W) \subset \text{Int}(\text{Cl}(\text{Int}((g \circ f)^{-1}(\text{Cl}(W)))))$. Since f is open and continuous, we have $f(\text{Int}(\text{Cl}(\text{Int}(A)))) \subset \text{Int}(\text{Cl}(\text{Int}(f(A))))$ for every subset A of X. Moreover, since f is surjective, we obtain $g^{-1}(W) \subset \text{Int}(\text{Cl}(\text{Int}(g^{-1}(\text{Cl}(W)))))$. It follows from Theorem 2.4 that g is w.α.c.

By the function $f : (X, \tau) \to (X, \sigma)$ in Example 5.4, we observe that the restriction of a w.α.c. function to a closed set is not necessarily w.α.c. However, we have

THEOREM 3.4. Let $f : (X, \tau) \to (Y, \sigma)$ be w.α.c. and A a subset of X. If either $A \in PO(X, \tau)$ or $A \in SO(X, \tau)$, then the restriction $f|A : (A, \tau/A) \to (Y, \sigma)$ is w.α.c.

PROOF. Since either $A \in PO(X, \tau)$ or $A \in SO(X, \tau)$, it follows from [10, Lemma 1.1] and [17, Lemma 2] that $\tau^\alpha/A \subset (\tau/A)^\alpha$. Since f is w.α.c., for each $x \in A$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau^\alpha$ containing x such that $f(U) \subset \text{Cl}(V)$. Put $U_A = U \cap A$, then we have $x \in U_A \in (\tau/A)^\alpha$ and $(f|A)(U_A) \subset \text{Cl}(V)$. This indicates that $f|A$ is w.α.c.

COROLLARY 3.5. The restriction of a w.α.c. function to an open set is w.α.c.

PROOF. Since every open set is semi-open and pre-open, this is an immediate consequence of Theorem 3.4.

THEOREM 3.6. A function $f : (X, \tau) \to (Y, \sigma)$ is w.α.c. if and only if the graph function $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ defined by $g(x) = (x, f(x))$ for every $x \in X$ is w.α.c.

PROOF. *Necessity*. Suppose that f is w.α.c. Let $x \in X$ and $g(x) \in W \in \tau \times \sigma$. There exist $U_1 \in \tau$ and $V \in \sigma$ such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is w.α.c., there exists $U_2 \in \tau^\alpha$ containing x such that $f(U_2) \subset \text{Cl}_\sigma(V)$. Put $U = U_1 \cap U_2$, then we have $x \in U \in \tau^\alpha$ and $g(U) \subset \text{Cl}_{\tau\times\sigma}(W)$. This indicates that g is w.α.c.

Sufficiency. Suppose that g is w.α.c. Let $x \in X$ and $f(x) \in V \in \sigma$. Then $g(x) \in X \times V \in \tau \times \sigma$ and there exists $U \in \tau^\alpha$ containing x such that $g(U) \subset \text{Cl}_{\tau\times\sigma}(X \times V) = X \times \text{Cl}_\sigma(V)$. Therefore, we obtain $f(U) \subset \text{Cl}_\sigma(V)$ and hence f is w.α.c.

Let $\{(X_\lambda, \tau_\lambda) \mid \lambda \in \Lambda\}$ and $\{(Y_\lambda, \sigma_\lambda) \mid \lambda \in \Lambda\}$ be any two families of spaces with the same index set Λ. Let $f_\lambda : (X_\lambda, \tau_\lambda) \to (Y_\lambda, \sigma_\lambda)$ be a function for each $\lambda \in \Lambda$. Let $f : (\Pi X_\lambda, \Pi\tau_\lambda) \to (\Pi Y_\lambda, \Pi\sigma_\lambda)$ denote the product function defined by

$f(\{x_\lambda\}) = \{f_\lambda(x_\lambda)\}$ for every $\{x_\lambda\} \in \Pi X_\lambda$. Moreover, let $p_\mu : \Pi X_\lambda \to X_\mu$ and $q_\mu : \Pi Y_\lambda \to Y_\mu$ be the natural projections. Then, we have

THEOREM 3.7. The product function $f : (\Pi X_\lambda, \Pi\tau_\lambda) \to (\Pi Y_\lambda, \Pi\sigma_\lambda)$ is w.α.c. if and only if $f_\lambda : (X_\lambda, \tau_\lambda) \to (Y_\lambda, \sigma_\lambda)$ is w.α.c. for each $\lambda \in \Lambda$.

