ON EXTENDING ENDOMORPHISMS TO AUTOMORPHISMS

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ABSTRACT.   A monic endomorphism of a structure A can be extended to an
automorphism of a larger structure A^*. We investigate which properties
are preserved by this process.

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1.   INTRODUCTION.

It is known [1, Thm. VII.3.4] that monic endomorphisms of an algebraic
structure A can be extended to automorphisms of a larger structure A^* of
the same type. In a recent paper [3], Jordan studied the relationship between
the two structures in the case in which A is an associative ring. The
constructions of A^* in [1] and [3] are quite different.

In this paper, we investigate what can be said in general about the
relationship between the properties of A and the properties of A^* . Our
main result is that all of the "universal formulas" satisfied by the
operations and relations on A which are compatible with the given
endomorphisms will also be satisfied by the extensions of the operations and
relations to A^* . In comparison to [1], we slightly weaken the hypotheses
and broaden the scope of the investigation to include relations, universal
first-order formulas, and relationships involving more than one structure.
Our results apply to non-algebraic structures, such as metric spaces.

In Section 2, we start with a monoid M acting on a set A , and
construct a map from A to a "universal" A^* on which M acts by
bijections. We show that A^* is uniquely determined by certain of its
properties. After developing some additional properties, we turn in Section 3
to the question of extending operations and relations from A to A^* . We
also indicate how the analogous procedure may be carried out when

relationships with other structures are involved. We conclude the paper with three examples, showing how our techniques may be employed.

2. UNIVERSAL M-SETS.

Recall that a monoid is a set M, together with an associative binary operation $M \times M \to M$ (which we denote by juxtaposition) and an identity element (denoted 1). A (right) M-set is a set A, together with a map $A \times M \to A$ such that $a1 = a$ and $(a\alpha)\beta = a(\alpha\beta)$ for all $a \in A$ and $\alpha, \beta \in M$. A map of M-sets is a function $\zeta: A \to B$ on their underlying sets for which $\zeta(a\alpha) = \zeta(a)\alpha$ for all $a \in A$, $\alpha \in M$. The M-set A is an M-subset of the M-set B if A is a subset of B and the inclusion map $A \to B$ is a map of M-sets.

We say that the monoid M acts on the M-set A by injections (bijections) if for each $\alpha \in M$ the map $\theta_\alpha^A: A \to A$ given by $a\theta_\alpha^A = a\alpha$ ($a \in A$) is injective (bijective).

A monoid M is directed by left divisibility if for each $\alpha, \beta \in M$ there exist $\lambda, \mu \in M$ with $\alpha\lambda = \beta\mu$. The monoid M satisfies left cancellation if for $\alpha, \beta, \gamma \in M$ we have that $\gamma\alpha = \gamma\beta$ implies $\alpha = \beta$.

THEOREM 1. Let M be a monoid which is directed by left divisibility and which satisfies left cancellation. Let A be an M-set. Then there exists an M-set A^* and a map $\iota: A \to A^*$ with the following properties.

(1) M acts on A^* by bijections.

(2) Suppose that $h: A \to B$ is an M-set map, and M acts on B by bijections. Then there exists a unique M-set map $\zeta: A^* \to B$ with $\zeta \circ \iota = h$.

If $(A^*)'$ and $\iota': A \to (A^*)'$ both satisfy (1) and (2), then there exists a unique isomorphism $\zeta: A^* \to (A^*)'$ with $\zeta \circ \iota = \iota'$.

The M-set A^* (or any isomorphic $(A^*)'$, as above) also satisfies

(3) For all $a^* \in A^*$ there exists $\alpha \in M$ and $a \in A$ with $a^*\alpha = \iota(a)$.

(4) For $a_1, a_2 \in A$ we have $\iota(a_1) = \iota(a_2)$ if and only if there exists $\alpha \in M$ with $a_1\alpha = a_2\alpha$. In particular, if M acts on A by injections, then ι is injective.

Moreover, if A^* and $\iota: A \to A^*$ satisfy (1), (3), and (4), then they also satisfy (2).

