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TWO DIMENSIONAL LAPLACE TRANSFORMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

R.S. DAHIYA

Department of Mathematics
Iowa State University
Ames, Iowa 50011

I.H. JOWHAR

Department of Mathematics
Florida State University
Tallahassee, Florida 32306

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ABSTRACT. The object of this paper is to obtain new operational relations between the original and the image functions that involve generalized hypergeometric G-functions.

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1. INTRODUCTION.

The integral equation

$$\phi(p,q) = pq \int_0^\infty \int_0^\infty \exp(-px-qy) f(x,y)\, dy\, dx, \quad \operatorname{Re}(p,q) > 0 \tag{1.1}$$

represents the classical Laplace transform of two variables and the functions $\phi(p,q)$ and $f(x,y)$ related by (1.1), are said to be operationally related to each other. $\phi(p,q)$ is called the image and $f(x,y)$ the original.

Symbolically we can write

$$\phi(p,q) \doteqdot f(x,y) \quad \text{or} \quad f(x,y) \doteqdot \phi(p,q), \tag{1.2}$$

and the symbol $\doteqdot$ is called "operational".

Meijer's G-function [5] is defined by a Mellin-Barnes type integral

$$G_{u,v}^{m,n}\left(z \,\middle|\, \begin{matrix} a_u \\ b_v \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{u} \Gamma(1-b_j+s) \prod_{j=n+1}^{v} \Gamma(a_j - s)} z^s\, ds \tag{1.3}$$

where m, n, u, v are integers with $v \geq 1$; $0 \leq n \leq u$; $0 \leq m \leq v$, the parameters a_j and b_j are such that no poles of $\Gamma(b_j-s)$; $j = 1,2,\ldots,m$ coincides with any pole of $\Gamma(1-a_k+s)$; $k = 1,2,\ldots,n$. Thus (a_k-b_j) is not a positive integer. The path L goes from $-i\infty$ to $+i\infty$ so that all poles of integrand must be simple and those of $\Gamma(b_j-s)$; $j = 1,2,\ldots,m$ lie on one side of the contour L and those of $\Gamma(1-a_k+s)$; $k = 1,2,\ldots,n$ must lie on the other side. The integrand converges if $u+v < 2(m+n)$ and $|\arg z| < (m + n - \frac{1}{2} u - \frac{1}{2} v)\pi$. For sake of brevity a_u denotes $a_1, a_2, \ldots, a_u$.

In the present paper, we propose to establish a couple of formulae for calculating Laplace transform pairs of two dimensions that involve Meijer's G-function.

2. THE MAIN RESULTS.

(i) $\bar{\delta} = m + n - \frac{1}{2}(u+v) > 0, \quad |\arg \alpha| < \bar{\delta}\pi,$

(ii) $0 < n < u, \quad 0 < m < v, \quad v > 1,$

(iii) $\mathrm{Re}(b_j+\xi) > 0, \quad j = 1,2,\ldots,m$

(iv) $\mathrm{Re}\left(a_k + \xi - \frac{\sigma}{2} - \frac{r}{2}\right) < 0, \quad k = 1,2,\ldots,n,$

(v) a_k-b_k is not a positive integer, $j = 1,2,\ldots,m, \quad k = 1,2,\ldots,n,$

(vi) r represents the non-negative integer, $0,1,2,3,\ldots,$ then

$$x^r(xy)^{\sigma-\xi-1}\, G^{m,n+1}_{u+2,v}\left(\frac{\alpha}{xy}\,\middle|\,\begin{matrix}1-\xi,\ a_u,\ \sigma+r-\xi\\ b_v\end{matrix}\right)$$

$$\doteqdot p^{-r}(pq)^{\xi-\sigma+1}\, G^{m+1,n+1}_{u+1,v+1}\left(\alpha\, pq\,\middle|\,\begin{matrix}1-\xi,\ a_u\\ \sigma-\xi,\ b_v\end{matrix}\right) \tag{2.1}$$

and

$$x^{\delta-r-1}\left(\frac{x}{y}\right)^{\xi-\sigma+1} G^{m,n+1}_{u+2,v}\left(\frac{\alpha x}{y}\,\middle|\,\begin{matrix}1-\xi,\ a_u,\ \sigma+r-\xi\\ b_v\end{matrix}\right)$$

$$\doteqdot p^{r-\delta-1}\left(\frac{q}{p}\right)^{\xi-\sigma+1} G^{m+1,n+2}_{u+3,v+1}\left(\frac{\alpha q}{p}\,\middle|\,\begin{matrix}\sigma-\xi+r-\delta,\ 1-\xi,\ a_u,\ \sigma-\xi+r\\ \sigma-\xi,\ b_v\end{matrix}\right) \tag{2.2}$$

valid under the conditions:

(a) $\mathrm{Re}(\delta + \xi - \sigma - r + b_j) > -1, \quad j = 1,2,\ldots,m,$

(b) $\mathrm{Re}\left(\frac{\delta}{2} + \xi - \sigma - r + a_k\right) < \frac{1}{4}, \quad k = 1,2,\ldots,n,$

(c) along with (i), (ii), (v) and (vi).

