Internat. J. Math. & Math. Sci.
VOL. 11 NO. 2 (1988) 275-284

SOLVABILITY OF A FOURTH ORDER BOUNDARY VALUE PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

CHAITAN P. GUPTA

Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115

(Received May 28, 1987 and in revised form July 16, 1987)

ABSTRACT. Fourth order boundary value problems arise in the study of the equilibrium of an elastaic beam under an external load. The author earlier investigated the existence and uniqueness of the solutions of the nonlinear analogues of fourth order boundary value problems that arise in the equilibrium of an elastic beam depending on how the ends of the beam are supported. This paper concerns the existence and uniqueness of solutions of the fourth order boundary value problems with periodic boundary conditions.

KEY WORDS AND PHRASES. fourth order boundary value problem, periodic boundary conditions, linear eigenvalue problem, Leray-Schauder continuation theorem, equilibrium of an elastic beam, non-trivial kernel.
AMS SUBJECT CLASSIFICATION: 34B15, 34C25

1. INTRODUCTION

Fourth order boundary value problems arise in the study of the equilibrium of an elastic beam under an external load, (e.g., see [1], [2], [3]) where the existence, uniqueness and iterative methods to construct the solutions have been studied extensively. The purpose of this paper is to study the fourth order boundary value problem with periodic boundary conditions:

$$\frac{d^4u}{dx^4} + f(u)u' + g(x,u) = e(x),\ x \in [0, 2\pi],$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) \tag{1.1}$$
$$= u'''(0) - u'''(2\pi) = 0,$$

where $f: \mathbb{R} \to \mathbb{R}$ is continuous and $g: [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory's conditions with $e \in L^1[0,2\pi]$.

We note that the fourth order linear eigenvalue problem

$$\frac{d^4u}{dx^4} = \lambda u, \tag{1.2}$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

has $\lambda = n^4$, $n = 0, 1, 2, \ldots$ as eigenvalues. Now the problem (1.1) is at resonance since the linear operator $Lu = \frac{d^4u}{dx^4}$ with $D(L) = \{u \in H^3(0,2\pi) \mid u(0) = u(2\pi),$

$u'(0) = u'(2\pi)$, $u''(0) = u''(2\pi)$, $u'''(0) = u'''(2\pi)\}$ has a non-tirival kernel. (See end of this introduction for the definition of $H^3(0,2\pi)$.) We shall prove that the boundary value problem (1.1) has at least one solution if $\int_0^{2\pi} e(x)dx = 0$, and there exists a constant $\rho > 0$ such that $g(x,u)u \geq 0$ for $|u| \geq \rho$. To prove the existence of a solution for the boundary value problem

$$- \frac{d^4u}{dx^4} + \alpha u' + g(x,u) = e(x), \; x \in [0,2\pi],$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0, \tag{1.3}$$

we also need to assume that

$$\limsup_{|u|\to\infty} \frac{g(x,u)}{u} = \beta < 1, \text{ uniformly for a.e. } x \in [0,2\pi].$$

This is because the second eigenvalue $\lambda = 1$ of the linear eigenvalue problem (1.2) interferes with the non-linearity $g(x,u)$ in (1.3). The question of asymptotic conditions in which non-linearity $g(x,u)$ in (1.3) can interact with infinitely many eigenvalues of the eigenvalue problem (1.2) will be studied in a forthcoming paper [4].

To obtain the existence of solutions for (1.1) and (1.3), we use Mawhin's version of Leray Schauder continuation theorem as given in [5], [6], [7]. We also show that in case $f = \alpha$, where α is a constant, any two solutions of the boundary value problem (1.1), (respectively, (1.3)), differ by a constant and have a unique solution when, for example, $g(x,u)$ is strictly increasing in u for a.e. x in $[0,2\pi]$.