PROOF. We first show uniqueness. If $\iota: A \to A^*$ and $\iota': A \to (A^*)'$ both have properties (1) and (2), then there exist unique M-set maps $\zeta: A^* \to (A^*)'$ and $\zeta': (A^*)' \to A^*$ with $\zeta \circ \iota = \iota'$ and $\zeta' \circ \iota' = \iota$. Then $\zeta' \circ \zeta \circ \iota = \iota$ and $\zeta \circ \zeta' \circ \iota' = \iota'$. By uniqueness, $\zeta' \circ \zeta$ is the identity on A^*, and $\zeta \circ \zeta'$ is the identity on $(A^*)'$. Thus ζ is an isomorphism.

Now we construct A^*. We first define a relation $\sim$ on $A \times M$ by: $(a,\alpha) \sim (b,\beta)$ if there exist $\lambda, \mu \in M$ with $\alpha\lambda = \beta\mu$ and $a\lambda = b\mu$. This relation is clearly reflexive and symmetric. Suppose that $(a,\alpha) \sim (b,\beta)$ and $(b,\beta) \sim (c,\gamma)$. We have $\lambda, \mu \in M$ with $\alpha\lambda = \beta\mu$ and $a\lambda = b\mu$, and also $\lambda', \mu' \in M$ with $\beta\lambda' = \gamma\mu'$ and $b\lambda' = c\mu'$. Select $\delta, \delta' \in M$ with

$\mu\delta = \lambda'\delta'$. Then $\alpha\lambda\delta = \beta\mu\delta = \beta\lambda'\delta' = \gamma\mu'\delta'$, and $\alpha\lambda\delta = b\mu\delta = b\lambda'\delta' = c\mu'\delta'$, so $(a,\alpha)\sim(c,\gamma)$. Thus $\sim$ is an equivalence relation. We let A^* be the set of equivalence classes, and denote the equivalence class of (a,α) as $[a,\alpha]$.

Given $(a,\alpha) \in A\times M$ and $\beta \in M$, we would like to define $(a,\alpha)\beta \in A^*$ by selecting $\lambda,\mu \in M$ with $\alpha\lambda = \beta\mu$ and setting $(a,\alpha)\beta = [a\lambda,\mu]$. Suppose that we also have $\alpha\lambda' = \beta\mu'$. Select $\gamma,\gamma' \in M$ with $\mu\gamma = \mu'\gamma'$. Then $\alpha\lambda\gamma = \beta\mu\gamma = \beta\mu'\gamma' = \alpha\lambda'\gamma'$, so $\lambda\gamma = \lambda'\gamma'$. We now have $\mu\gamma = \mu'\gamma'$ and $(a\lambda)\gamma = (a\lambda')\gamma'$, so $[a\lambda,\mu] = [a\lambda',\mu']$. Thus our definition of $(a,\alpha)\beta$ does not depend upon the choice of λ,μ . Suppose that $(a,\alpha)\sim(a',\alpha')$. Then there exist $\gamma,\gamma' \in M$ with $\alpha\gamma = \alpha'\gamma'$ and $a\gamma = a'\gamma'$. Say $\gamma\delta_1 = \lambda\delta_2$ for $\delta_1,\delta_2 \in M$. Then $\alpha'\gamma'\delta_1 = \alpha\gamma\delta_1 = \alpha\lambda\delta_2 = \beta\mu\delta_2$, so $(a',\alpha')\beta = [a'\gamma'\delta_1,\mu\delta_2] = [a\gamma\delta_1,\mu\delta_2] = [a\gamma\delta_2,\mu\delta_2] = [a\lambda,\mu] = (a,\alpha)\beta$. Thus there is a well-defined function $A^* \times M \to A^*$ given by $[a,\alpha]\beta = [a\lambda,\mu]$, where $\alpha\lambda = \beta\mu$.