From (2.1) and (2.2), we propose to prove the following relations.

$$x^r(xy)^{\sigma-1}\, E\left(b_1,\ldots,b_v : a_1,\ldots,a_u,\sigma+r : \frac{\alpha}{xy}\right)$$

$$\doteqdot p^{-r}(pq)^{1-\sigma}\, E\left(\sigma,b_1,\ldots,b_v : a_1,\ldots a_u : \alpha\, pq\right), \tag{2.3}$$

where $\mathrm{Re}(\sigma) > 0$, $v > u+1$ and r is a positive integer.

$$x^{\delta-r-1}\left(\frac{x}{y}\right)^{1-\sigma} E\left(\xi+b_1,\ldots,\xi+b_v : \xi+a_1,\ldots,\xi+a_u,\ \sigma+r : \frac{x}{y}\right)$$

$$\doteqdot pq^{r-\delta}\, E\big(1+\delta-r,\ 1+\xi+\delta-\sigma-r+b_1,\ \ldots,\ 1+\xi+\delta-\sigma-r+b_v :$$

$$2+\delta-\sigma-r,\ 1+\xi+\delta-\sigma-r+a_1,\ \ldots,\ 1+\xi+\delta-\sigma-r+a_u,\ 1+\delta : \frac{q}{p}\big) \tag{2.4}$$

valid under the same conditions as (2.3).

The function appearing in (2.3) and (2.4) is MacRobert's E-function, whose properties are given in [6], pp. 433-434, and [8].

PROOF: The generalized Stieltjes transform of a G-function is given by (see [6], p. 237)

$$\int_0^\infty \frac{x^{\xi-1}}{(x+\beta)^\sigma} G^{m,n}_{p,q}\left(\alpha x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dx = \frac{\beta^{\xi-\sigma}}{\Gamma(\sigma)} G^{m+1,n+1}_{p+1,q+1}\left(\alpha\beta \,\middle|\, \begin{matrix} 1-\xi,\ a_p \\ \sigma-\xi,\ b_q \end{matrix}\right) \tag{2.5}$$

On writing pq for β and multiplying both the sides of (2.5) by $p^{1-r}q$, we have

$$\int_0^\infty \frac{p^{1-r}q}{(pq+t)^\sigma} \cdot t^{\xi-1} G^{m,n}_{u,v}\left(\alpha t \,\middle|\, \begin{matrix} a_u \\ b_v \end{matrix}\right) dt$$

$$= \frac{(pq)^{\xi-\sigma+1}\, p^{-r}}{\Gamma(\sigma)} G^{m+1,n+1}_{u+1,v+1}\left(\alpha pq \,\middle|\, \begin{matrix} 1-\xi,\ a_u \\ \sigma-\xi,\ b_v \end{matrix}\right) \tag{2.6}$$

Now interpreting with the help of the known result ([4], result (2.83), p. 137), it follows

$$\frac{y^{\sigma-1}}{\Gamma(\sigma)} \left(\frac{x}{y}\right)^{\frac{\sigma+r-1}{2}} \int_0^\infty t^{\xi-\frac{\sigma}{2}-\frac{r}{2}-\frac{1}{2}} J_{\sigma+r-1}(2\sqrt{txy}) G^{m,n}_{u,v}\left(\alpha t \,\middle|\, \begin{matrix} a_u \\ b_v \end{matrix}\right) dt$$

$$\doteqdot \frac{(pq)^{\xi-\sigma+1}}{\Gamma(\sigma)p^r} G^{m+1,n+1}_{u+1,v+1}\left(\alpha pq \,\middle|\, \begin{matrix} 1-\xi,\ a_u \\ \sigma-\xi,\ b_v \end{matrix}\right). \tag{2.7}$$

Evaluating the left-hand side integral of (2.7), we get

$$x^r (xy)^{\sigma-\xi-1} G^{m,n+1}_{u+2,v}\left(\frac{\alpha}{xy} \,\middle|\, \begin{matrix} 1-\xi,\ a_u,\ \sigma+r-\xi \\ b_v \end{matrix}\right)$$

$$\doteqdot p^{-r}(pq)^{\xi-\sigma-1} G^{m+1,n+1}_{u+1,v+1}\left(\alpha pq \,\middle|\, \begin{matrix} 1-\xi,\ a_u \\ \sigma-\xi,\ b_v \end{matrix}\right). \tag{2.8}$$

The following result will be used in the proof of (2.2).

