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CONVERGENCE THEOREMS FOR BANACH SPACE VALUED INTEGRABLE MULTIFUNCTIONS

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ABSTRACT. In this work we generalize a result of Kato on the pointwise behavior of a weakly convergent sequence in the Lebesgue-Bochner spaces $L_X^p(\Omega)$ $(1 \leq p \leq \infty)$. Then we use that result to prove Fatou's type lemmata and dominated convergence theorems for the Aumann integral of Banach space valued measurable multifunctions. Analogous convergence results are also proved for the sets of integrable selectors of those multifunctions. In the process of proving those convergence theorems we make some useful observations concerning the Kuratowski-Mosco convergence of sets.

KEY WORDS AND PHRASES. Convergence, measurable multifunctions, nonatomic.
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1. INTRODUCTION.

In [1] Schmeidler motivated from problems in mathematical economics, proved a set valued version of Fatou's lemma, for multifunctions taking values in $\mathbb{R}^n$ A different proof and some additional results in this direction were obtained later by Hildenbrand and Mertens [2].

Finally Artstein in [3] provided the sharpest version of that result. However all the above authors apparently were unaware of an earlier analogous result of Kato [4], for Banach space valued functions. The purpose of this note is to significantly extend the result of Kato [4], use that extension to prove a Fatou's lemma for Banach space valued multifunctions, extending this way the works of Schmeidler [1], Hildenbrand-Mertens [2] and Artstein [3] and finally prove a dominated convergence theorem for Banach space valued multifunctions. Then we obtain analogous convergence results for the sets of Bochner integrable selectors of the multifunctions. Our results can have important applications in optimization, optimal control, differential inclusions, abstract evolution equations and mathematical economics.

2. PRELIMINARIES.

Let (Ω,Σ,μ) be a complete, σ-finite measure space and X a separable Banach space, with X^* being its topological dual. We will use the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{nonempty, closed, (convex)}\}$$

$P_{wk(c)}(x) = \{A \subseteq X : \text{nonempty, w-compact, (convex)}\}$

For $A \in 2^X\backslash\{\phi\}$, we set $|A| = \sup_{a\in A}||a||$, by $d_A(\cdot)$ we denote the distance function from A i.e. for all $x \in X$, $d_A(x) = \inf_{a\in A}||x-a||$ and by $\sigma_A(\cdot)$ the support function of A i.e. for all $x^* \in X^*, \sigma_A(x^*) = \sup_{a\in A}(x^*,a)$.

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if it satisfies any of the following equivalent conditions:

i) for all $x \in X$, $\omega \to d_{F(\omega)}(x)$ is measurable

ii) there exists a sequence $\{f_n(\cdot)\}_{n\geq 1}$ of measurable functions s.t. $F(\omega) = cl\{f_n(\omega)\}_{n\geq 1}$ for all $\omega \in \Omega$ (Castaing's representation).

iii) $GrF = \{(\omega,x) \in \Omega xX : x \in F(\omega)\} \in \Sigma xB(X)$, where $B(X)$ is the Borel σ-field of X (graph measurability).

We denote by S_F^1 the set of all selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L_X^1(\Omega)$ i.e. $S_F^1 = \{f(\cdot) \in L_X^1(\Omega) : f(\omega) \in F(\omega)\mu\text{-a.e.}\}$. It is easy to see that this set is closed and it is nonempty if and only if $\inf_{x\in F(\omega)}||x|| \in L_+^1$. We say that $F : X \to P_f(X)$ is integrably bounded if it is measurable and $|F(\cdot)| \in L_+^1$. Using the set S_F^1, we can define a set valued integral for $F(\cdot)$ as follows:

$\int_\Omega F(\omega)d\mu(\omega) = \{\int_\Omega f(\omega)d\mu(\omega) : f(\cdot) \in S_F^1\}$. This integral is known as Aumann's integral.

If $\{A_n\}_{n\geq 1}$ are nonempty subsets of X, we define

$$s - \underline{\lim}_{n\to\infty} A_n = \{x \in X : x = s - \lim x_n,\ x_n \in A_n, n \geq 1\}$$

and $w - \overline{\lim}_{n\to\infty} A_n = \{x \in X : x - \lim x_k,\ x_k \in A_{n_k},\ k \geq 1\}$.

