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# IDENTIFICATION OF WAITING TIME DISTRIBUTION OF M/G/1, $M^X$/G/1, $GI^r$/M/1 QUEUEING SYSTEMS

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ABSTRACT. This paper brings out relations among the moments of various orders of the waiting time of the 1st customer and a randomly selected customer of an arrival group for bulk arrivals queueing models, and as well as moments of the waiting time (in queue) for M/G/1 queueing system. A numerical study of these relations has been developed in order to find the $(\beta_1, \beta_2)$ measures of waiting time distribution in a comutable form. On the basis of these measures one can look into the nature of waiting time distribution of bulk arrival queues and the single server M/G/1 queue.

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## 1. THE M/G/1 SYSTEM

The Laplace - Steiltjes transform (L.S.T.) of the waiting time distribution of M/G/1 system has been earlier obtained by several methods by many authors and is also known as Pollaczek - Khintchine (P-K) transform formula, namely

$$\bar{w}_q(\alpha) = \frac{\alpha(1-\rho)}{\alpha - \lambda + \lambda\bar{B}(\alpha)} \tag{1.1}$$

where $\bar{w}_q(\alpha)$ and $\bar{B}(\alpha)$ are respectively L.S.T. of the waiting time and service time of d.f.s'.

Takacs [1] obtained a recursive relation for this type of system to estimate the moments of various orders of the waiting time distribution as follows:

$$u_k'(w_q) = \frac{\lambda}{(1-\rho)} \sum_{j=1}^{k} \binom{k}{j} \frac{u_{j+1}'(s)\; u_{k-j}'(w_q)}{(j+1)} \tag{1.2}$$

where $u_k'(w_q) = \int_0^\infty t^k \, d\, w_q(t)$ is the $k_{th}$ moment (about the origin) of the waiting time distribution, and $u_o'(w_q) = 1$.

One can also obtain the above moments by an alternate approach which is as follows: Rewriting the P-K formula (1.1), we have

$$\bar{w}_q(\alpha) = \frac{(1-\rho)}{1-\rho[1-\bar{B}(\alpha)/\alpha\bar{S}]} \qquad (1.3)$$

where $\bar{S} = u_1'(s)$ is the first moment about the origin of the service time (s). We can recognize the brackted terms in the denominator of the equation (1.3) as the L.S.T. of the residual service time density functions as follows:

$$\hat{\bar{B}}(\alpha) = [1-\bar{B}(\alpha)] / \alpha\bar{S} \qquad (1.4)$$

where the residual service time has a distribution i.e. pr(residual service time ≥ y) = 1 - B(y), and $\hat{b}(y)$ is to be a density function. We have

$$\int_0^\infty \hat{b}(y)dy = \int_0^\infty c[1-B(y)]\,dy = 1$$

implying that $c = 1/E(y) = 1/\bar{S}$ and hence $\hat{b}(y) = c[1-B(y)] = \frac{1-B(y)}{\bar{S}}$.

Rewriting (1.2) as follows:

$$\bar{w}_q(\alpha) = \frac{1-\rho}{1-\rho\,\hat{\bar{B}}(\alpha)} \qquad (1.5)$$

whereas (1.5) can be expanded in the form of a power series

$$\text{i.e.}\quad \bar{w}_q(\alpha) = (1-\rho)\sum_{k=0}^{\infty} \rho^k [\hat{\bar{B}}(\alpha)]^k \qquad (1.6)$$

Now making use of the fact $u_k'(w_q) = (-1)^k\, d^k\bar{w}_q(\alpha)/d\alpha^k$, the different moments of the waiting time (in queue) are obtained from (1.6).