We note that in addition to using the classical spaces $C([0,2\pi])$, $C^k([0,2\pi])$, and $L^k(0,2\pi)$ and $L^\infty(0,2\pi)$ of continuous, k-times continuously differentiable, measurable real-valued functions whose k-th power of the absolute value is Lebesgue integrable or measurable functions which are essentially-bounded on $[0,2\pi]$ we shall use the Sobolev space $H^3(0,2\pi)$ defined by

$$H^3(0,2\pi) = \{u\colon [0,2\pi] \to \mathbb{R} \mid u, u', u'' \text{ abs. cont. on } [0,2\pi], \; u''' \in L^2(0,2\pi)\}.$$

Also for $u \in L^1(0,2\pi)$ we define $\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(x)dx$.

2. MAIN RESULTS

Let X, Y denote the Banach spaces $X = C^1[0,2\pi]$, $Y = L^1(0,2\pi)$ with usual norms and let H denote the Hilbert space $L^2(0,2\pi)$. Let Y_2 be the subspace of Y defined by

$$Y_2 = \{u \in Y \mid u = \text{constant a.e. on } [0,2\pi]\},$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. ($\oplus$ denotes the direct sum.) We note that for $u \in Y$ we can write

$$u(x) = \left(u(x) - \frac{1}{2\pi}\int_0^{2\pi} u(t)dt\right) + \frac{1}{2\pi}\int_0^{2\pi} u(t)dt, \; x \in [0,2\pi].$$

We define the canonical projection operators $P\colon Y \to Y_1$; $Q\colon Y \to Y_2$ as follows

$$P(u) = u(x) - \frac{1}{2\pi}\int_0^{2\pi} u(t)dt,$$

$$Q(u) = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt,$$

for $u \in Y$. Clearly, $Q = I - P$, where I denotes the identity mapping on Y. and the projection operators P and Q are continous. Now let $X_2 = X \cap Y_2$.

Clearly X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P|X: X \to X_1$, $Q|X: X \to X_2$ are continuous. Similarly, we obtain $H = H_1 \oplus H_2$ and continuous projections $P|H: H \to H_1$, $Q|H: H \to H_2$. In the following, X, Y, H, P, Q, etc. will refer to Banach spaces, Hilbert space and the projections as defined above and we shall not distinguish between P, $P|X$ $P|H$ (resp. Q, $Q|X$, $Q|H$) and depend on the context for proper meaning.

Also for $u \in X$, $v \in Y$ let $(u,v) = \frac{1}{2\pi}\int_0^{2\pi} u(x)v(x)dx$ denote the duality pairing between X and Y. We note that for $u \in X$, $v \in Y$ where $u = Pu + Qu$, $v = Pv + Qv$, we have

$$(u,v) = (Pu,Pv) + (Qu,Qv).$$

Define a linear operator $L: D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in H^3(0,2\pi)\,|\,u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi)\} \tag{2.1}$$

and for $u \in D(L)$,

$$Lu = \frac{d^4u}{dx^4}. \tag{2.2}$$

Now, for $u \in D(L)$ we see using integration by parts and Wirtinger's inequality ([8]) that

$$(Lu,u) = \int_0^{2\pi} \frac{d^4u}{dx^4}\,u\,dx = \int_0^{2\pi} u''^2 dx \geq \int_0^{2\pi} [(Pu)(x)]^2 dx \geq 0. \tag{2.3}$$

LEMMA 2.1: - For a given $\alpha \in \mathbb{R}$ and $h \in Y_1$, i.e. $h \in L^1(0,2\pi)$ with $\bar{h} = Qh = 0$, the linear boundary value problem

$$\begin{gathered}\frac{d^4u}{dx^4} + \alpha u' = h(x),\ x \in [0,2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi),\end{gathered} \tag{2.4}$$

has a unique solution $u(x)$ with $\bar{u} = Qu = 0$.