We say that the A_n's converge to A in the Kuratowski-Mosco sense (denoted by $A_n \xrightarrow{K-M} A$) if and only if $w - \overline{\lim} A_n = A = s - \underline{\lim} A_n$. For more details we refer to the nice works of Mosco [5], [6] and of Salinetti and Wets [7], [8] and [9].

3. CONVERGENCE RESULTS FOR THE AUMANN INTEGRAL.

In this section our goal is to prove a Fatou's lemma and a dominated convergence theorem for the Aumann integral. We start with an interesting observation concerning the w-$\overline{\lim}$ of a sequence of nonempty sets. Assume that X is a Banach space.

PROPOSITION 3.1. If for all $n \geq 1$ $A_n \neq \phi$ and $A_n \subseteq G$ where $G \in P_{wk}(x)$

then for all $x^* \in X^*$, $\overline{\lim}\, \sigma_{A_n}(X^*) \leq \sigma_{w-\overline{\lim} A_n}(x^*)$.

PROOF. Fix $x^* \in X^*$ and let $x_n \in A_n$ s.t. $(x^*,x_n) = \sigma_{A_n}(x^*)$. Let $\{x_k\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$ s.t. $(x^*,x_k) \to \overline{\lim}\, \sigma_{A_n}(x^*)$ as $k \to \infty$. Since $\{x_n\}_{n\geq 1} \subseteq G$, invoking the Eberlein-Smulian theorem and by passing to a subsequence if necessary, we may assume that $x_k \xrightarrow{w} x$.

Then $x \in w\text{-}\overline{\lim}\, A_n \Rightarrow (x^*, x) \le \sigma_{w\text{-}\overline{\lim}\, A_n}(x^*) \Rightarrow \overline{\lim}\, \sigma_{A_n}(x^*) \le \sigma_{w\text{-}\overline{\lim}\, A_n}(x^*)$.

Q.E.D.

This leads us to the following interesting theorem that generalizes significantly an earlier result of Kato [4], who had X to be reflexive with a uniformly convex dual, $1 < p < \infty$ and the sequence of vector valued functions was uniformly bounded.

Here (Ω, Σ, μ) is a measure space, X a Banach space and $1 \le p < \infty$.

THEOREM 3.1. If $\{f_n(\cdot), f(\cdot)\}_{n\ge 1} \subseteq L^p_X(\Omega)$, $f_n(\cdot) \xrightarrow{w\text{-}L^p_X} f(\cdot)$ and $f_n(\omega) \in G(\omega)\, \mu\text{-a.e.}$ where $G(\omega) \in P_{wk}(X)\, \mu\text{-a.e.}$

<u>then</u> $f(\omega) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\, \{f_n(\omega)\}_{n\ge 1}\, \mu\text{-a.e.}$

PROOF. From Mazur's lemma we know that for all $k \ge 1$

$$f(\omega) \in \overline{\text{conv}} \bigcup_{n\ge k} f_n(\omega)\, \mu\text{-a.e.}$$

Let $x^* \in X^*$. Then for all $k \ge 1$ we have:

$$(x^*, f(\omega)) \le \sigma_{\overline{\text{conv}} \bigcup_{n\ge k} f_n(\omega)}(x^*) = \sigma_{\bigcup_{n\ge k} f_n(\omega)}(x^*) = \sup_{n\ge k}(x^*, f_n(\omega))\, \mu\text{-a.e.}$$

$$\Rightarrow (x^*, f(\omega)) \le \overline{\lim}\, (x^*, f_n(\omega)) = \overline{\lim}\, \sigma_{\{f_n(\omega)\}}(x^*)\ \ \mu\text{-a.e.}$$

Using proposition 3.1 we get that

$$\overline{\lim}\, \sigma_{\{f_n(\omega)\}}(x^*) \le \sigma_{w\text{-}\overline{\lim}\, \{f_n(\omega)\}_{n\ge 1}}(x^*) \quad \mu\text{-a.e.}$$

$$\Rightarrow (x^*, f(\omega)) \le \sigma_{w\text{-}\overline{\lim}\, \{f_n(\omega)\}_{n\ge 1}}(x^*) \quad \mu\text{-a.e.}$$

$$\Rightarrow f(\omega) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\, \{f_n(\omega)\}_{n\ge 1}\, \mu\text{-a.e.}$$

Q.E.D.