Clearly $[a,\alpha]1 = [a,\alpha]$. Given $[a,\alpha] \in A^*$ and $\beta_1,\beta_2 \in M$, we have $[a,\alpha]\beta_1 = [a\lambda_1,\mu_1]$ where $\alpha\lambda_1 = \beta_1\mu_1$, and $[a\lambda_1,\mu_1]\beta_2 = [a\lambda_1\lambda_2,\mu_2]$ where $\mu_1\lambda_2 = \beta_2\mu_2$. Then $\alpha\lambda_1\lambda_2 = \beta_1\mu_1\lambda_2 = \beta_1\beta_2\mu_2$, so $[a,\alpha]\beta_1\beta_2 = [a\lambda_1\lambda_2,\mu_2] = ([a,\alpha]\beta_1)\beta_2$. Thus A^* is an M-set. Since for $[a,\alpha] \in A^*$ and $\beta \in M$ we have $[a,\alpha] = [a,\beta\alpha]\beta$, the action of each $\beta \in M$ on A^* is surjective. Suppose that $[a_1,\alpha_1]\beta = [a_2,\alpha_2]\beta$. Then $[a_1\lambda_1,\mu_1] = [a_2\lambda_2,\mu_2]$, where $\alpha_1\lambda_1 = \beta\mu_1$ and $\alpha_2\lambda_2 = \beta\mu_2$. So there exists $\gamma_1,\gamma_2 \in M$ with $\mu_1\gamma_1 = \mu_2\gamma_2$ and $a_1\lambda_1\gamma_1 = a_2\lambda_2\gamma_2$. Since $\alpha_1\lambda_1\gamma_1 = \beta\mu_1\gamma_1 = \beta\mu_2\gamma_2 = \alpha_2\lambda_2\gamma_2$, we have $[a_1,\alpha_1] = [a_2,\alpha_2]$. Thus M acts on A^* by bijections, and we have shown (1).

The map $\iota: A \to A^*$ given by $\iota(a) = [a,1]$ is clearly a map of M-sets. Note that for $a^* = [a,\alpha] \in A^*$ we have $a^*\alpha = [a,\alpha]\alpha = [a,1] = \iota(a)$, verifying (3). For $a_1,a_2 \in A$ we have $\iota(a_1) = \iota(a_2)$ if and only if there exists $\alpha_1,\alpha_2 \in M$ with $1\alpha_1 = 1\alpha_2$ and $a_1\alpha_1 = a_2\alpha_2$. Since we must have $\alpha_1 = \alpha_2$, we have verified (4).

Let us now assume that we are given A^* and $\iota: A \to A^*$ satisfying (1), (3), and (4). Let h: $A \to B$ be an M-set map from A to an M-set B on which M acts by bijections. Suppose that there exists an M-set map $\zeta: A^* \to B$ with $\zeta\circ\iota = h$. Let $a^* \in A^*$. Select $\alpha \in M$ and $a \in A$ with $a^*\alpha = \iota(a)$. Then $\zeta(a^*)\alpha = \zeta(a^*\alpha) = \zeta(\iota(a)) = h(a)$, so $\zeta(a^*) = h(a)(\theta_\alpha^B)^{-1}$. Thus ζ , if it exists, is unique. To obtain existence, we must show that for $a^* \in A^*$ the expression $\zeta(a^*) = h(a)(\theta_\alpha^B)^{-1}$ does not depend upon the choice of $\alpha \in M$ and $a \in A$ with $a^*\alpha = \iota(a)$, and yields an M-set map with $\zeta\circ\iota = h$.

Let us first note that if $\iota(a_1) = \iota(a_2)$ then by (4) we can find $\beta \in M$ with $a_1\beta = a_2\beta$, and so $h(a_1) = h(a_1)\beta(\theta_\beta^B)^{-1} = h(a_1\beta)(\theta_\beta^B)^{-1} = h(a_2\beta)(\theta_\beta^B)^{-1} = h(a_2)\beta(\theta_\beta^B)^{-1} = h(a_2)$. Now let $a^* \in A^*$, and suppose we have $\alpha_1,\alpha_2 \in M$ and $a_1,a_2 \in A$ with $a^*\alpha_1 = \iota(a_1)$ and $a^*\alpha_2 = \iota(a_2)$. Select $\lambda_1,\lambda_2 \in M$ with $\alpha_1\lambda_1 = \alpha_2\lambda_2$. Then $\iota(a_1\lambda_1) = a^*\alpha_1\lambda_1 = a^*\alpha_2\lambda_2 = \iota(a_2\lambda_2)$, so $h(a_1\lambda_1) = h(a_2\lambda_2)$. Thus $h(a_1)(\theta_{\alpha_1}^B)^{-1} = h(a_1)\lambda_1(\theta_{\lambda_1}^B)^{-1}(\theta_{\alpha_1}^B)^{-1} = h(a_1\lambda_1)(\theta_{\alpha_1\lambda_1}^B)^{-1} =$