If $F(p,q) \doteqdot f(x,y)$, then

$$x^{\delta-1} f\left(\frac{1}{x}, y\right) \doteqdot \int_0^\infty \left(\frac{\lambda}{p}\right)^{\frac{\delta}{2}-1} J_\delta(2\sqrt{p\lambda}) F(\lambda, q)\, d\lambda. \tag{2.9}$$

From (2.8) and (2.9), we have

$$x^{\delta-r-\sigma+\xi}\, y^{\sigma-\xi-1} G^{m,n+1}_{u+2,v}\left(\frac{\alpha x}{y} \,\middle|\, \begin{matrix} 1-\xi,\ a_u,\ \sigma+r-\xi \\ b_v \end{matrix}\right)$$

$$\doteqdot \frac{q^{\xi-\sigma+1}}{p^{\delta/2-1}} \int_0^\infty \lambda^{\frac{\delta}{2}+\xi-\sigma-r} J_\delta(2\sqrt{p\lambda}) G^{m+1,n+1}_{u+1,v+1}\left(\alpha q\lambda \,\middle|\, \begin{matrix} 1-\xi,\ a_u \\ \sigma-\xi,\ b_v \end{matrix}\right) d\lambda. \tag{2.10}$$

On evaluating the right hand side integral and after some simplification, we obtain the desired result (2.2).

In (2.1) and (2.2), reducing the Meijer's G-function to MacRobert's E-function to obtain (2.3) and (2.4).

3. PARTICULAR CASES.

On specializing the parameters, the G-function can be reduced to MacRobert's E-function, generalized hypergeometric function and other higher transcendental functions. Therefore, the results (2.1) and (2.2) leads to many new operational relations not listed in [4,9,2,3] and other literature.

3(a). Named image functions expressed in terms of the G-function.

The following results are obtained by using the known results from [7] on pages 226-334.

$$p^{-r}(pq)^{\frac{1}{2}(\xi-\sigma+b+1)} \exp\left(\frac{pq}{2}\right) W_{\frac{1}{2}(1-\xi-\sigma-b),\ \frac{1}{2}(\sigma-\xi-b)}(pq)$$

$$\doteqdot [\Gamma(\sigma)\Gamma(b+\xi)]^{-1} x^r (xy)^{\sigma-\xi-1} G^{1,1}_{2,1}\left(\frac{1}{xy}\ \middle|\ {1-\xi,\ \sigma+r-\xi \atop b}\right), \tag{3.1}$$

$$\mathrm{Re}(\sigma) > 0, \quad \mathrm{Re}(b+\xi) > 0.$$

$$p^{-r}(pq)^{\frac{3}{2}-\sigma} \exp\left(\frac{pq}{2}\right) K_{\frac{1}{2}(\sigma-\xi-b)}\left(\frac{pq}{2}\right)$$

$$\doteqdot \pi^{-1/2} \cos\left(\frac{\sigma-\xi-b}{2}\right)\pi\, x^r (xy)^{\sigma-\xi-1} G^{1,1}_{2,1}\left(\frac{1}{xy}\ \middle|\ {1-\xi,\ \sigma+r-\xi \atop b}\right) \tag{3.2}$$

$$p^{-r}(pq)^{\xi}[H_{1-2\xi}(2\sqrt{pq}) - Y_{1-2\xi}(2\sqrt{pq})]$$

$$\doteqdot \pi^{-2} \cos 2\pi\left(\frac{1}{2} - \xi\right) x^r (xy)^{-\xi} G^{2,1}_{2,2}\left(\frac{1}{xy}\ \middle|\ {1-\xi,\ 1+r-\xi \atop \xi-\frac{1}{2},\ \frac{1}{2}-\xi}\right) \tag{3.3}$$

$$p^{-r}(pq)^{\frac{1}{2}(b+\xi)+\frac{3}{4}} [I_{\frac{1}{2}-\xi-b}(2\sqrt{pq}) - L_{\frac{1}{2}-\xi-b}(2\sqrt{pq})]$$