Having this theorem we can have the $w\text{-}\overline{\lim}$ version of Fatou's lemma for the Aumann integral.

Now (Ω, Σ, μ) is a nonatomic, σ-finite, complete measure space and X a separable Banach space.

THEOREM 3.2. *If* $F_n : \Omega \to P_f(X)$ are measurable multifunctions s.t. for all $n \geq 1$, $F_n(\omega) \subseteq G(\omega)\mu$-a.e. where $G : \Omega \to P_{wkc}(X)$ is integrably bounded and $\omega \to w\text{-}\overline{\lim}\, F_n(\omega)$ is measurable

$$\underline{\text{then}}\quad w\text{-}\overline{\lim} \int_\Omega F_n(\omega)d\mu(\omega) \subseteq cl \int_\Omega w\text{-}\overline{\lim}\, F_n(\omega)d\mu(\omega).$$

PROOF. Let $x \in w\text{-}\overline{\lim} \int_\Omega F_n(\omega)d\mu(\omega)$. Then there exist $x_k \in \int_\Omega F_{n_k}(\omega)d\mu(\omega)$ s.t. $x_k \xrightarrow{w} x$. From the definition of the Aumann integral, we know that there exist $f_k(\cdot) \in S^1_{F_{n_k}}$ s.t. $x_k = \int_\Omega f_k(\omega)d\mu(\omega)$. But $S^1_{F_{n_k}} \subseteq S^1_G$ and the latter is w-compact in $L^1_X(\Omega)$ [10]. So by passing if necessary to a further subsequence, we may assume that $f_k(\cdot) \xrightarrow{w\text{-}L^1_X} f(\cdot) \in S^1_G$. Hence $x = \int_\Omega f(\omega)d\mu(\omega)$. But from theorem 3.1 we know that $f(\omega) \in \overline{conv}\, w\text{-}\overline{\lim}\, \{f_n(\omega)\}_{n\geq 1}\mu$-a.e. $\Rightarrow f(\omega) \in \overline{conv}\, w\text{-}\overline{\lim}\, F_n(\omega)\mu$-a.e. $\Rightarrow x \in \int_\Omega \overline{conv}\, w\text{-}\overline{\lim}\, F_n(\omega)d\mu(\omega)$. Since by hypothesis $\omega \to w\text{-}\overline{\lim}\, F_n(\omega)$ is graph measurable and $\mu(\cdot)$ is nonatomic, we have that $cl \int_\Omega w\text{-}\overline{\lim}\, F_n(\omega)d\mu(\omega) = \int_\Omega \overline{conv}\, w\text{-}\overline{\lim}\, F_n(\omega)d\mu(\omega)$. Thus finally we have that $x \in cl \int_\Omega w\text{-}\overline{\lim}\, F_n(\omega)d\mu(\omega)$, which proves Fatou's lemma for the weak limit superior.

Q.E.D.

Next we will prove the s-$\underline{\lim}$ version of Fatou's lemma. This can be achieved under less restrictive hypotheses on the sequence $\{F_n(\cdot)\}_{n\geq 1}$.

Here (Ω,Σ,μ) is a complete, σ-finite measure space and X a separable Banach space.

THEOREM 3.3. *If* $F_n : \Omega \to 2^X\setminus\{\phi\}$ are integrably bounded and $\{|F_n(\cdot)|\}_{n\geq 1}$ is uniformly integrable

$$\underline{\text{then}} \int_\Omega s\text{-}\underline{\lim}\, F_n f(\omega)d\mu(\omega) \subseteq s\text{-}\lim \int_\Omega F_n(\omega)d\mu(\omega).$$