PROOF:- Let us set $\omega = \operatorname{Cos}\frac{2\pi}{3} + i \operatorname{Sin}\frac{2\pi}{3}$, $i = \sqrt{-1}$, so that $\alpha^{1/3}$, $\omega\alpha^{1/3}$, $\omega^2\alpha^{1/3}$ are the three cube roots of $\alpha \in \mathbb{R}$. For $x \in [0,2\pi]$, we , define

$$v_1(x) = \int_0^x h(t)dt,\quad v_2(x) = e^{-\alpha^{1/3}\omega x}\int_0^x v_1(t)e^{\alpha^{1/3}\omega t}dt,$$

$$v_3(x) = e^{-\alpha^{1/3}\omega^2 x}\int_0^x v_2(t)e^{\alpha^{1/3}\omega^2 t}dt,\quad v(x) = e^{-\alpha^{1/3}x}\int_0^x e^{\alpha^{1/3}t}v_3(t)dt.$$

Then $u(x) = C_1 + C_2e^{-\alpha^{1/3}x} + C_3e^{-\alpha^{1/3}\omega x} + C_4e^{-\alpha^{1/3}\omega^2 x} + v(x)$ is such that $Rl(u(x))$ is a general solution of the equation (2.4).

Next, we compute C_1, C_2, C_3, C_4 using the boundary conditions in (2.4) and and the condition $\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(x)dx = 0$. C_2, C_3, C_4 are computed uniquely from the three linearly indpendent equations

$$C_2 + C_3 + C_4 = C_2e^{-\alpha^{1/3}2\pi} + C_3e^{-\alpha^{1/3}2\pi\omega} + C_4e^{-\alpha^{1/3}2\pi\omega^2} + v(2\pi),$$

$$C_2 + \omega C_3 + \omega^2 C_4 = C_2 e^{-\alpha^{1/3}2\pi} + \omega C_3 e^{-\alpha^{1/3}2\pi\omega} + \omega^2 C_4 e^{-\alpha^{1/3}2\pi\omega^2} - \alpha^{-1/3} v'(2\pi),$$

$$C_2 + \omega^2 C_3 + \omega C_4 = C_2 e^{-\alpha^{1/3}2\pi} + \omega^2 C_3 e^{-\alpha^{1/3}2\pi\omega} + \omega C_4 e^{-\alpha^{1/3}2\pi\omega^2} + \alpha^{-2/3} v''(2\pi).$$

The constant C_1 is computed uniquely using the condition $\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(x)dx = 0$. In this way we get Rl u(x) as the unique solution for (2.4).//

For $h \in Y_1$, i.e. $h \in L^1(0,2\pi)$ with $\bar{h} = \frac{1}{2\pi}\int_0^{2\pi} h(x)dx = 0$; let $u = Kh$ be the unique solution of the problem

$$\frac{d^4u}{dx^4} = h(x), \ x \in [0,2\pi],$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi),$$

such that $\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt = 0$. It is immediate that the linear mapping $K\colon Y_1 \to X_1$ is bounded and for $u \in Y$,

$$KP(u) \in D(L),\ LK\,P(u) = P(u), \text{ and } (KP(u), P(u)) \geq 0. \tag{2.5}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $g\colon [0,2\pi] \times \mathbb{R} \to \mathbb{R}$, $(x,u) \to g(x,u)$ be such that $g(.,u)$ is measurable on $[0,2\pi]$ for each $u \in \mathbb{R}$ and $g(x,.)$ is continuous on $\mathbb{R}$ for almost each $x \in [0,2\pi]$. Assume, moreoever, that for each $r > 0$ there exists an $\alpha_r \in L^1(0,2\pi)$ such that $|g(x,u)| \leq \alpha_r(x)$ for a.e. $x \in [0,2\pi]$ and all $u \in [-r,r]$. Such a g will be said to satisfy Caratheodory's conditions. Now define $N\colon X \to Y$ by setting

$$(Nu)(x) = f(u(x))\,u'(x) + g(x,u(x)),\ x \in [0,2\pi],$$

for $u \in X$. It follows easily from Arzela-Ascoli theorem that $KPN\colon X \to X_1$ is a well-defined compact mapping and $QN\colon X \to X_2$ is bounded.

For $e(x) \in Y = L^1(0,2\pi)$, the boundary value probelm (1.1) now reduces to the functional equation

$$Lu + Nu = e, \tag{2.6}$$

in X with $e \in Y$, given.