$h(a_2\lambda_2)(\Theta^B_{\alpha_2\lambda_2})^{-1} = h(a_2)(\Theta^B_{\alpha_2})^{-1}$, so the function $\zeta: A^* \to B$ is well-defined.

Let $\hat{a}^{*2} \in A^*$, $\beta \in M$.^{2} Select $\alpha \in M$, $a \in A$ with $a^*\alpha = \iota(a)$. Select $\lambda,\mu \in M$ with $\alpha\lambda = \beta\mu$. Note that $\Theta^B_\alpha\Theta^B_\lambda = \Theta^B_\beta\Theta^B_\mu$, so $\Theta^B_\lambda(\Theta^B_\mu)^{-1} = (\Theta^B_\alpha)^{-1}\Theta^B_\beta$. Since $(a^*\beta)\mu = a^*\alpha\lambda = \iota(a\lambda)$, we have $\zeta(a^*\beta) = h(a\lambda)(\Theta^B_\mu)^{-1} = h(a)\Theta^B_\lambda(\Theta^B_\mu)^{-1} = h(a)(\Theta^B_\alpha)^{-1}\Theta^B_\beta = \zeta(a^*)\beta$. Thus ζ is an M-set map. Clearly $\zeta \circ \iota = h$, and we are done.

Let M be a monoid which is directed by left divisibility and which satisfies left cancellation. Let M denote the category of M-sets, and let M^* denote the category of M-sets on which M acts by bijections. We associate $A^* \in Ob(M^*)$ to $A \in Ob(M)$. Let $\iota_A: A \to A^*$ be the map given by Theorem 1. Now let $g: A \to B$ be any M-set map. Since $\iota_B \circ g$ is a map from A to an M-set B^* on which M acts by bijections, we have by Theorem 1 that there is a unique map $g^*: A^* \to B^*$ with $g^* \circ \iota_A = \iota_B \circ g$. If $h: B \to C$ is another M-set map, then $h^* \circ g^* \circ \iota_A = h^* \circ \iota_B \circ g = \iota_C \circ h \circ g$, so $h^* \circ g^* = (h \circ g)^*$ by uniqueness. Thus we have the following.

THEOREM 2. Let M be a monoid which is directed by left divisibility and which satisfies left cancellation. Then there is a functor $()^*$ from the category M of M-sets to the category M^* of M-sets on which M acts by bijections, which associates to each M-set A the "universal" M-set A^* on which M acts by bijections, and to each M-set map $g: A \to B$ the unique M-set map $g^*: A^* \to B^*$ for which $g^* \circ \iota_A = \iota_B \circ g$.

We shall refer to g^* as the <u>extension</u> of g .

In the next section, we will need an additional property of the functor $()^*$. Let $A_1,\ldots,A_r$ be M-sets. Then $A_1 \times \ldots \times A_r$ is an M-set, with M acting via $(a_1,\ldots,a_r)\alpha = (a_1\alpha,\ldots,a_r\alpha)$ for $(a_1,\ldots,a_r) \in A_1 \times \ldots \times A_r$ and $\alpha \in M$. This M-set is the product of $A_1,\ldots,A_r$ in the category M . It is easy to see that M acts by bijections on $A_1^* \times \ldots \times A_r^*$, and thus that $A_1^* \times \ldots \times A_r^*$ is the product of $A_1^*,\ldots,A_r^*$ in M^* .

PROPOSITION 3. The functor $()^*$ preserves finite products. More explicitly, for any M-sets $A_1,\ldots,A_r$, the unique map ζ from $(A_1 \times \ldots \times A_r)^*$ to $A_1^* \times \ldots \times A_r^*$ such that $\zeta \circ \iota_{A_1 \times \ldots \times A_r} = \iota_{A_1} \times \ldots \times \iota_{A_r}$ is an isomorphism.