PROOF. Let $x \in \int_\Omega s\text{-}\underline{\lim}\, F_n f(\omega)d\mu(\omega)$. Then $x = \int_\Omega f(\omega)d\mu(\omega)$ with $f(\cdot) \in S^1_{s\text{-}\underline{\lim}\, F_n}$. Now consider the multifunctions $L_n(\omega) = \{x \in F_n(\omega) : d_{F_n(\omega)}(f(\omega)) \leq ||x-f(\omega)|| + \frac{1}{n}\}$. Because the function $(\omega,x) \to d_{F_n(\omega)}(x)$ is Caratheodory, it is superpositionally measurable and so $\omega \to d_{F_n(\omega)}((f(\omega))$ is measurable. Then $(\omega,x) \to d_{F_n(\omega)}(f(\omega)) - ||x-f(\omega)||$ is a Caratheodory function and so jointly measurable. So $\{(\omega,x) \in \Omega x X : d_{F_n(\omega)}(f(\omega)) - ||x-f(\omega)|| \leq \frac{1}{n}\} \in \Sigma x B(X)$. Now note that $GrL_n = \{(\omega,x) \in \Omega x X : d_{F_n(\omega)}(f(\omega)) - ||x-f(\omega))|| \leq \frac{1}{n}\} \cap GrF_n \in \Sigma x B(X)$. Apply Aumann's selection theorem to find $f_n : \Omega \to X$ measurable s.t. $f_n(\omega) \in L_n(\omega)$ for all $\omega \in \Omega$. From the definition of $s\text{-}\underline{\lim}\, F_n(\omega)$ [11] we know that $d_{F_n(\omega)}(f(\omega)) \to 0$ μ-a.e. So $||f_n(\omega) - f(\omega)|| \to 0$ μ-a.e. $\Rightarrow \int_\Omega f_n(\omega)d\mu(\omega) = x_n \xrightarrow{s} \int_\Omega f(\omega)d\mu(\omega) = x$ and $x_n \in \int_\Omega F_n(\omega)d\mu(\omega) \Rightarrow x \in s\text{-}\underline{\lim} \int_\Omega F_n(\omega)d\mu(\omega)$. Hence Fatou's lemma follows.

Q.E.D.

REMARK. From Kuratowski [11], we know that an equivalent definition of $s\text{-}\underline{\lim}\, F_n(\omega)$ is: $s\text{-}\underline{\lim}\, F_n(\omega) = \{x \in X : \lim_{n\to\infty} d_{F_n(\omega)}(x) = 0\}$ and that $s\text{-}\underline{\lim}\, F_n(\omega)$ is a closed set. Note that $(\omega,x) \to d_{F_n(\omega)}(x)$ being Caratheodory it is jointly measurable and then so is $\lim_{n\to\infty} d_{F_n(\omega)}(x)$. Hence $\{(\omega,x) \in \Omega x X : \lim_{n\to\infty} d_{F_n(\omega)}(x) = 0\} \in \Sigma x B(X) \Rightarrow Gr(s\text{-}\underline{\lim}\, F_n(\cdot)) \in \Sigma x B(X) \Rightarrow \omega \to s\text{-}\underline{\lim}\, F_n(\omega)$ is measurable.

Combining the two Fatou's lemmata we can have a dominated convergence theorem for the Aumann integral.

So assume that (Ω,Σ,μ) is nonatomic, complete, σ-finite mesure space and X a separable Banach space.

THEOREM 3.4. *If* $F_n : \Omega \to P_f(X)$ are measurable multifunctions s.t. $F_n(\omega) \subseteq G(\omega)$ μ-a.e. with $G : \Omega \to P_{wkc}(X)$ integrably bounded and $F_n(\omega) \xrightarrow{K-M} F(\omega)$ μ-a.e.

$$\text{\underline{then}} \int_\Omega F_n(\omega)d\mu(\omega) \xrightarrow{K-M} cl \int_\Omega F(\omega)d\mu(\omega).$$

PROOF. This follows from theorem 3.2 and 3.3 if we recall that $F_n(\omega) \to F(\omega)$ μ-a.e. $\Leftrightarrow w\text{-}\overline{\lim}\, F_n(\omega) = F(\omega) = s\text{-}\underline{\lim}\, F_n(\omega)$ and $F(\cdot)$ is closed valued and measurable.

Q.E.D.

REMARK. If we assume that $F(\cdot)$ is convex valued (which is the case if the F_n's are) then we have that $\int_\Omega F_n(\omega)d\mu(\omega) \xrightarrow{K-M} \int_\Omega F(\omega)d\mu(\omega)$ [10]. Furthermore in this case we can relax the nonatomicity hypothesis on $\mu(\cdot)$.