$$\doteqdot \pi^{-1} x^r (xy)^{-\frac{1}{2}-\xi} G^{1,1}_{2,2}\left(\frac{1}{xy}\ \middle|\ {1-\xi,\ \frac{1}{2}+r-\xi \atop 1-\xi,\ b}\right) \tag{3.4}$$

$$p^{-r}(pq)^{\xi} S_{1-2\xi,2b}(2\sqrt{pq}) \doteqdot 2^{-2\xi} [\Gamma(\xi-b)\Gamma(\xi+b)]^{-1} x^r (xy)^{-\xi}$$

$$\cdot\ G^{2,1}_{2,2}\left(\frac{1}{xy}\ \middle|\ {1-\xi,\ 1+r-\xi \atop b,\ -b}\right) \tag{3.5}$$

$$p^{-r}(pq)^{\frac{3}{2}-\sigma} H^{(1)}_{\sigma-\frac{1}{2}}(\sqrt{pq}) H^{(2)}_{\sigma-\frac{1}{2}}(\sqrt{pq}) \doteqdot 2\pi^{-5/2} \cos\left(\sigma-\frac{1}{2}\right)\pi\, x^r (xy)^{\sigma-\xi-1}$$

$$\cdot\ G^{2,1}_{2,2}\left(\frac{1}{xy}\ \middle|\ {1-\xi,\ \sigma+r-\xi \atop 1-\sigma-\xi,\ \frac{1}{2}-\xi}\right) \tag{3.6}$$

$$p^{-r}\,(pq)^{\frac{1}{2}-m}\,W_{k,m}(2i\sqrt{pq})W_{K,m}(-2i\sqrt{pq})$$

$$\doteq \frac{x^{r}(xy)^{m+\frac{a}{2}-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-K+m)\Gamma(\frac{1}{2}-K-m)} \cdot G^{3,1}_{3,3}\left(\frac{1}{xy}\,\middle|\,\begin{matrix}1+\frac{a}{2}+K,\ 1+\frac{a}{2}-K,\ m+\frac{a}{2}+r+\frac{1}{2}\\ \frac{a+1}{2}-m,\ \frac{a}{2}+1,\ \frac{a+1}{2}\end{matrix}\right) \tag{3.7}$$

$$p^{-r}(pq)^{1-\frac{u}{2}-\frac{v}{2}}\left[e^{i\pi(v-u)/2}\,H^{(1)}_{v}(\sqrt{pq})\,H^{(2)}_{u}(\sqrt{pq}) - e^{i\pi(u-v)/2}\,H^{(1)}_{u}(\sqrt{pq})\,H^{(2)}_{v}(\sqrt{pq})\right]$$

$$\doteq 2\pi^{-5/2}(\cos u\pi - \cos v\pi)x^{r}(xy)^{\frac{1}{2}(u+v+w)-1} \cdot G^{3,1}_{3,3}\left(\frac{1}{xy}\,\middle|\,\begin{matrix}\frac{w}{2},\ \frac{w+1}{2},\ \frac{w+u+v}{2}+r\\ \frac{w-u+v}{2},\ \frac{w+u-v}{2},\ \frac{w-u-v}{2}\end{matrix}\right) \tag{3.8}$$

The following four results are obtained from (2.1) by using Carlson's results [1]:

$$p^{1-r}u[\psi(1-2a,1-2c;2i\sqrt{pq}) + \psi(1-2a,1-2c;-2i\sqrt{pq})]$$

$$\doteq \frac{2^{2c-2a}\,x^{r-1}\,y^{-1}}{\sqrt{\pi}\,\Gamma(1-2a)\,\Gamma(1+2c-2a)}\,G^{2,2}_{3,3}\left(\frac{1}{xy}\,\middle|\,\begin{matrix}a,\ a+\frac{1}{2},\ r\\ c,\ c+\frac{1}{2},\ \frac{1}{2}\end{matrix}\right) \tag{3.9}$$

$$\sqrt{pq}\;p^{-r}[\psi(1-2a,1-2c;2i\sqrt{pq}) - \psi(1-2a,1-2c;-2i\sqrt{pq})]$$

$$\doteq \frac{-i\,2^{2c-2a}\,x^{r-\frac{1}{2}}\,y^{-\frac{1}{2}}}{\Gamma(1-2a)\,\Gamma(1+2c-2a)}\,G^{2,2}_{3,3}\left(\frac{1}{xy}\,\middle|\,\begin{matrix}a,\ a+\frac{1}{2},\ r+\frac{1}{2}\\ c,\ c+\frac{1}{2},\ 0\end{matrix}\right) \tag{3.10}$$