PROOF. By Theorem 1, it suffices to show that $A_1^* \times \ldots \times A_r^*$ and $\iota_{A_1} \times \ldots \times \iota_{A_r}$ satisfy (1), (3), and (4). We have already observed that it is easy to see that (1) holds. Given $(a_1^*,\ldots,a_r^*) \in A_1^* \times \ldots \times A_r^*$, there exist $\alpha_j \in M$ and $a_j \in A_j$ $(j=1,\ldots,r)$ with $a_j\alpha_j = \iota_{A_j}(a_j)$, each j . Select $\gamma,\lambda_1,\ldots,\lambda_r \in M$ so that $\alpha_j\lambda_j = \gamma$, all j . Then $a_j^*\gamma = \iota_{A_j}(a_j\lambda_j)$, all j , so $(a_1^*,\ldots,a_r^*)\gamma = (\iota_{A_1} \times \ldots \times \iota_{A_r})(a_1\lambda_1,\ldots,a_r\lambda_r)$, and we have verified (3). For $(a_1,\ldots,a_r),(a_1',\ldots,a_r') \in A_1 \times \ldots \times A_r$ we have $\iota_{A_1} \times \ldots \times \iota_{A_r}((a_1,\ldots,a_r)) = \iota_{A_1} \times \ldots \times \iota_{A_r}((a_1',\ldots,a_r'))$ if and only if there exist $\alpha_1,\ldots,\alpha_r \in M$ with $a_j\alpha_j = a_j'\alpha_j$, $j = 1,\ldots,r$. In this case we can find a single $\gamma = \alpha_1\lambda_1 = \ldots = \alpha_r\lambda_r$ with $a_j\gamma = a_j'\gamma$, all j , so $(a_1,\ldots,a_r)\gamma = (a_1',\ldots,a_r')\gamma$. As the reverse implication is trivial, we have verified (4), and we are done.

Let us also observe that for any M-set A and any $r > 0$ there is an

M-set map $\delta_A^{(r)}: A \to A^{(r)}$ (where $A^{(r)}$ denotes the product of r copies of A) given by $\delta_A^{(r)}(a) = (a,\ldots,a)$ for all $a \in A$, and that we have $(\delta_A^{(r)})^* = \delta_{A^*}^{(r)}$.

3. EXTENSION OF OPERATIONS AND RELATIONS

In this section, we show how the results of Section 2 can be used to extend operations, functions, and relations from A to A^* .

We refer to elements of A as <u>constants</u>. A constant c is <u>admissible</u> if $c\alpha = c$ for all $\alpha \in M$. Note that there is a <u>trivial M-set</u> $\{\cdot\}$, for which $\cdot\alpha = \cdot$ for all $\alpha \in M$, and that we may view an admissible constant of A as being the same thing as an M-set map from $\{\cdot\}$ to A . For $r > 0$, an r-place <u>operation</u> on A is a function $f: A^{(r)} \to A$. We say that the operation is <u>admissible</u> if it is a map of M-sets. More generally, an r-place <u>function on A</u> , with <u>values</u> in the set B , is a function $f: A^{(r)} \to B$. The function f is <u>admissible</u> if B is an M-set, and f is a map of M-sets. (Any set B may be given the <u>trivial</u> M-structure, with $b\alpha = b$ for all $b \in B$, $\alpha \in M$.) An r-place <u>relation</u> on A is a subset R of $A^{(r)}$. We may view R as being a function $R: A^{(r)} \to T$, where $T = \{0,1\}$ is our set of "truth values". Then for $a = (a_1,\ldots,a_r) \in A^{(r)}$, we say that $a \in R$ iff $R(a) = 1$. The relation R is <u>admissible</u> if for all $a \in A^r$ and $\alpha \in M$, we have $a \in R$ if and only if $a\alpha \in R$. If we consider R as a function with values in the set T , with T given the trivial M-structure, then this condition is simply that R be an admissible function, that is, an M-set map.