We will close this section with a dominated convergence theorem for the Hausdorff metric $h(\cdot,\cdot)$ on $P_f(X)$.

Let (Ω,Σ,μ) be a complete, σ-finite measure space and X a separable Banach space.

THEOREM 3.5. *If* $F_n : \Omega \to P_f(X)$ are measurable multifunctions, $\{|F_n(\cdot)|\}_{n\geq 1}$ is uniformly integrable and $F_n(\omega) \xrightarrow{h} F(\omega)$ in measure

$$\underline{\text{then}}\ \mathrm{cl}\int_\Omega F_n(\omega)d\mu(\omega) \xrightarrow{h} \mathrm{cl}\int_\Omega F(\omega)d\mu(\omega).$$

PROOF. Recall that $h(\mathrm{cl}\int_\Omega F_n(\omega)d\mu(\omega), \mathrm{cl}\int_\Omega F(\omega)d\mu(\omega)) \leq \int_\Omega h(F_n(\omega), F(\omega))d\mu(\omega)$. Also $h(F_n(\omega), F(\omega)) \leq |F_n(\omega)| + |F(\omega)|$. Then using the extended dominated convergence theorem [12], we get $\int_\Omega h(F_n(\omega), F(\omega))d\mu(\omega) \to 0 \Rightarrow h(\mathrm{cl}\int_\Omega F_n(\omega)d\mu(\omega), \mathrm{cl}\int F(\omega)d\mu(\omega)) \to 0$ as $n \to \infty$.

Q..E.D.

4. CONVERGENCE RESULTS FOR THE SETS OF INTEGRABLE SELECTORS.

In this section we prove analogous convergence theorems for the sets $S^1_{F_n}$.

As before we will start with two Fatou's type theorems. But first we need the following auxiliary result about the Kuratowski-Mosco convergence of sets.

Here X is any Banach space.

PROPOSITION 4.1. <u>If</u> for all $x^* \in X^*$, $\overline{\lim}\, \sigma_{A_n}(x^*) \leq \sigma_A(x^*)$

$$\underline{\text{then}}\ w\text{-}\overline{\lim}\, A_n \subseteq \overline{\mathrm{conv}}\, A.$$

PROOF. Let $x \in w\text{-}\overline{\lim}\, A_n$. Then there exist $x_k \in A_{n_k}$ s.t. $x_k \xrightarrow{w} x$. So for all $x^* \in X^*$ $(x^*, x_k) \to (x^*, x) \Rightarrow (x^*, x) \leq \overline{\lim}\, \sigma_{A_n}(x^*) \leq \sigma_A(x^*) \Rightarrow x \in \overline{\mathrm{conv}}\, A$.

Q.E.D.

Now we are ready for the first Fatou's type convergence result. So let (Ω, Σ, μ) be a complete, σ-finite measure space and X a separable Banach space.

THEOREM 4.1. <u>If</u> $F_n : \Omega \to P_f(X)$ are measurable multifunctions s.t. $\{|F_n(\cdot)|\}_{n\geq 1}$ is uniformly integrable and $s\text{-}\underline{\lim}\, F_n(\omega) \neq \phi$ μ-a.e.

$$\underline{\text{then}}\ S^1_{s\text{-}\underline{\lim}\, F_n} \subseteq s\text{-}\underline{\lim}\, S^1_{F_n}.$$

PROOF. Let $u(\cdot) \in L^1_X(\Omega)$. Then we have:

$$d_{S^1_{F_n}}(u) = \inf_{f\in S^1_{F_n}} ||f-u||_1 = \inf_{f\in S_{F_n}} \int_\Omega ||f(\omega) - u(\omega)||d\mu(\omega) = \int_\Omega \inf_{x\in F_n(\omega)} ||x-u(\omega)||d\mu(\omega)$$

$$= \int_\Omega d_{F_n(\omega)}(u(\omega))d\mu(\omega).$$

So using Fatou's lemma [12] we get that:

$$\overline{\lim}\, d_{S^1_{F_n}}(u) = \overline{\lim} \int_\Omega d_{F_n(\omega)}(u(\omega))d\mu(\omega) \leq \int_\Omega \overline{\lim}\, d_{F_n(\omega)}(u(\omega))d\mu(\omega).$$