PROOF. *Necessity*. Suppose that f is w.α.c. Let μ be an arbitrarily fixed index of Λ. Since q_μ is continuous, by Theorem 3.1 $q_\mu \circ f = f_\mu \circ p_\mu$ is w.α.c. Moreover, p_μ is an open continuous surjection and hence by Theorem 3.3 f_μ is w.α.c.

Sufficiency. Suppose that f_λ is w.α.c. for each $\lambda \in \Lambda$. Let $x = \{x_\lambda\} \in \Pi X_\lambda$ and $f(x) \in W \in \Pi\sigma_\lambda$. There exists a basic open set ΠV_λ such that

$$f(x) \in \Pi V_\lambda \subset W \text{ and } \Pi V_\lambda = \prod_{i=1}^{n} V_{\lambda_i} \times \prod_{\lambda \neq \lambda_i} Y_\lambda,$$

where $V_\lambda \in \sigma_\lambda$ for each $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$. Since f_λ is w.α.c., for each $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$ there exists $U_\lambda \in \tau_\lambda^\alpha$ containing x_λ such that $f_\lambda(U_\lambda) \subset \mathrm{Cl}(V_\lambda)$. Now, let us put

$$U = \prod_{i=1}^{n} U_{\lambda_i} \times \prod_{\lambda \neq \lambda_i} X_\lambda,$$

then we have $x \in U \in (\Pi\tau_\lambda)^\alpha$ and $f(U) \subset \mathrm{Cl}(W)$. This indicates that f is w.α.c.

THEOREM 3.8. A function $f : (X, \tau) \to (\Pi Y_\lambda, \Pi\sigma_\lambda)$ is w.α.c. if and only if $q_\lambda \circ f : (X, \tau) \to (Y_\lambda, \sigma_\lambda)$ is w.α.c. for each $\lambda \in \Lambda$.

PROOF. This follows immediately from Lemma 2.2 and the fact that a function $f : X \to \Pi Y_\lambda$ is weakly continuous if and only if $q_\lambda \circ f : X \to Y_\lambda$ is weakly continuous for each $\lambda \in \Lambda$ [16, Theorem 2].

4. WEAK α-CONTINUITY AND SUBWEAK CONTINUITY.

Rose [1] introduced and studied the concept of subweakly continuous functions. In [16], he also obtained further properties of such functions. In [3], Baker obtained several properties of subweak continuity which are analogous to results in [18]. In this section, we investigate similarities and dissimilarities between weak α-continuity and subweak continuity.

DEFINITION 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *subweakly continuous* [1] if there exists an open basis Σ for σ such that for every $V \in \Sigma$ $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$.

REMARK 4.2. In Example 5.4 (below), f is w.α.c. but not subweakly continuous since $\{a\}$ belongs to every open basis for σ and $\mathrm{Cl}(f^{-1}(\{a\})) \not\subset f^{-1}(\mathrm{Cl}(\{a\}))$. Moreover, the following example indicates that subweak continuity does not imply weak α-continuity in general. Consequently, we observe that weak α-continuity and subweak continuity are independent of each other.

EXAMPLE 4.3. Let X be the set of all real numbers, τ the countable complement topology for X and σ the discrete topology for X. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is subweakly continuous but not w.α.c.

In [18, Theorem 3], the present author showed that connectedness is preserved under weakly continuous surjections. By Example 4.3, we observe that subweakly continuous surjections need not preserve connected spaces. However, w.α.c. surjections preserve connected spaces.