We see that admissible operations, functions, and relations on M-sets may be viewed as M-set maps. By Theorem 2, each has a unique extension from M to M^* . Moreover, any relationship between certain M-sets which may be expressed as saying that a certain composite of M-set maps into the M-set T has constant value 1 will hold when we pass from M to M^* ; by Proposition 3, a composite defined in terms of a product in M will be defined in the corresponding manner in M^* .

This result is quite powerful. We shall work out some consequences for the case in which a single M-set is involved.

We shall describe what we mean by a "universal formula" for an M-set A . The basic building blocks for the formula are the admissible constants of A and the variables $x_1, x_2, \ldots$. The formula will be a finite expression; let us say that the variables that occur in it are among $x_1, \ldots x_n$. We first explain, recursively, what we mean by a term of the formula. Each term is an M-set map from $A^{(n)}$ to A . A constant c is the map sending $a \in A^{(n)}$ to $c \in A$. A variable x_i is the map sending $(a_1,\ldots,a_n) \in A^{(n)}$ to $a_i \in A$. If $t_1,\ldots,t_r$ are terms and f is an r-place operation on A , then $f(t_1,\ldots,t_r)$ is the term sending $a \in A^{(n)}$ to $f(t_1(a),\ldots,t_r(a)) \in A$. We next explain, recursively, how to create formulas. Each formula will be an M-set map from $A^{(n)}$ to the trivial M-set $T = \{0,1\}$. If R is an r-place relation on M and $t_1,\ldots,t_r$ are terms, then the composite $R(t_1,\ldots,t_r)$,

defined by $R(t_1,\ldots,t_r)(a) = R(t_1(a),\ldots,t_r(a))$ for $a \in A^{(n)}$, is a
formula. If A is a formula, then $(\neg A)$ is the formula given by $(\neg A)(a) = 1$
if $A(a) = 0$, $(\neg A)(a) = 0$ if $A(a) = 1$. Similarly, if A and B are
formulas, then $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are the formulas obtained by
viewing the logical operations "$\vee$" (or), "$\wedge$" (and), "$\rightarrow$" (implies), and "$\leftrightarrow$"
(iff) as M-set maps $L: T \times T \rightarrow T$, and composing with the map $A^{(n)} \rightarrow T \times T$
sending a to $(A(a),B(a))$. Note that this latter map is $(A \times B) \circ \delta_{A(n)}^{(2)}$.

Thus a universal formula for A, involving variables from $x_1,\ldots,x_n$,
is interpreted as an M-set map from $A^{(n)}$ to T. We say that the formula
"holds" if, as a function, it is constant with value 1. We call the formula
"universal" because, formally, we interpret it as holding if and only if it
holds "with a universal quantifier for each variable that appears."

By Theorem 2, Proposition 3, and the observation following Proposition 3,
we have the following.

THEOREM 4. Let M be a monoid directed by left divisibility and
satisfying left cancellation. Let A be an M-set. Then every admissible
operation and relation on A has a unique extension to A^*. These
extensions satisfy the same universal formulas in A^* that were satisfied by
the original operations and relations in A.

Let us now discuss the uses of this result, as well as its limitations.

The most important limitation has to do with the "equals" relation. We
often want to express the idea that two composites are equal. For example, if
"+" is a binary operation on A, then the formula $x_1 + x_2 = x_2 + x_1$ means that
$=((a_1+a_2),(a_2+a_1))$ is 1 for all $a_1,a_2 \in A$. In order to be able to apply
the theorem, $=$ must be an admissible relation.

PROPOSITION 5. The equals relation is admissible iff M acts on A by
injections. In this case, $=^*$ is the equals relation on A^*.

PROOF. For equals to be admissible we require $a_1 = a_2$ iff $a_1\alpha = a_2\alpha$,
for all $a_1,a_2 \in A$, $\alpha \in M$. This is clearly the condition that M act on A
by injections. In this case, suppose that $a_1^* =^* a_2^*$ for $a_1^*,a_2^* \in A^*$. There
exists $\gamma \in M$ with $a_1^*\gamma \in A$ and $a_2^*\gamma \in A$. (This makes sense, since ι_A is
injective.) Then $a_1^*\gamma = a_2^*\gamma$, so $a_1^* = (a_1^*\gamma)(\theta_\gamma^{A^*})^{-1} = (a_2^*\gamma)(\theta_\gamma^{A^*})^{-1} = a_2^*$.