THEOREM 2.2:- Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $g\colon [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions. Assume that there exist real numbers a, A, r and R with $a \leq A$ and $r < 0 < R$ such that

$$g(x,u) \geq A, \tag{2.7}$$

for a.e. $x \in [0,2\pi]$ and all $u \leq R$; and

$$g(x,u) \leq a, \tag{2.8}$$

for a.e. $x \in [0,2\pi]$ and all $u \leq r$. Then the boundary value problem (1.1) has at least one solution for each given $e \in L^1(0,2\pi)$ with

$$a \leq \bar{e} \leq A. \tag{2.9}$$

PROOF:- Define $g_1\colon [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ by $g_1(x,u) = g(x,u) - \frac{1}{2}(a + A)$ and $e_1 \in L^1(0,2\pi)$ by $e_1(x) = e(x) - \frac{1}{2}(a + A)$, so that, for a.e. $x \in [0,2\pi]$, using (2.7), (2.8), (2.9) we have

$$g_1(x,u) \geq \frac{1}{2}(A - a) \geq 0 \quad \text{if} \quad u \geq R, \tag{2.10}$$

$$g_1(x,u) \leq \frac{1}{2}(a - A) \leq 0 \quad \text{if} \quad u \leq r, \tag{2.11}$$

and

$$\frac{1}{2}(a - A) \leq \bar{e}_1 \leq \frac{1}{2}(A - a). \tag{2.12}$$

Clearly, the boundary value problem (1.1) is equivalent to

$$\begin{gathered}\frac{d^4u}{dx^4} + f(u)u' + g_1(x,u(x)) = e_1(x), \ x \in [0,2\pi], \\ u(0) = u(2\pi), \ u'(0) = u'(2\pi), \ u''(0) = u''(2\pi), \ u'''(0) = u'''(2\pi).\end{gathered} \tag{2.13}$$

Let $N: X \to Y$ be defined by

$$(Nu)(x) = f(u(x))u'(x) + g_1(x,u(x)), \ x \in [0,2\pi], \tag{2.14}$$

for $u \in X$. We then see, as above, that $KPN: X \to X_1$ is a well-defined compact mapping. $QN: X \to X_2$ is bounded and the boundary value problem (2.13) is equivalent to the functional equation,

$$Lu + Nu = e_1, \tag{2.15}$$

in X with $e_1 \in Y$. Setting, $\tilde{e}_1 = KPe$, we see that to solve the functional equation (2.15) it suffices to solve the system of equations

$$\begin{aligned} Pu + KPNu &= \tilde{e}_1, \\ QNu &= \bar{e}_1, \end{aligned} \tag{2.16}$$

$u \in X$. Indeed, if $u \in X$ is a solution of (2.16) then $u \in D(L)$ and

$$\begin{aligned} LPu + LKPNu = Lu + PNu &= L\tilde{e}_1 = Pe_1, \\ QNu &= \bar{e}_1 = Qe_1, \end{aligned}$$

which gives on adding that $Lu + Nu = e_1$.

Now, (2.16) is clearly equivalent to the single equation

$$Pu + QNu + KPNu = \tilde{e}_1 + \bar{e}_1, \tag{2.17}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can therefore apply the version given in [6] (Theorem 1, Corollary 1) or [5] (Theorem IV.4) or [7] of the Leray-Schauder Continuation theorem which ensures the existence of a solution for (2.17) if the set of solutions of the family of equations,

$$Pu + (1-\lambda)Qu + \lambda QNu + \lambda KPNu = \lambda\tilde{e}_1 + \lambda\bar{e}_1, \ \lambda \in (0,1), \tag{2.18}$$

is, a priori, bounded in X by a constant independent of λ. Notice that (2.18) is equivalent to the system of equations,

$$\begin{aligned} Pu + \lambda KPNu &= \lambda\tilde{e}_1, \\ (1-\lambda)Qu + \lambda QNu &= \lambda\bar{e}_1, \ \lambda \in (0,1). \end{aligned} \tag{2.19}$$