LEMMA 4.4. Let $f : (X, \tau) \to (Y, \sigma)$ be a w.α.c. function. If V is a clopen set of (Y, σ), then $f^{-1}(V)$ is clopen in (X, τ).

PROOF. It follows from (c) and (d) of Lemma 2.2 that $f^{-1}(V)$ is α-closed and α-open. Therefore, we have $Cl(Int(Cl(f^{-1}(V)))) \subset f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V))))$. This fact implies that $f^{-1}(V)$ is clopen in (X, τ).

THEOREM 4.5. If $f : (X, \tau) \to (Y, \sigma)$ is a w.α.c. surjection and (X, τ) is a connected space, then (Y, σ) is connected.

PROOF. Assume that (Y, σ) is not connected. There exist nonempty V_1, $V_2 \in \sigma$ such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Therefore, V_1 and V_2 are clopen in (Y, σ) and by Lemma 4.4 $f^{-1}(V_i) \in \tau$ for $i = 1, 2$. Moreover, we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_i)$ is nonempty for $i = 1, 2$. This indicates that (X, τ) is not connected. This is a contradiction.

In [18, Theorem 6], the present author showed that if $f : X \to Y$ is a weakly continuous injection and Y is Urysohn, then X is Hausdorff. Baker [3] pointed out that "subweakly continuous" can not be substituted for "weakly continuous" in the preceding result.

THEOREM 4.6. If $f : (X, \tau) \to (Y, \sigma)$ is a w.α.c. injection and (Y, σ) is Urysohn, then (X, τ) is Hausdorff.

PROOF. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$ and there exist $V_1, V_2 \in \sigma$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $Cl(V_1) \cap Cl(V_2) = \emptyset$. It follows from Theorem 2.4 that $x_i \in f^{-1}(V_i) \subset Int(Cl(Int(f^{-1}(Cl(V_i)))))$ for $i = 1, 2$. Since $f^{-1}(Cl(V_1))$ and $f^{-1}(Cl(V_2))$ are disjoint, we obtain

$$Int(Cl(Int(f^{-1}(Cl(V_1))))) \cap Int(Cl(Int(f^{-1}(Cl(V_2))))) = \emptyset.$$

This shows that (X, τ) is Hausdorff.

DEFINITION 4.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(a) *semi-continuous* [6] if $f^{-1}(V) \in SO(X, \tau)$ for every $V \in \sigma$;

(b) *almost continuous* [5] if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, $Cl(f^{-1}(V))$ is a neighborhood of x.

LEMMA 4.8 (Rose [1]). A function $f : (X, \tau) \to (Y, \sigma)$ is almost continuous (resp. semi-continuous) if and only if $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ (resp. $f^{-1}(V) \subset Cl(Int(f^{-1}(V))))$ for every $V \in \sigma$.

In [1, Theorem 10], Rose showed that every almost continuous and subweakly continuous function is weakly continuous. Similarly, we have

THEOREM 4.9. If : $(X, \tau) \to (Y, \sigma)$ is almost continuous and w.α.c., then it is weakly continuous.

PROOF. Let V be any open set of (Y, σ). Since f is w.α.c., by Theorem 2.4 $Cl(Int(Cl(f^{-1}(V)))) \subset f^{-1}(Cl(V))$. Since f is almost continuous, by Lemma 4.8 $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ and hence $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$. It follows from [13, Theorem 1] or [1, Theorem 7] that f is weakly continuous.

In [3, Corollary 1], Baker showed that if Y is Hausdorff, $f_1 : X \to Y$ is continuous, $f_2 : X \to Y$ is subcontinuous, and $f_1 = f_2$ on a dense subset of X, then $f_1 = f_2$ on X. On the other hand, in [4, Corollary 2], Janković showed that if Y is Hausdorff, $f_1 : X \to Y$ is weakly continuous, $f_2 : X \to Y$ is semi-continuous and $f_1 = f_2$ on a dense subset of X, then $f_1 = f_2$ on X. These two results are improved as follows:

THEOREM 4.10. Let (Y, σ) be Hausdorff and $f_1 : (X, \tau) \to (Y, \sigma)$ be semi-continuous. If $f_2 : (X, \tau) \to (Y, \sigma)$ is either w.α.c. or subweakly continuous, and if $f_1 = f_2$ on a dense subset D of X, then $f_1 = f_2$ on X.