When M acts on A by injections, then from Theorem 4 and Proposition 5
we can often get that A^* is the same type of algebraic object as A. For
example, the operation $+$ on A is associative iff it satisfies $(x_1+x_2)+x_3$
$= x_1+(x_2+x_3)$, and in this case $(x_1 \overset{*}{+} x) \overset{*}{+} x_3 = x_1 \overset{*}{+} (x_2 \overset{*}{+} x_3)$, so $\overset{*}{+}$ is an
associative operation on A^*.

Sometimes some reformulation is required. Suppose that A is a field,
and that M acts injectively on A as field endomorphisms. For $a \in A$, set
$\nu(a) = 0$ if $a = 0$, $\nu(a) = a^{-1}$ if $a \neq 0$. Then ν is an admissible
operation on A, and A satisfies the formula $(x_1=0) \vee (x_1 \cdot \nu(x_1)=1)$.
Since A^* satisfies the same formula, we obtain that A^* is a field.

Our next result provides another method for obtaining properties of A^*
in terms of properties of A.

Let f be an admissible r-place operation on A, and let f^* be its extension to A^*. We first note that f^* is defined on $\iota(A)$ by $f^*(\iota(a_1),\ldots,\iota(a_r)) = \iota f(a_1,\ldots,a_r)$ for all $(a_1,\ldots,a_r) \in A^{(r)}$. Now let $a_1^*,\ldots,a_r^*$ be elements of A^*. There exists an element γ of M and elements $a_1,\ldots,a_r$ of A with $a_j^*\gamma = \iota(a_j)$ for $j = 1,\ldots,r$. We have $f^*(a_1^*,\ldots,a_r^*) = f^*(\iota(a_1)(\theta_\gamma^{A^*})^{-1},\ldots,(\iota(a_r)(\theta_\gamma^{A^*})^{-1}) = f^*(\iota(a_1),\ldots,\iota(a_r))(\theta_\gamma^{A^*})^{-1} = (\iota f(a_1,\ldots,a_r))(\theta_\gamma^{A^*})^{-1}$. Since the same argument goes through for relations, we have the following result.

PROPOSITION 6. Let M be a monoid divided by left divisibility and satisfying left cancellation. Let A be an M-set. Then A^* is the directed union of the sets $\iota(A)(\theta_\gamma^{A^*})^{-1}$, $\gamma \in M$. The admissible operations and relations on A are extended from $\iota(A)$ to $\iota(A)(\theta_\gamma^{A^*})^{-1}$ via $\theta_\gamma^{A^*}$; thus $\iota(A)(\theta_\gamma^{A^*})^{-1}$ is "isomorphic" to $\iota(A)$.

Note, however, that $\iota(A)(\theta_\gamma^{A^*})^{-1}$ may not be closed under the action of M if M is not commutative.

We now present our three examples. Note that [1, Thm. VII.3.4] does not apply in these cases, either because the "locally K" conclusion of that theorem does not allow us to infer that certain relations and formulas extend from A to A^*, or because we need to consider two M-sets simultaneously, together with a map between them.

EXAMPLE 7. Let N denote the monoid of natural numbers under addition. Let A be the set $N \times N$. Define a relation $<$ on A by: $(n_1,n_2) < (m_1,m_2)$ iff $n_1+n_2 < m_1+m_2$. Then $<$ is a strict order on A — that is, it satisfies the universal formulas $\neg(x_1<x_1)$; $(x_1<x_2) \rightarrow \neg(x_2<x_1)$; and $((x_1<x_2)\wedge(x_2<x_3)) \rightarrow (x_1<x_3)$. We give A an N-set structure by $(n_1,n_2)\cdot n = (0,n_1+n_2+n)$ for all $(n_1,n_2) \in A$, $n \in N$. The relation $<$ is admissible. By Theorem 4, the universal N-set A^* has a strict order $<^*$ which extends $<$. Let the set $\mathbb{Z}$ of integers be an N-set under addition. Define $j: A \rightarrow \mathbb{Z}$ by $j((n_1,n_2)) = n_1+n_2$. Then j is an N-set map, with $\mathbb{Z}$ and j satisfying (1), (3), and (4) of Theorem 1, so we may take A^* and ι to be $\mathbb{Z}$ and j. We see that $<^*$ is the usual order on $\mathbb{Z}$.