$$p^{-r}(pq)^{1+c}[\exp(2i\sqrt{pq}+ic\pi)\Gamma(-2c,2i\sqrt{pq}) + \exp(-2i\sqrt{pq}-ic\pi)\Gamma(-2c,-2i\sqrt{pq})]$$

$$\doteq \pi^{-\frac{1}{2}}\,[\Gamma(1+2c)]^{-1}\,x^{r-1}\,y^{-1}\,G^{2,1}_{2,2}\left(\frac{1}{xy}\,\middle|\,\begin{matrix}0,\ r\\ c,\ c+\frac{1}{2}\end{matrix}\right) \tag{3.11}$$

where $\Gamma(a,x)$ denotes incomplete gamma function.

$$p^{-r}[\psi(2,2-2c;2i\sqrt{pq}) - \psi(2,2-2c;-2i\sqrt{pq})]$$

$$\doteq -i\pi^{-\frac{1}{2}}\,2^{2c}[\Gamma(1+2c)]^{-1}\,x^{r}\,G^{2,1}_{2,2}\left(\frac{1}{xy}\,\middle|\,\begin{matrix}0,\ 1+r\\ c,\ c+\frac{1}{2}\end{matrix}\right) \tag{3.12}$$

In the last six relations, $G^{3,1}_{3,3}$ or $G^{2,2}_{3,3}$ provides an interpretation for the symbol ${}_3F_2$, as does ψ for ${}_2F_0$. So does $G^{2,1}_{2,2}$ or $G^{1,1}_{2,2}$ for ${}_2F_1$.

3(b). The G-function expressed as a named original function.

The following results are obtained from (2.1) by using the known results [7] on pages 228-234.

$$p^{\sigma-2K-1}(pq)^{K-\sigma-\frac{a}{2}}\, G^{4,1}_{1,5}\left(pq \,\Big|\, \begin{matrix} \frac{a}{2}-K+1 \\ \sigma-K+\frac{a}{2},\ \frac{a+1}{2}+m,\ \frac{a}{2}+1,\ \frac{a+1}{2},\ \frac{a+1}{2}-m \end{matrix}\right)$$

$$\doteqdot \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}+K+m)}{\Gamma(2m+1)}\, x^{2K-\sigma+1}(xy)^{\sigma-K-1} M_{K,m}\left(\frac{2}{\sqrt{xy}}\right) W_{-K,m}\left(\frac{2}{\sqrt{xy}}\right) \tag{3.13}$$

$$p^{2K+\sigma-1}(pq)^{1-\sigma-K-\frac{a}{2}}\, G^{5,1}_{1,5}\left(pq \,\Big|\, \begin{matrix} \frac{a}{2}+K+1 \\ \sigma+K+\frac{a}{2},\ \frac{a+1}{2}+m,\ \frac{a+1}{2}-m,\ \frac{a}{2}+1,\ \frac{a+1}{2} \end{matrix}\right)$$

$$\doteqdot \sqrt{\pi}\,\Gamma(\tfrac{1}{2}-K+m)\,\Gamma(\tfrac{1}{2}-K-m)\, x^{1-\sigma-2K}(xy)^{\sigma+K-1} \cdot W_{K,m}\left(\frac{2i}{\sqrt{xy}}\right) W_{K,m}\left(\frac{-2i}{\sqrt{xy}}\right) \tag{3.14}$$

$$p^{\sigma-\frac{3}{2}}(pq)^{2-\sigma-\frac{w}{2}}\, G^{5,1}_{1,5}\left(pq \,\Big|\, \begin{matrix} \frac{w}{2} \\ \sigma+\frac{w}{2}-1,\ \frac{w+u+v}{2},\ \frac{w-u+v}{2},\ \frac{w+u-v}{2},\ \frac{w-u-v}{2} \end{matrix}\right)$$

$$\doteqdot \frac{-i\,\pi^{5/2}}{2(\cos u\pi - \cos v\pi)}\, x^{\frac{3}{2}-\sigma}(xy)^{\sigma-2} \cdot \left[e^{i\pi(v-u)/2} H^{(1)}_{v}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{u}\left(\frac{1}{\sqrt{xy}}\right) - e^{i\pi(u-v)/2} H^{(1)}_{u}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{v}\left(\frac{1}{\sqrt{xy}}\right)\right] \tag{3.15}$$