But from theorem 2.2 (i) of Tsukada [13] we have that for all $\omega \in \Omega$

$$\overline{\lim}\, d_{F_n(\omega)}(u(\omega)) \leq d_{s\text{-}\underline{\lim}\, F_n(\omega)}(u(\omega))$$

$$\Rightarrow \int_\Omega \overline{\lim}\, d_{F_n(\omega)}(u(\omega)) \leq \int_\Omega d_{s\text{-}\underline{\lim}\, F_n(\omega)}(u(\omega))d\mu(\omega) = d_{S^1_{s\text{-}\underline{\lim}\, F_n}}(u)$$

$$\Rightarrow \overline{\lim}\, d_{S^1_{F_n}}(u) \leq d_{S^1_{s\text{-}\underline{\lim}\, F_n}}(u) \quad .$$

Note that $s\text{-}\underline{\lim}\, F_n(\omega) \in P_{fc}(X)\mu$-a.e. So $S^1_{s\text{-}\underline{\lim}\, F_n} \in P_{fc}(L^1_X)$ and since $u(\cdot) \in L^1_X(\Omega)$ was arbitrary we can apply theorem 2.2 (ii) of Tsukada [13] and conclude that $S^1_{s\text{-}\underline{\lim}\, F_n} \subseteq s\text{-}\underline{\lim}\, S^1_{F_n}$.

Q.E.D.

We have the analogous result for $w\text{-}\overline{\lim}$. The assumptions on the spaces (Ω,Σ,μ) and X remain the same.

THEOREM 4.2. If $F_n : \Omega \to P_{fc}(X)$ are measurable multifunctions s.t. for all $n \geq 1$ $F_n(\omega) \subseteq G(\omega)\phi$-a.e. where $G : \Omega \to P_{wkc}(X)$ is integrably bounded and $\omega \to w\text{-}\overline{\lim}\, F_n(\omega)$ is graph measurable

$$\text{then} \quad w\text{-}\overline{\lim}\, S^1_{F_n} \subseteq \overline{\text{conv}}\, S^1_{w\text{-}\overline{\lim}\, F_n}.$$

If in addition $w\text{-}\overline{\lim}\, F_n(\omega) \in P_{fc}(X)$ for all $\omega \in \Omega$

$$\text{then} \quad w\text{-}\overline{\lim}\, S^1_{F_n} \subseteq S^1_{w\text{-}\overline{\lim}\, F_n}.$$

PROOF. From the Dinculeanu-Foias theorem [14], we know that $(L^1_X)^* = L^\infty_{X^*_{w^*}}$. Let $u(\cdot) \in L^\infty_{X^*_{w^*}}$. Then we have:

$$\sigma_{S^1_{F_n}}(u) = \sup_{f(\cdot)\in S^1_{F_n}} \int_\Omega (u(\omega),f(\omega))d\mu(\omega) =$$

$$\int_\Omega \sup_{x\in F_n(\omega)} (u(\omega),x)d\mu(\omega) = \int_\Omega \sigma_{F_n(\omega)}(u(\omega))d\mu(\omega).$$

Then using Fatou's lemma we get that

$$\overline{\lim}\, \sigma_{S^1_{F_n}}(u) = \overline{\lim} \int_\Omega \sigma_{F_n(\omega)}(u(\omega))d\mu(\omega) \leq \int_\Omega \overline{\lim}\, \sigma_{F_n(\omega)}(u(\omega))d\mu(\omega).$$

But from proposition 3.1 we know that for all $\omega \in \Omega$

$$\overline{\lim}\, \sigma_{F_n(\omega)}(u(\omega)) \leq \sigma_{w\text{-}\overline{\lim}\, F_n(\omega)}(u(\omega))$$

and since $w\text{-}\overline{\lim}\, F_n(\omega)$ is by hypothesis graph measurable we have that:

$$\int_\Omega \sigma_{w\text{-}\overline{\lim}\, F_n(\omega)}(u(\omega))d\mu(\omega) = \sigma_{S^1_{w\text{-}\overline{\lim}\, F_n}}(u).$$

So finally we have that:

$$\overline{\lim}\, \sigma_{S^1_{F_n}}(u) \leq \sigma_{S^1_{w\text{-}\overline{\lim}\, F_n}}(u).$$