Let for $\lambda \in (0,1)$, $u_\lambda \in X$ be a solution of (2.19) so that

$$Pu_\lambda + \lambda KPNu_\lambda = \lambda\tilde{e}_1,$$
$$(1-\lambda)Qu_\lambda + \lambda QNu_\lambda = \lambda\,\bar{e}_1. \qquad (2.20)$$

The second equation in (2.20) can now be written as

$$(1-\lambda)\cdot\frac{1}{2\pi}\int_0^{2\pi} u_\lambda(x)dx + \frac{\lambda}{2\pi}\int_0^{2\pi} g_1(x,u_\lambda(x))dx = \lambda\,\bar{e}_1.$$

So, if $u_\lambda(x) \geq R$ for $x \in [0,2\pi]$ we have, using (2.10), (2.12) that

$$0 < (1-\lambda)\,R + \frac{\lambda}{2}(A-a) \leq \frac{\lambda}{2}(A-a),$$

i.e.

$$0 < (1-\lambda)\,R \leq 0, \quad \text{a contradiction.}$$

Similarly if $u_\lambda(x) \leq r$ for $x \in [0,2\pi]$ leads to a contradiction. Hence, there exists a $\tau_\lambda \in [0,2\pi]$ such that

$$r < u_\lambda(\tau_\lambda) < R. \qquad (2.21)$$

Now, for $x \in [0,2\pi]$ we have

$$u_\lambda(x) = u_\lambda(\tau_\lambda) + \int_{\tau_\lambda}^{x} u'_\lambda(s)ds,$$

so that

$$|u_\lambda(x)| \leq \max(R,-r) + (2\pi)^{1/2}\left(\int_0^{2\pi}(u'_\lambda(s))^2ds\right)^{1/2}$$
$$\leq \max(R,-r) + (2\pi)^{1/2}\left(\int_0^{2\pi}(u''_\lambda(s))^2ds\right)^{1/2}$$

since $u_\lambda \in D(L)$, Wirtinger's inequality applies. Thus,

$$\|u_\lambda\|_X \leq C_1\|u''_\lambda\|_H + C_2, \qquad (2.22)$$

for some constants C_1, C_2 independent of λ.

Next, the first equation in (2.20) gives that

$$LPu_\lambda + \lambda LKPNu_\lambda = \lambda L\tilde{e}_1,$$

i.e.

$$Lu_\lambda + \lambda PNu_\lambda = \lambda Pe_1. \qquad (2.23)$$

From (2.23) and the second equation in (2.20), we get

$$(Lu_\lambda,Pu_\lambda) + \lambda(PNu_\lambda,Pu_\lambda) = \lambda(Pe_1,Pu_\lambda),$$
$$(1-\lambda)(Qu_\lambda,Qu_\lambda) + \lambda(QNu_\lambda,Qu_\lambda) = \lambda(\bar{e}_1,\,Qu_\lambda). \qquad (2.24)$$

We next note that our assumptions on g_1 and (2.10), (2.12) imply that there is a constant C_3, indpendent of λ such that for $u \in X$,

$$(Nu,u) \geq -C_3,$$

and $(Lu_\lambda,Pu_\lambda) = (Lu_\lambda,u_\lambda) = \int_0^{2\pi}(u''_\lambda)^2 = \|u''_\lambda\|_H^2$ since (2.3) holds. Using this we get on adding the equations in (2.24) that

$$\|u_\lambda''\|_H^2 - C_3 \le (Lu_\lambda, u_\lambda) + (1-\lambda)(Qu_\lambda, Qu_\lambda) + \lambda(Nu_\lambda, u_\lambda)$$
$$= \lambda(Pe_1, Pu_\lambda) + \lambda(\overline{e}_1, Qu_\lambda)$$
$$\le C_4\|u_\lambda\|_X$$
$$\le C_4C_1\|u_\lambda''\|_H + C_4C_2,$$

where C_4 is a constant independent of λ. Accordingly, there is a constant C_5, independent of λ, such that

$$\|u_\lambda''\|_H \le C_5,$$

which implies, using (2.22) that

$$\|u_\lambda\|_X \le C_1C_5 + C_2 \equiv C.$$

We have thus proved that the set of solutions of the family of equations (2.18) is bounded in X by a constant independent of $\lambda \in (0,1)$. Hence the theorem.//

REMARK 2.3:- If we take $a = A = 0$ in Theorem 2.2, then we immediately obtain the assertion made in the introduction concerning the boundary value problem (1.1).