$$p^{\sigma-\frac{1}{2}}(pq)^{\frac{3}{2}-\frac{w}{2}-\sigma}\, G^{5,1}_{1,5}\left(pq \,\Big|\, \begin{matrix} \frac{w+1}{2} \\ \sigma+\frac{w-1}{2},\ \frac{w+u+v}{2},\ \frac{w-u+v}{2},\ \frac{w+u-v}{2},\ \frac{w-u-v}{2} \end{matrix}\right)$$

$$\doteqdot \frac{\pi^{\frac{5}{2}}\, x^{\frac{1}{2}-\sigma}}{2(\cos u\pi + \cos v\pi)}(xy)^{\sigma-\frac{3}{2}} \left[e^{i\pi(u-v)/2} H^{(1)}_{v}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{u}\left(\frac{1}{\sqrt{xy}}\right) + e^{i\pi(u-v)/2} H^{(1)}_{u}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{v}\left(\frac{1}{\sqrt{xy}}\right)\right] \tag{3.16}$$

$$p^{\sigma-\frac{3}{4}}(pq)^{2-a-\sigma}\, G^{3,2}_{1,5}\left(pq \,\Big|\, \begin{matrix} a \\ \sigma+a-b,\ b,\ c,\ 2a-c,\ 2a-b \end{matrix}\right)$$

$$\doteqdot 2\sqrt{\pi}\, x^{\frac{3}{2}-\sigma}(xy)^{\sigma-2}\, I_{b+c+2a}\left(\frac{1}{\sqrt{xy}}\right) K_{b-c}\left(\frac{1}{\sqrt{xy}}\right) \tag{3.17}$$

$$p^{\sigma-\frac{3}{2}} (pq)^{2-\sigma} G^{5,1}_{1,5} \left(pq \,\middle|\, \begin{matrix} 0 \\ \sigma-1,\ a,\ b,\ -b,\ -a \end{matrix}\right)$$

$$\doteq \frac{i\,\pi^{5/2}}{4 \sin a\pi \sin b\pi} x^{\frac{3}{2}-\sigma} (xy)^{\sigma-2} \cdot \Big[e^{-i\pi b} H^{(1)}_{a-b}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{a+b}\left(\frac{1}{\sqrt{xy}}\right)$$

$$- e^{i\pi b} H^{(1)}_{a+b}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{a-b}\left(\frac{1}{\sqrt{xy}}\right)\Big] \tag{3.18}$$

$$p^{\sigma-\frac{1}{2}} (pq)^{\frac{3}{2}-\sigma} G^{5,1}_{1,5} \left(pq \,\middle|\, \begin{matrix} \frac{1}{2} \\ \sigma-\frac{1}{2},\ a,\ b,\ -b,\ -a \end{matrix}\right)$$

$$\doteq \frac{\pi^{\frac{5}{2}} x^{\frac{1}{2}-\sigma}}{4 \cos a\pi \cos b\pi} (xy)^{\sigma-\frac{3}{2}} \cdot \Big[e^{-i\pi b} H^{(1)}_{a-b}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{a+b}\left(\frac{1}{\sqrt{xy}}\right)$$

$$- e^{i\pi b} H^{(1)}_{a+b}\left(\frac{1}{\sqrt{xy}}\right) H^{(2)}_{a-b}\left(\frac{1}{\sqrt{xy}}\right)\Big] \tag{3.19}$$

$$p^{\sigma+2a-1} (pq)^{\frac{3}{2}-a-\sigma} G^{5,1}_{1,5} \left(pq \,\middle|\, \begin{matrix} \frac{1}{2}+a \\ \sigma+a-\frac{1}{2},\ 0,\ \frac{1}{2},\ b,\ -b \end{matrix}\right)$$

$$\doteq \sqrt{\pi}\, \Gamma\left(\frac{1}{2}+b-a\right) \Gamma\left(\frac{1}{2}-b-a\right) x^{1-2a-\sigma} (xy)^{\sigma+a-1} W_{a,b}\left(\frac{2i}{\sqrt{xy}}\right) W_{a,b}\left(\frac{-2i}{\sqrt{xy}}\right) \tag{3.20}$$

The following three results are obtained by using Carlson's results [1] on page 239 in (2.1).