PROOF. First, suppose that f_2 is w.α.c. Let $A = \{x \in X |\ f_1(x) = f_2(x)\}$ and assume that $x \in X - A$. Then $f_1(x) \neq f_2(x)$ and there exist $V_1, V_2 \in \sigma$ such that $f_1(x) \in V_1$, $f_2(x) \in V_2$ and $V_1 \cap V_2 = \emptyset$; hence $V_1 \cap Cl(V_2) = \emptyset$. Since f_1 (resp. f_2) is semi-continuous (resp. w.α.c.), there exists $U_1 \in SO(X, \tau)$ (resp. $U_2 \in \tau^\alpha$) containing x such that $f_1(U_1) \subset V_1$ (resp. $f_2(U_2) \subset Cl(V_2)$). Therefore, we obtain $x \in U_1 \cap U_2 \in SO(X, \tau)$ [9, Prop. 1] and $(U_1 \cap U_2) \cap A = \emptyset$. Since $\emptyset \neq U_1 \cap U_2 \in SO(X, \tau)$, we have $Int(U_1 \cap U_2) \neq \emptyset$ and $Int(U_1 \cap U_2) \cap A = \emptyset$. On the other hand, since $f_1 = f_2$ on D, we have $D \subset A$ and hence $X = Cl(D) \subset Cl(A)$. This is a contradiction. Thus, $A = X$ and $f_1 = f_2$ on X. Next, suppose that f_2 is subweakly continuous. By Theorem 2 of [3], the graph $G(f_2)$ is closed and hence it follows from [4, Corollary 1] that $f_1 = f_2$ on X.

5. COMPARISONS.

DEFINITION 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(a) *α-continuous* [10] if $f^{-1}(V) \in \tau^\alpha$ for every $V \in \sigma$;

(b) *almost weakly continuous* [8] if $f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V))))$ for every $V \in \sigma$;

(c) *weakly quasi continuous* [7] if for each $x \in X$, each $G \in \tau$ containing x and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau$ such that $\emptyset \neq U \subset G$ and $f(U) \subset Cl(V)$.

We shall investigate the interrelations between the weak forms of continuity previously stated. It is shown in [11, Theorem 3.2] that a function is α-continuous if and only if it is semi-continuous and almost continuous. It will be shown that weak continuity, semi-continuity and almost continuity are respectively independent.

LEMMA 5.2. If $f : (X, \tau) \to (Y, \sigma)$ is α-continuous, then it is weakly continuous.

PROOF. Let V be any open set of (Y, σ). Then $f^{-1}(V) \in \tau^\alpha$ and $f^{-1}(Cl(V))$ is α-closed in (X, τ). Therefore, we have

$$f^{-1}(V) \subset Int(Cl(Int(f^{-1}(V)))) \subset Cl(Int(Cl(f^{-1}(Cl(V))))) \subset f^{-1}(Cl(V)).$$

Thus, we obtain $f^{-1}(V) \subset Int(f^{-1}(Cl(V)))$. It follows from [2, Theorem 1] that f is weakly continuous.

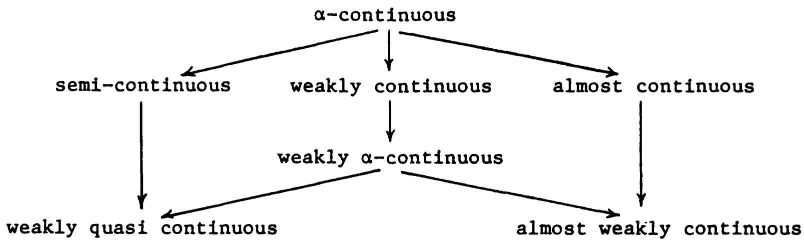
LEMMA 5.3. A function $f : (X, \tau) \to (Y, \sigma)$ is weakly quasi continuous if and only if $f^{-1}(V) \subset Cl(Int(f^{-1}(Cl(V))))$ for every $V \in \sigma$.