Now, to study the boundary value problem (1.3) we define, for a given $\alpha \in \mathbb{R}$, a linear operator $L_\alpha: D(L_\alpha) \subset X \to Y$ by setting

$$D(L_\alpha) = \{u \in H^3(0,2\pi) \mid u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi)\} \tag{2.25}$$

and for $u \in D(L_\alpha)$,

$$L_\alpha u = -\frac{d^4u}{dx^4} + \alpha u'. \tag{2.26}$$

It follows, using integration by parts and Wirtinger's inequality, ([8]), that

$$\begin{aligned}(L_\alpha u,\ u) &= -\int_0^{2\pi} \frac{d^4u}{dx^4}\, u dx + \alpha\int_0^{2\pi} u'u\, dx \\ &= -\int_0^{2} (u'')^2 dx \ge -\int_0^{2\pi} \left(\frac{d^4u}{dx^4}\right)^2 dx \\ &\ge -\|L_\alpha u\|_H^2 .\end{aligned} \tag{2.27}$$

We, next, use lemma 2.1 to define a bounded linear mapping $K_\alpha : Y_1 \to X_1$ by setting $u = K_\alpha h$ for a given $h \in Y_1$, where $u \in X_1$ (so that $\overline{u} = Qu = 0$) is the unique solution of the boundary value problem

$$-\frac{d^4u}{dx^4} + \alpha u' = h(x),\ x \in [0,\ 2\pi], \tag{2.28}$$
$$u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi).$$

The bounded linear mapping $K_\alpha : Y_1 \to X_1$ defined in this way has the following properties:

(i) for $u \in Y$, $K_\alpha P(u) \in D(L_\alpha)$, $L_\alpha K_\alpha P(u) = P(u)$ and

$$(K_\alpha P(u),\ P(u)) \ge -\|Pu\|_H^2, \text{ (in view of (2.27))}; \tag{2.29}$$

(ii) if $g : [0,\ 2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory's conditions and $N : X \to Y$ is defined by setting

$$(Nu)(x) = g(x,u(x)),\ x \in [0,\ 2\pi]$$

then $K_\alpha PN : X \to X_1$ is a well-defined compact mapping and $QN : X \to X_2$ is bounded.

Theorem 2.4: Let $\alpha \in \mathbb{R}$ be given and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions. Assume that there exist real numbers a, A, r, R with $a \leq A$, and $r < 0 < R$ such that

$$g(x, u) \geq A, \tag{2.30}$$

for a.e. $x \in [0, 2\pi]$, and all $u \geq R$; and

$$g(x, u) \leq a, \tag{2.31}$$

for a.e. $x \in [0, 2\pi]$, and all $u \leq r$. Suppose, further, that

$$\limsup_{|u|\to\infty} \left|\frac{g(x,u)}{u}\right| = \beta < 1 \tag{2.32}$$

uniformly for a.e. $x \in [0, 2\pi]$. Then the boundary value problem (1.3) has at least one solution for each given $e \in L^2[0, 2\pi]$ with

$$a \leq \bar{e} \leq A. \tag{2.33}$$

Proof:- As in the proof of Theorem 2.2, define $g_1 : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ by $g_1(x,u) = g(x,u) - \frac{1}{2}(a + A)$ and $e_1 \in L^2(0, 2\pi)$ by $e_1(x) = e(x) - \frac{1}{2}(a+A)$. Then for a.e. $x \in [0,2\pi]$,