Since this is true for every $u(\cdot) \in L^\infty_{X^*_{w^*}} = (L^1_X)^*$, from proposition 4.1 we conclude that

$$w\text{-}\overline{\lim}\, S^1_{F_n} \subseteq \overline{\text{conv}}\, S^1_{w\text{-}\overline{\lim}\, F_n}.$$

If in addition, $w\text{-}\overline{\lim}\, F_n(\cdot)$ is $P_{fc}(X)$-valued, then $S^1_{w\text{-}\overline{\lim}\, F_n}$ is convex and of course closed and so $w\text{-}\overline{\lim}\, S^1_{F_n} \subseteq S^1_{w\text{-}\overline{\lim}\, F_n}$.

Q.E.D.

Combining theorems 4.1 and 4.2 we can have a dominated convergence theorem for the sequence $\{S^1_{F_n}\}_{n\geq 1}$. Our assumptions on (Ω,Σ,μ) and X remain as before.

THEOREM 4.4. <u>If</u> $F_n : \Omega \to P_{fc}(X)$ are measurable multifunctions s.t. for all $n \geq 1$ $F_n(\omega) \subseteq G(\omega)\mu$-a.e. where $G : \Omega \to P_{wkc}(X)$ is integrably bounded and $F_n(\omega) \xrightarrow{K-M} F(\omega)\mu$-a.e.

<u>then</u> $S^1_{F_n} \xrightarrow{K-M} S^1_F$.

PROOF. Note that because for all $n \geq 1$ $F_n(\omega) \subseteq G(\omega)\mu$-a.e. with $G(\omega) \in P_{wkc}(X)$ $w\text{-}\overline{\lim}\, F_n(\omega) \neq \phi$ μ-a.e. But $w\text{-}\overline{\lim}\, F_n(\omega) = F(\omega) \neq \phi$ μ-a.e. Also since $s\text{-}\underline{\lim}\, F_n(\omega) = F(\omega)\mu$-a.e., we have that $F(\omega) \in P_{fc}(X)\mu$-a.e. and $\omega \to F(\omega)$ is measurable (recall $\mu(\cdot)$ is complete). So using theorems 4.1 and 4.2, it is easy to see that $S^1_{F_n} \xrightarrow{K-M} S^1_F$.

Q.E.D.

We would like to have such a dominated convergence theorem for the Hausdorff mode of convergence. In this direction we have the following rusult. The spaces (Ω,Σ,μ) and X remain as before.

THEOREM 4.5. If $F_n : \Omega \to P_{fc}(X)$ are measurable multifunctions s.t. $\{|F_n(\cdot)|\}_{n\geq 1}$ is uniformly integrable and $F_n(\omega) \xrightarrow{h} F(\omega)$ in measure then $F : \Omega \to P_{fc}(X)$ is integrably bounded and $S^1_{F_n} \xrightarrow{h} S^1_F$.

PROOF. First note that since $(P_{fc}(X), h)$ is a complete, metric space, we have that $F(\omega) \in P_{fc}(X)$ μ-a.e. By modifying $F(\cdot)$ on a μ-null set we can have $F(\omega) \in P_{fc}(X)$ for all $\omega \in \Omega$ and since $\mu(\cdot)$ is complete, the modified multifunction is still going to be measurable. Also from the properties of the Hausdorff metric we have that $\big||F_n(\omega)| - |F(\omega)|\big| \leq h(F_n(\omega), F(\omega))$ μ-a.e. $\Rightarrow |F_n(\omega)| \to |F(\omega)|$ in measure and since by hypothesis $\{|F_n(\ )|\}_{n\geq 1}$ is uniformly integrable, we deduce that $|F(\cdot)| \in L^1_+$ i.e. $F(\cdot)$ is integrably bounded as claimed by the theorem.