$$p^{\sigma-\frac{3}{2}} (pq)^{2-\sigma} G^{3,1}_{1,5} \left(pq \,\middle|\, \begin{matrix} 0 \\ \sigma-1,\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2} \end{matrix}\right)$$

$$\doteq \frac{\Gamma(1+2c)\, \pi^{-\frac{1}{2}} x^{\frac{3}{2}-\sigma}}{\Gamma(1+2c-2d)\, 2^{2d+1}} (xy)^{\sigma-c-2} \Big[e^{ic\pi} {}_1F_1\left(1+2c;\ 1+2c-2d;\ \frac{-2i}{\sqrt{xy}}\right)$$

$$+ e^{-ic\pi} {}_1F_1\left(1+2c;\ 1+2c-2d;\ \frac{2i}{\sqrt{xy}}\right)\Big] \tag{3.21}$$

$$p^{\sigma-\frac{3}{2}} (pq)^{2-\sigma} G^{5,1}_{1,5} \left(pq \,\middle|\, \begin{matrix} 0 \\ \sigma-1,\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2} \end{matrix}\right)$$

$$\doteq \sqrt{\pi}\, 2^{-2d}\, \Gamma(1+2c)\Gamma(1+2d) \cdot x^{\frac{3}{2}-\sigma} (xy)^{\sigma-c-2} \Big[e^{ic\pi} \psi\left(1+2c,\ 1+2c-2d;\ \frac{2i}{\sqrt{xy}}\right)$$

$$+ e^{-ic\pi} \psi\left(1+2c,\ 1+2c-2d;\ \frac{-2i}{\sqrt{xy}}\right)\Big] \tag{3.22}$$

$$p^{\sigma-\frac{1}{2}}(pq)^{\frac{3}{2}-\sigma}\,G^{5,1}_{1,5}\left(pq \,\middle|\, \begin{matrix}\frac{1}{2}\\ \sigma-\frac{1}{2},\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2}\end{matrix}\right)$$

$$\doteqdot i\,2^{-2d}\sqrt{\pi}\,\Gamma(1+2c)\Gamma(1+2d)\,x^{\frac{1}{2}-\sigma}(xy)^{\sigma-c-\frac{3}{2}}$$

$$\cdot\left[e^{ic\pi}\psi\left(1+2c,\ 1+2c-2d;\ \frac{2i}{\sqrt{xy}}\right)-e^{-ic\pi}\psi\left(1+2c,\ 1+2c-2d;\ \frac{-2i}{\sqrt{xy}}\right)\right] \qquad (3.23)$$

3(c). Particular cases of (2.2).

The following results are obtained by using the known results from [7] on pages 228-230 in (2.2).

$$p^{2\kappa-\sigma-\delta+2}\left(\frac{q}{p}\right)^{\kappa-\sigma-\frac{a}{2}+1}G^{4,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}\kappa+\frac{a}{2}-\delta+1,\ 1-\kappa+\frac{a}{2},\ \kappa+\frac{a}{2}+1\\ \sigma-\kappa+\frac{a}{2},\ \frac{a+1}{2}+m,\ \frac{a}{2}+1,\ \frac{a+1}{2}-m\end{matrix}\right)$$

$$\doteqdot \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\kappa+m)}{\Gamma(1+2m)}\,x^{\delta+\sigma-2\kappa-2}\left(\frac{x}{y}\right)^{\kappa-\sigma+1}M_{\kappa,m}\left(2\sqrt{\frac{x}{y}}\right)W_{-\kappa,m}\left(2\sqrt{\frac{x}{y}}\right) \qquad (3.24)$$

$$p^{2-2\kappa-\sigma-\delta}\left(\frac{q}{p}\right)^{1-\sigma-\kappa-\frac{a}{2}}G^{5,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}1+\kappa-\delta+\frac{a}{2},\ 1+\kappa+\frac{a}{2},\ a-\kappa+\frac{a}{2}\\ \sigma+\kappa+\frac{a}{2},\ \frac{a+1}{2}+m,\ \frac{a+1}{2}-m,\ \frac{a+1}{2},\ \frac{a}{2}+1\end{matrix}\right)$$

$$\doteqdot \sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\kappa+m\right)\Gamma\left(\frac{1}{2}-\kappa-m\right)x^{\delta+\kappa-1}\,y^{\kappa+\sigma-1}\,W_{\kappa,m}\left(2i\sqrt{\frac{x}{y}}\right)W_{\kappa,m}\left(-2i\sqrt{\frac{x}{y}}\right) \qquad (3.25)$$

$$p^{\frac{3}{2}-\sigma-\delta}\left(\frac{q}{p}\right)^{\frac{3}{2}-\frac{w}{2}-\sigma}G^{3,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}\frac{w}{2}-\delta,\ \frac{1+w}{2},\ \frac{w}{2}\\ \sigma+\frac{w}{2}-\frac{1}{2},\ \frac{w+u+v}{2},\ \frac{w-u-v}{2},\ \frac{w-u+v}{2},\ \frac{w+u-v}{2}\end{matrix}\right)$$