$$\text{i.e.}\quad u_1'(w_q) = \frac{\lambda}{(1-\rho)}\, u_2'(s)/2$$

$$u_2'(w_q) = \frac{\lambda}{(1-\rho)}\, u_1'(w_q)u_2'(s) + \frac{\lambda}{(1-\rho)}\, u_3'(s)/3$$

$$u_3'(w_q) = \frac{\lambda}{(1-\rho)}\,\frac{3}{2}\, u_2'(s)u_2'(w_q) + \frac{\lambda}{(1-\rho)}\, u_3'(s)u_1'(w_q) + \frac{\lambda}{(1-\rho)}\, u_4'\frac{(s)}{4}$$

$$u_4'(w_q) = \frac{\lambda}{(1-\rho)}\, 2u_2'(s)u_3'(w_q) + \frac{\lambda}{(1-\rho)}\, 2u_3'(s)u_2'(w_q) + \frac{\lambda}{(1-\rho)}\, u_4'(s)u_1'(w_q) +$$

$$+ \frac{\lambda}{(1-\rho)}\, u_5'(s)/5 \qquad (1.7)$$

and, so on. We can find that the formulae (1.7) are equivalent to (1.2).

## 2. RATIONALE FOR NUMERICAL ANALYSIS.

The moment relations (1.7) have useful applications, both for single server models of the type M/G/1 and bulk gueueing models with group arrivals. This is due to the fact that a statistical distribution can be identified in terms of the first four moments about the mean or about an arbitrary origin (see Davies [2], Elderton [3], Kendall and Stuart [4]).

A numerical study of the moment relations therefore has been made in order to find statistical measures $(\beta_1, \beta_2)$ of the waiting time distribution in a computable form. These measures can be defined as follows:

$$\beta_1 = \mu_3^2 \ / \ \mu_2^3 \quad \text{and} \quad \beta_2 = \mu_4 \ / \ \mu_2^2 . \tag{2.1}$$

The strategy for computing $\mu_1', \mu_2', \mu_3', \mu_4'$ for $w_q$ is as follows:

(a) For each system, we compute four moments from the relations (1.7). For M/G/1 system, we get the moments of waiting time $w_q$ of a random customer, but for bulk arrival systems they give moments of waiting time of the first customer of an arrival group (i.e.$w_{q1}$). For example;

(i) in case of $M^X/G/1$ System, the L.S.T. of the waiting time (in queue) distribution of the first member of an arrival group for is given by:

$$\bar{w}_{q1}(\alpha) = \frac{(1-\rho)\alpha}{\alpha - [1 - A(\bar{B}(\alpha))]} \tag{2.2}$$

Now, if we identify a single customer with a group whose total service time distribution has L.S.T. equal to $A(\bar{B}(\alpha))$, then $\bar{w}_{q1}(\alpha)$ given by (2.2) can be also obtained from an ordinary M/G/1 queue with $\bar{B}(\alpha)$ replaced by $A(\bar{B}(\alpha))$.

(ii) It is shown by Prabhu [5] that the waiting time (in queue) distribution for the system $GI/E_r/1$ is equivalent to the corresponding distribution for the first member of a group in $GI^r/M/1$. We can therefore obtain the first four moments for the first member of an arrival group in $GI^r/M/1$ system from the knowledge of the L.S.T. of waiting time of the $GI/E^r/1$ system. By keeping GI = M, the same is applicable to the systems $M^r/M/1$ and $M/E^r/1$ and hence the relations (1.7) are applicable for obtaining the moments of waiting time of the first customer of an arrival batch of size r.

In order to obtain the moments of waiting time ($w_q$) of a random customer for group arrival models, we therefore proceed to step (b) as follows:

(b) On the basis of (a) we obtain the four moments of waiting time for a random customer $w_q$ by using cumulant relations between $w_q$ and $w_{q1}$ for group arrival models. (These cumulants relations have been obtained in case of $M^X/G/1$ and $GI^r/M/1$ queueing systems as presented below).