EXAMPLE 8. Let A,N be as above. Change the action of N on A to $(n_1,n_2)\cdot n = (n_1+n, n_2+n)$ for $(n_1,n_2) \in A$, $n \in N$. Then $<$ is still an admissible relation, and now N acts on A by injections. Thus we can view A^* as being a set containing A. We can equip A with the metric d: $A \rightarrow \mathbb{R}$ given by $d((n_1,n_2), (n_1',n_2')) = \sqrt{(n_1-n_1')^2 + (n_2-n_2')^2}$ for $(n_1,n_2), (n_1',n_2') \in A$. We give $\mathbb{R}$ the trivial N-set structure; then d is a map of N-sets. That d is a metric is expressed by the universal formulas: $d(x_1,x_2) \geq 0$; $(d(x_1,x_2) = 0) \leftrightarrow (x_1=x_2)$; $d(x_1,x_2) = d(x_2,x_1)$; $d(x_1,x_3) \leq d(x_1,x_2) + d(x_1,x_2)$. Each formula may be interpreted as asserting that a certain composite of N-set maps from $A^{(2)}$ (or $A^{(3)}$) to $T = \{0,1\}$ has the constant value 1. By Theorem 2 and Proposition 3, the corresponding result holds for A^*. Thus we have that A^* is a metric space containing A, equipped with a strict order $<^*$, on which N acts by isometric automorphisms. Using Theorem

1 we see that A^* can be realized as $\mathbb{Z} \times \mathbb{Z}$, with d^* and $<^*$ given by the formal extension to $\mathbb{Z} \times \mathbb{Z}$ of the formulas defining d and $<$ respectively.

EXAMPLE 9. Let F be a field, and let $K = F(X_1, \ldots X_t)$ be the field of rational functions over F in t indeterminates. Let $\theta: K \to K$ be the F-algebra homomorphism sending X_i to X_i^2, for $1 \leq i \leq t$, and give K the N-set structure $(k,n) \to k\theta^n$ for $k \in K$, $n \in N$. Let $G = \mathbb{R} \cup \{\infty\}$. Extend the operation "+" and the relation "<" from $\mathbb{R}$ to G by setting $a + \infty = \infty = \infty + a = \infty + \infty$ and $a < \infty$ for all $a \in \mathbb{R}$. Make G into an N-set via $(a,n) \to a2^n$ for $a \in \mathbb{R}$, $n \in N$ and $(\infty, n) \to \infty$. Let $\alpha_1, \ldots, \alpha_t$ be rationally independent elements of R. Define a function $\nu: K \to G$ as follows: $\nu(0) = \infty$; $\nu(m)$, where $m = aX_1^{e_1} .. X_t^{e_t}$ is a nonzero monomial $(0 \neq a \in F)$, is $e_1\alpha_1 + \ldots + e_t\alpha_t$; $\nu(f)$, where $f = m_1 + \ldots + m_s$ is the unique representation of $0 \neq f \in F[X_1, \ldots, X_t]$ as a sum of distinct monomials m_i, is $\inf\{\nu(m_i)\}_{i=1}^s$, and $\nu(f/g) = \nu(f) - \nu(g)$ for $0 \neq f, g \in F[X_1, \ldots, X_t]$. The function ν satisfies $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K$, and $\nu(x + y) \leq \inf\{\nu(x), \nu(y)\}$ for all $x, y \in K$ [2, Theorem 18.3]; that is, ν is a valuation on the field K. It is easy to see that ν is a map of N-sets. We note that the ring operations on K are admissible, and that $+$, $<$, and inf are admissible on G. Arguing as in Example 8, we see that N acts as automorphisms of an extension field K^* of K. Since N already acts bijectively on G, we see that $G^* = G$, and that $\nu^*: K^* \to G$ is a valuation on K^*. The structure of K^* is tolerably complicated; it is an infinite algebraic extension of K.

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