$$g_1(x,u) \geq \frac{1}{2}(A - a) \geq 0 \quad \text{if} \quad u \geq R, \tag{2.34}$$

$$g_1(x,u) \leq \frac{1}{2}(a - A) \leq 0 \quad \text{if} \quad u \leq r, \tag{2.35}$$

$$\limsup_{|u|\to\infty} \frac{g_1(x,u)}{u} = \beta < 1, \tag{2.36}$$

uniformly, and

$$\frac{1}{2}(a - A) \leq \bar{e}_1 \leq \frac{1}{2}(A - a). \tag{2.37}$$

Also the boundary value problem (1.3) is equivalent to

$$\begin{gathered} -\frac{d^4u}{dx^4} + \alpha u' + g_1(x,u) = e_1(x),\ x \in [0,\ 2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi). \end{gathered} \tag{2.38}$$

Next, let $N : X \to Y$ be defined by

$$Nu(x) = g_1(x,u(x)),\ x \in [0,\ 2\pi],$$

for $u \in X$. Choosing, now, $\varepsilon > 0$ such that $\beta + \varepsilon < 1$, we see, using the fact that g_1 satisfies Caratheodory's conditions and (2.34), (2.35), (2.36), that there exists a constant $C(\varepsilon) > 0$ such that

$$(Nu,u) \geq \frac{1}{\beta+\varepsilon} \|Nu\|_H^2 - C(\varepsilon), \tag{2.39}$$

for $u \in X$. Also, $K_\alpha PN : X \to X_1$ is a well-defined compact mapping and $QN : X \to X_2$ is bounded.

Again, we see as in the proof of Theorem 2.2, that the boundary value problem (2.38) is equivalent to the system of equations

$$\begin{aligned} Pu + K_\alpha PNu &= \tilde{e}_1 = K_\alpha Pe_1, \\ QNu &= \bar{e}_1. \end{aligned} \tag{2.40}$$

Further, it suffices to prove that the set of solutions of the family of equations

$$\begin{aligned} Pu + \lambda K_\alpha PNu &= \lambda\tilde{e}_1 \\ (1-\lambda)Qu + \lambda QNu &= \lambda\,\bar{e}_1, \quad \lambda \in (0,1) \end{aligned} \tag{2.41}$$

is, a priori, bounded in X by a constant independent of $\lambda \in (0,1)$.

Let, now, for $\lambda \in (0,1)$, $u_\lambda \in X$ be a solution of (2.41) so that

$$\begin{aligned} Pu_\lambda + \lambda K_\alpha PNu_\lambda &= \lambda e_1, \\ (1-\lambda)Qu_\lambda + \lambda QNu_\lambda &= \lambda\bar{e}_1. \end{aligned} \tag{2.42}$$

It, now, follows from the second equation in (2.42), in a manner similar to deriving the estimate (2.22) in the proof of Theorem 2.2, that

$$\|Qu_\lambda\| \le \|u_\lambda\|_X \le C_1\ \|L_\alpha u_\lambda\|_H + C_2, \tag{2.43}$$

for some constants C_1, C_2 independent of $\lambda \in (0,1)$.

Also, we have from (2.42) that

$$(Pu_\lambda,\ PNu_\lambda) + \lambda(K_\alpha PNu_\lambda,\ PNu_\lambda) = \lambda(\tilde{e}_1,\ PNu_\lambda),$$

$$(1-\lambda)\ \|Qu_\lambda\|^2 + \lambda(Qu_\lambda,\ QNu_\lambda) = \lambda(\bar{e}_1,\ Qu_\lambda).$$

These equations then give us, in view of (2.29) and (2.39), that

$$\begin{aligned} \frac{1}{\beta+\varepsilon}\ \|Nu_\lambda\|_H^2 - \|PNu_\lambda\|_H^2 - C(\varepsilon) &\le (Nu_\lambda,u_\lambda) + \lambda(K_\alpha PNu_\lambda,\ PNu_\lambda) \\ &\le (\tilde{e}_1,\ PNu_\lambda) + (\bar{e}_1,\ Qu_\lambda). \end{aligned}$$