3. BULK ARRIVAL SYSTEM.

For a $M^X/G/1$ system, we have the L.S.T. of the waiting time distribution of a randomly selected customer of an arrival group which is given below in terms of the L.S.T. of the distribution of queueing time for the first member of the group service time distribution i.e. $A(\bar{B}(\alpha))$, and the L.S.T. of the service time distribution i.e. $\bar{B}(\alpha)$ (Ref. Chaudhry and Templeton [6]):

$$\text{i.e.} \quad \bar{w}_q(\alpha) = \frac{\bar{w}_{q1}(\alpha)}{\bar{a}} \left[ \frac{1 - A(\bar{B}(\alpha))}{1 - \bar{B}(\alpha)} \right] \tag{3.1}$$

where $A(z) = \sum_{m=1}^{\infty} a_m z^m$ and $\bar{a} = A^{(1)}(1)$.

If we define $\psi(\alpha) = \ln \bar{w}_q(\alpha)$, then we can obtain the cumulants from the relation:

$$k_m(w_q) = (-1)^m \quad \psi^m(0)$$

$$\text{i.e.} \quad K_1(w_q) = \frac{u_2'(\emptyset)}{u_1'(\emptyset)} - \frac{u_2'(\emptyset g)}{u_1'(\emptyset g)}$$

$$k_2(w_q) = k_2(w_{q1}) + [u_3'(\not{s})+2u_2'(\not{s})] / u_1'(\not{s}) - [u_3'(\not{s}_g)+ 2u_2'(\not{s}_g)] / u_1' (\not{s}_g)$$

$$k_3(w_q) = k_3(w_{q1}) - 2u_4'(\not{s}) / u_1'(\not{s})-21\ u_2'(\not{s})u_3'(\not{s}) / 2u_1'(\not{s})^2 - 2u_2'(\not{s})^3 / u_1'(\not{s})^3$$

$$+ 2u_4'(\not{s}_g) / u_1'(\not{s}_g) + 21u_2'(\not{s}_g)\ u_3'(\not{s}_g) / 2u_1'(\not{s}_g)^2 + 2u_2'(\not{s}_g)^3\ /\ u_1' (\not{s}_g)^3 \tag{3.2}$$

Similarly, we can work out the fourth order cumulant relation.

And, for a $GI^r/M/1$ system, we have the L.S.T. of the randomly selected customer of an arrival group which is given below in terms of the L.S.T. of the queueing time for the first member of a group (see Chaudhry and Templeton [6]).

$$\text{i.e.}\quad \bar{w}_q[(1/z - 1)\ \mu] = [\bar{w}_{q1}\ (1/z - 1)\ \mu]\ [1/r \sum_{j=0}^{r-1} (z)^j\ ] \tag{3.3}$$

Differentiating (3.3) w.r.t. z on both sides and putting $z = 1$, we get the cumultant relations between $w_q$ and $w_{q1}$ as follows:

$$\mu k_1(w_q) = \mu\ k_1(w_{q1}) + \frac{r-1}{2}$$

$$\mu^2 k_2(w_q) - 2\mu k_1(w_q) = \mu^2 k_2(w_{q1}) - 2\mu k_1(w_{q1}) + (r-1)(r-5)/12$$

$$\mu^3 k_3(w_q) - 6\ \mu^2 k_2(w_q) + 6\mu k_1(w_q) = \mu^3 k_3(w_{q1}) - 6\mu^2 k_2(w_{q1}) + 6\mu k_1(w_{q1}) - (r-1)(r-3)/4 \tag{3.4}$$

Similarly, we can work out the fourth order cumulant relation.