$$\doteqdot \sqrt{\pi}\left[2\cos(u+v)\frac{\pi}{2}\right]^{-1}x^{\delta}y^{\sigma-\frac{3}{2}}\left[J_u\left(\sqrt{\frac{x}{y}}\right)J_v\left(\sqrt{\frac{x}{y}}\right)+J_{-u}\left(\sqrt{\frac{x}{y}}\right)J_{-v}\left(\sqrt{\frac{x}{y}}\right)\right] \qquad (3.26)$$

$$p^{\frac{5}{2}-\sigma-\delta}\left(\frac{q}{p}\right)^{2-\sigma-\frac{w}{2}}G^{5,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}\frac{1}{2}+\frac{w}{2}-\delta,\ \frac{w}{2},\ \frac{w+1}{2}\\ \sigma+\frac{w}{2}-1,\ \frac{w+u+v}{2},\ \frac{w-u+v}{2},\ \frac{w+u-v}{2},\ \frac{w-u-v}{2}\end{matrix}\right)$$

$$\doteqdot \frac{\pi^{5/2}}{2i(\cos u\pi-\cos v\pi)}\,x^{\delta+\frac{1}{2}}y^{\sigma-2}\cdot\left[e^{i\pi(v-u)/2}H^{(1)}_v\left(\sqrt{\frac{x}{y}}\right)H^{(2)}_u\left(\sqrt{\frac{x}{y}}\right)\right.$$

$$\left.-\,e^{i\pi(u-v)/2}H^{(1)}_u\left(\sqrt{\frac{x}{y}}\right)H^{(2)}_v\left(\sqrt{\frac{x}{y}}\right)\right] \qquad (3.27)$$

The following three results are obtained by using the known results from [1] on page 239 in (2.2).

$$p^{\frac{5}{2}-\sigma-\delta}\left(\frac{p}{p}\right)^{2-\sigma} G^{3,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}\frac{1}{2}-\delta,\ 0,\ \frac{1}{2}\\ \sigma-1,\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2}\end{matrix}\right)$$

$$\doteqdot \frac{\Gamma(1+2c)\ \pi^{-1/2}}{\Gamma(1+2c-2d)\ 2^{2d+1}}\, x^{c+\delta-\frac{1}{2}}\, y^{\sigma-c-2}\cdot\left[e^{ic\pi}{}_1F_1\left(1+2c;\ 1+2c-2d;\ -2i\sqrt{\frac{x}{y}}\right)\right.$$

$$\left.+\ e^{-ic\pi}{}_1F_1\left(1+2c;\ 1+2c-2d;\ 2i\sqrt{\frac{x}{y}}\right)\right] \qquad (3.28)$$

$$p^{\frac{1}{2}-\delta}\, q^{2-\sigma}\, G^{5,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}\frac{1}{2}-\delta,\ 0,\ \frac{1}{2}\\ \sigma-1,\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2}\end{matrix}\right)$$

$$\doteqdot 2^{-2d}\sqrt{\pi}\ \Gamma(1+2c)\Gamma(1+2d)\ x^{\delta+c-\frac{1}{2}}\, y^{\sigma-c-2}\cdot\left[e^{ic\pi}\ \psi\left(1+2c,\ 1+2c-2d;\ 2i\sqrt{\frac{x}{y}}\right)\right.$$

$$\left.+\ e^{-ic\pi}\ \psi\left(1+2c;\ 1+2c-2d;\ -2i\sqrt{\frac{x}{y}}\right)\right] \qquad (3.29)$$

$$p^{\frac{3}{2}-\sigma-\delta}\left(\frac{q}{p}\right)^{\frac{3}{2}-\sigma} G^{5,2}_{3,5}\left(\frac{q}{p}\,\middle|\,\begin{matrix}-\delta,\ \frac{1}{2},\ 0\\ \sigma-\frac{1}{2},\ c,\ c+\frac{1}{2},\ d,\ d+\frac{1}{2}\end{matrix}\right)$$

$$\doteqdot \frac{\sqrt{\pi}i}{2^{2d}}\ \Gamma(1+2c)\Gamma(1+2d)x^{c+\delta}\, y^{\sigma-c-\frac{3}{2}}\cdot\left[e^{ic\pi}\ \psi\left(1+2c,\ 1+2c-2d;\ 2i\sqrt{\frac{x}{y}}\right)\right.$$

$$\left.-\ e^{-ic\pi}\ \psi\left(1+2c,\ 1+2c-2d;\ -2i\sqrt{\frac{x}{y}}\right)\right] \qquad (3.30)$$

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