PROOF. *Necessity*. Suppose that f is weakly quasi continuous. Let V be any open set of (Y, σ) and $x \in f^{-1}(V)$. For any $G \in \tau$ containing x, there exists $U \in \tau$ such that $\emptyset \neq U \subset G$ and $f(U) \subset Cl(V)$. Therefore, we have $U \subset f^{-1}(Cl(V))$ and hence $U \subset Int(f^{-1}(Cl(V)))$. Since $\emptyset \neq U \subset G \cap Int(f^{-1}(Cl(V)))$, $x \in Cl(Int(f^{-1}(Cl(V))))$ and hence $f^{-1}(V) \subset Cl(Int(f^{-1}(Cl(V))))$.

Sufficiency. Let $x \in X$, $x \in G \in \tau$, and $f(x) \in V \in \sigma$. Then, by hypothesis $x \in f^{-1}(V) \subset Cl(Int(f^{-1}(Cl(V))))$ and hence $G \cap Int(f^{-1}(Cl(V))) \neq \emptyset$. Now, put $U = G \cap Int(f^{-1}(Cl(V)))$, then we have $\emptyset \neq U \subset G$ and $f(U) \subset Cl(V)$. This shows that f is weakly quasi continuous.

By Definition 5.1, Lemmas 4.8, 5.2 and 5.3, and Theorem 2.4, we obtain the following diagram.

DIAGRAM



In the sequel, we shall show that none of the implications in DIAGRAM is reversible.

EXAMPLE 5.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is w.α.c. but it is not weakly continuous.

EXAMPLE 5.5. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a function defined as follows: $f(a) = c$, $f(b) = d$, $f(c) = b$ and $f(d) = a$. Then f is weakly continuous [19, Example] and hence it is weakly quasi continuous and almost weakly continuous. However, f is neither semi-continuous nor almost continuous.

EXAMPLE 5.6. Let X be the set of all real numbers, τ the indiscrete topology for X and σ the discrete topology for X. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is almost continuous but it is not weakly quasi continuous.

EXAMPLE 5.7. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is semi-continuous but it is not almost weakly continuous.

REMARK 5.8. (1) By Examples 5.4 - 5.7, we observe that none of the implications in DIAGRAM is reversible.

(2) Examples 5.5 - 5.7 indicate that weak continuity, semi-continuity and almost continuity are respectively independent.

It follows from Example 5.6 (resp. Example 5.7) that an almost continuous (resp. semi-continuous) function into a regular space need not be continuous. However, we have

THEOREM 5.9. If $f : (X, \tau) \to (Y, \sigma)$ is w.α.c. and (Y, σ) is regular, then f is continuous.

PROOF. Let x be any point of X and V any open set of (Y, σ) containing $f(x)$. Since (Y, σ) is regular, there exists $W \in \sigma$ such that $f(x) \in W \subset Cl(W) \subset V$. Since f is w.α.c., there exists $U \in \tau^{\alpha}$ containing x such that $f(U) \subset Cl(W) \subset V$. Therefore, f is α-continuous [10, Theorem 1] and hence it is continuous [10, Remark].

In [20], Reilly and Vamanamurthy defined a space (X, τ) to be *α-pseudo compact* if every real valued α-continuous function on (X, τ) is bounded. It is obvious that every α-pseudo compact space is pseudo compact and that (X, τ) is α-pseudo compact if and only if (X, τ^{α}) is pseudo compact. In a letter of correspondence, Reilly conjectured that (X, τ) is pseudo compact if and only if (X, τ^{α}) is pseudo compact. By Theorem 5.9, every real valued α-continuous function is continuous and Reilly's conjecture is true.

COROLLARY 5.10 (Reilly). A space (X, τ) is α-pseudo compact if and only if it is pseudo compact.

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