## 4. NUMERICAL RESULTS.

Identification is based on the evaluation of $\beta_1$ and $\beta_2$ on the basis of criteria developed by Elderton [3] (see also Kendall and Stuart [4]). From the numerical values of the set $(\beta_1,\beta_2)$ we can identify the nature of the waiting time distribution function for $w_q$ on the basis of Pearson's analysis. The results for $M/E_n/1$, $M^x/E_n/1$, $M^r/M/1$ queueing systems have been obtained in this direction as given below:

For $M/E_n/1$ queueing system,(particularly for traffic density exceeding 0.5) it is found on the basis of Pearson's analysis that the form of the distribution function for $w_q$ (waiting time in queue) can be approximated by Pearson type III because all the set points $(\beta_1,\beta_2)$ lie very near to the line $2\beta_2-3\beta_1- 6 = 0$. (See Table Values, Fig. 1)

$$\text{i.e.}\quad d\ w(t) = \text{const.}\ e^{-\alpha t}\ t^{n-1}\ dt;\ \alpha \geq 0,\ n \geq 1,\ 0 < t < \infty, \tag{4.1}$$

on the range $[0, \infty)$ with a concentration at zero, and in a special case, when $n = 1$

$$d\ w\ (t) = \text{const.}\ e^{-\alpha t}\ dt\ ;\quad 0 < t < \infty$$

which is a exponential d.f.

While comparing the observed values of $\beta_1$ and $\beta_2$ in case of $M^x/E_n/1$ queueing system, we find that all values for $(w_{q1})$ lie very near to the line $2\beta_2-3\beta_1-6 = 0$ in the $(\beta_1,\beta_2)$ diagram. Accordingly, in such cases also the form of distribution function of $(w_{q1})$ in Pearson type III. (see Table values, Fig. 2).

The set $(\beta_1,\beta_2)$ values for $(w_q)$ lie above the line $2\beta_2-3\beta_1-6 = 0$ in the region of Pearson type VII distribution (See Table Values, Fig. 3). Accordingly, in such a case the form of the distribution function of $w_q$ (a randomly selected customer of an arrival group) which can be approximated by Pearson type VII distribution truncated on the range $[0,\infty)$ is given by:

$$d\,w(t) = \frac{1}{a\,\beta(\frac{1}{2},m-\frac{1}{2})}\left(1-\frac{t^2}{a^2}\right)^{-m} dt \tag{4.2}$$

where a and m are constants.

**Experimental Values for $M/E_n/1$ System**

| Experimental Data: | | | Statistical Measures for Waiting Time Distribution | |
|---|---|---|---|---|
| n | λ | u | $\beta_1$ | $\beta_2$ |
| 2 | 10 | 15 | 4.95 | 10.08 |
| 2 | 25 | 40 | 5.24 | 10.42 |
| 2 | 30 | 50 | 5.44 | 10.65 |
| 3 | 5 | 6 | 4.19 | 9.20 |
| 3 | 30 | 140/3 | 4.93 | 10.04 |
| 3 | 10 | 20 | 6.09 | 11.41 |
| 4 | 10 | 15 | 4.71 | 9.78 |
| 4 | 20 | 25 | 4.24 | 9.25 |
| 4 | 40 | 75 | 5.57 | 10.77 |

Fig. 1

Experimental Values for $M^x/E_n/1$ System: x Following a Geometric Distribution with p=.50 s.t. pr(x=j)= $p(1-p)^j$ ; j=0, 1.2...

| Experimental | Data | | Statistical Measures for Waiting Time Dist. of the first Member of an Arrival Group. | |
|---|---|---|---|---|
| n | λ | u | $\beta_1$ | $\beta_2$ |
| 1 | 5 | 8 | 4.60 | 9.70 |
| 1 | 20 | 30 | 4.45 | 9.51 |
| 2 | 10 | 15 | 4.58 | 9.67 |
| 2 | 25 | 40 | 4.76 | 9.90 |
| 3 | 10 | 50/3 | 4.80 | 10.05 |
| 3 | 20 | 30 | 4.58 | 9.66 |
| 4 | 5 | 10 | 5.51 | 10.82 |
| 4 | 10 | 15 | 4.55 | 9.63 |

x Following Geometric Dist.with p=.25 s.t.pr(x=j)=$p(1-p)^j$; j=0,1,2..