Next note that $\{S^1_{F_n}, S^1_F\}_{n\geq 1}$ are convex, closed and bounded subsets of $L^1_X(\Omega)$. So recalling that $(L^1_X)^* = L^\infty_{X^*_{w^*}}$ and using Hörmander's formula we have that

$$h(S^1_{F_n}, S^1_F) = \sup_{\|u\|_\infty \leq 1} |\sigma_{S^1_{F_n}}(u) - \sigma_{S^1_F}(u)|$$

$$= \sup_{\|u\|_\infty \leq 1} \Big| \int_\Omega (\sigma_{F_n(\omega)}(u(\omega)) - \sigma_{F(\omega)}(u(\omega)))d\mu(\omega) \Big|$$

$$\leq \sup_{\|u\|_\infty \leq 1} \int_\Omega |\sigma_{F_n(\omega)}(u(\omega)) - \sigma_{F(\omega)}(u(\omega))| d\mu(\omega)$$

$$= \int_\Omega \sup_{\|x^*\| \leq 1} |\sigma_{F_n(\omega)}(x^*) - \sigma_{F(\omega)}(x^*)| d\mu(\omega)$$

$$= \int_\Omega h(F_n(\omega), F(\omega)) d\mu(\omega).$$

Since by hypothesis $\{|F_n(\cdot)|\}_{n\geq 1}$ is uniformly integrable and $F_n(\omega) \xrightarrow{h} F(\omega)$ in measure then $\int_\Omega h(F_n(\omega), F(\omega))d\mu(\omega) \to 0 \Rightarrow h(S^1_{F_n}, S^1_F) \to 0$.

Q.E.D.

We will conclude our work with an important observation about the Kuratowski-Mosco convergence of closed, convex sets. It is a very useful necessary condition for K - M convergence of such sets.

Assume that X is a reflexive Banach space.

THEOREM 4.6. If $\{A_n\}_{n\geq 1} \subseteq P_{fc}(X)$, $\sup_{n\geq 1} |A_n| < \infty$ and $A_n \xrightarrow{K-M} A$

then $A \neq \phi$ and for all $x^* \in X^*$, $\sigma_{A_n}(x^*) \to \sigma_A(x^*)$.

PROOF. Let $M = \sup_{n\geq 1} |A_n|$ and let $B_M(0)$ be the M-ball centered at the origin. Then $B_M(0)$ is weakly compact and by the Eberlein-Smulian theorem sequentially w-compact. Let $x_n \in A_n$, $n \geq 1$. Then $\{x_n\}_{n\geq 1} \subseteq B_m(0)$ and so we can find a subsequence $x_k \xrightarrow{w} x$. Then $x \in w\text{-}\overline{\lim}\, A_n = A \Rightarrow A \neq \phi$. Next fix $x^* \in X^*$ and let

$x_n \in A_n$ s.t. $(x^*, x_n) = \sigma_{A_n}(x^*)$. By passing to an appropriate subsequence $\{x_k\}_{k\geq 1}$ we can assume that $(x^*, x_k) \to \overline{\lim}\, \sigma_{A_n}(x^*)$ and $x_k \xrightarrow{w} x \in A$. Then $(x^*, x) \leq \sigma_A(x^*) \Rightarrow \overline{\lim}\, \sigma_{A_n}(x^*) \leq \sigma_A(x^*)$. On the other hand from Mosco [6] we know that $\sigma_{A_n}(\cdot) \xrightarrow{\tau} \sigma_A(\cdot)$ i.e. epi $\sigma_{A_n} \xrightarrow{K-M}$ epi $\sigma_A(\cdot)$ and this implies that $\underline{\lim}\, \sigma_{A_n}(x^*) \geq \sigma_A(x^*)$ [6] and [7]. So finally we have that $\sigma_{A_n}(\cdot) \to \sigma_A(\cdot)$.

Q.E.D.

REMARK. The converse of the above result is not true. Namely pointwise convergence of the support functions does not imply the Kurtowski-Mosco convergence of the corresponding closed, convex sets. Here is a counter example. Let $\{x_n\}_{n\geq 1} \subseteq X$ and assume that $x_n \xrightarrow{w} x$ but it does not converge strongly. So $\{x_n\}$ do not converge to $\{x\}$ in the $K - M$ sense. On the other hand for every $x^* \in X^*$ $(x^*, x_n) = \sigma_{\{x_n\}}(x^*) \to (x^*, x) = \sigma_{\{x\}}(x^*)$. So in corollary 2E of [10], it must be added that X is finite dimensional or otherwise the result is not true as the previous counterexample illustrated.

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