Using, now, the facts that $\|Pv\|_H \le \|v\|_H$ for $v \in X$, and $\beta + \varepsilon < 1$, we see that these exist constants C_3, C_4 independent of $\lambda \in (0,1)$ such that

$$\|Nu_\lambda\|_H \le C_3\ \|Qu_\lambda\|_H^{1/2} + C_4. \tag{2.44}$$

Now, the first equation in (2.42) gives that

$$L_\alpha u_\lambda + \lambda PNu_\lambda = \lambda Pe_1,$$

so that

$$\begin{aligned} \|L_\alpha u_\lambda\|_H \le \lambda\|Pe_1 - PNu_\lambda\|_H &\le \|Pe_1\|_H + \|PNu_\lambda\|_H \\ &\le \|Pe_1\|_H + \|Nu_\lambda\|_H \\ &\le C_3\|Qu_\lambda\|^{1/2} + C_4 + \|Pe_1\|_H. \end{aligned}$$

(2.43) and (2.45) now imply that there exist a constant C_5, independent of $\lambda \in (0,1)$, such that

$$||Qu_\lambda|| \leq C_5$$

and

$$||u_\lambda||_X \leq C_1 C_3 \sqrt{C_5} + C_1 C_4 + C_1 ||Pe_1||_H + C_2 = C.$$

This completes the proof of the theorem //.

Remark 2.5:- The analogue of Theorem 2.4 when α in (1.3) is replaced by $f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function will be treated in a forthcoming paper [4].

Remark 2.6:- If $f(u) \equiv \alpha$, $\alpha \in \mathbb{R}$ given and $g(x,u)$ is strictly increasing in u for a.e $x \in [0, 2\pi]$ then it is easy to see that the boundary value problem (1.1) has exactly one solution. Similarly if $g(x,u)$ is strictly increasing in u and there is a $\beta < 1$, such that

$$(g(x,u_1) - g(x,u_2))(u_1 - u_2) \geq \beta(g(x,u_1) - g(x,u_2))^2,$$

for a.e. x in $[0, 2\pi]$, then the boundary value problem (1.3) has exactly one solution.

Remark 2.7:- If we take $a = A = 0$ in Theorem 2.4, we immediately obtain the assertion concerning the boundary value problem (1.3) in the introduction.

Acknowledgement:- The author is grateful to the referee for his suggestions to add references [1], [2], [3] to the bibliography and for editorial improvement of this paper.

REFERENCES

1. AGARWAL, R.P. Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore, Philadelphia, PA, 1986.
2. AGARWAL, R.P., CHOW, Y.M. Iterative Methods for a Fourth Order Boundary Value Problem, Jour. Comp. Appl. Math. 10 (1984), 203-207.
3. USMANI, R.A. Solving Boundary Value Problems in Plate Deflection Theory, Simulation, December 1981, 195-206.
4. GUPTA, C.P. Asymptotic conditions for the solvability of a fourth order boundary value problem with periodic boundary conditions. (Under preparation).
5. MAWHIN, J. Topological degree methods in nonlinear boundary value problems. CBMS - Regional Conference Series in Maths. No. 40, (1979) American Mathematical Society, Providence, R.I.
6. MAWHIN, J. Landesman - Lazer type problems for nonlinear equations. Conf. Sem. Mat. Univ. Bari no. 147, (1977).
7. MAWHIN, J. Compacite', monotonie, et convexite dans l'etude de problems aux limites semi-lineaires. Sem. Anal. Moderne No. 19, (1981) Universite' de Sherbrooke, Quebec, CANADA.
8. HARDY, G.H., LITTLEWOOD, J.E., and POLYA, G. Inequalities, Cambridge University Press, London and New York, 1952.
9. GUPTA, C.P. Existence and uniqueness theorems for the bending of an elastic beam equation. Applicable Analysis (to appear).
10. GUPTA, C.P. Existence and uniqueness theorems for the bending of an elastic beam equation at resonance. Jour. Math. Anal. & Appl. (to appear).