| n | λ | u | $\beta_1$ | $\beta_2$ |
|---|---|---|---|---|
| 1 | 5 | 30 | 5.83 | 11.32 |
| 1 | 10 | 50 | 5.03 | 10.25 |
| 2 | 10 | 50 | 5.95 | 11.49 |
| 2 | 20 | 90 | 5.31 | 10.62 |
| 3 | 5 | 30 | 8.14 | 14.54 |
| 3 | 10 | 40 | 4.95 | 10.16 |
| 4 | 12 | 60 | 6.78 | 12.59 |
| 4 | 20 | 350/4 | 5.70 | 11.12 |

Fig. 2

Experimental Values for $M^x/E_n/1$ System: x Following Geometric Dist. with p=.50 s.t. $pr(x=j) = p(1-p)^j$ ; j =0.1.2......

| Experimental | Data: | | Statistical Measures for Waiting Time Dist.of a Random Customer of An Arrival Group. | |
|---|---|---|---|---|
| n | λ | u | $\beta_1$ | $\beta_2$ |
| 1 | 15 | 30 | 0.17 | 4.88 |
| 1 | 10 | 20 | 0.38 | 6.21 |
| 2 | 15 | 30 | 0.11 | 4.21 |
| 2 | 12 | 25 | 0.19 | 4.64 |
| 3 | 5 | 20/3 | 0.44 | 7.58 |
| 3 | 10 | 50/3 | 0.11 | 5.44 |
| 4 | 5 | 10 | 0.90 | 8.62 |
| 4 | 12 | 20 | 0.60 | 4.87 |

x Following Geometric Dist. with p=.25 s.t.$pr(x=j)=p(1-p)^j$; j=0,1,2...

| | | | | |
|---|---|---|---|---|
| 1 | 12 | 70 | 0.10 | 3.83 |
| 1 | 10 | 50 | 0.13 | 4.36 |
| 2 | 5 | 30 | 0.31 | 4.48 |
| 2 | 10 | 50 | 0.07 | 3.60 |
| 3 | 5 | 30 | 0.17 | 4.01 |
| 3 | 10 | 40 | 0.03 | 3.58 |
| 4 | 5 | 25 | 0.18 | 4.10 |
| 4 | 10 | 50 | 0.03 | 3.32 |

Fig. 3

Experimental Values for $M^r/M/1$ System

| Experimental | | Data: | Statistical Measures for Waiting Time Distribution | |
|---|---|---|---|---|
| r | λ | u | $\beta_1$ | $\beta_2$ |
| 2 | 20 | 70 | 4.75 | 11.02 |
| 2 | 5 | 20 | 5.02 | 12.06 |
| 2 | 12 | 50 | 5.11 | 12.43 |
| 3 | 10 | 60 | 4.32 | 11.53 |
| 3 | 20 | 100 | 4.22 | 10.40 |
| 4 | 5 | 40 | 3.99 | 11.20 |
| 4 | 10 | 60 | 4.02 | 9.79 |
| 10 | 10 | 180 | 3.64 | 10.12 |
| 10 | 15 | 400 | 3.11 | 12.04 |
| 10 | 20 | 400 | 3.52 | 10.54 |

Fig. 4

While in case of $M^r/M/1$ queueing system, the above experimental values (fig. 4) while displayed in $(\beta_1,\beta_2)$ diagram show that the form of the distribution function of $w_q$ (waiting time of a random customer of an arriving batch size of r) can be approximated by Pearson type IV distribution truncated on the range $[0,\infty)$ and is given by:

$$d\ w(t) = k\ (1+ \frac{t^2}{a^2})^{-m} \exp\ [-v \text{ arc tan } (t/a)]\ dt$$

where K, a, v are constants.

However, all these conclusions are limited to the extent that for general service time mechanisms', and Erland or Negative Exponential distribution has been assumed. Also, a few experimental results are given in the Tables above, although a number of experiments have been made in these directions in order to find the statistical measures $(\beta_1,\beta_2)$ for waiting time distribution.

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