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A NOTE ON CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. The object of the present paper is to show a result for functions belonging to class $P'(1-\alpha,0)$ which is a subclass of close-to-convex functions in the unit disk U.

KEY WORDS AND PHRASES. Close-to-Convex of order α, Class $P'(\alpha)$, Class $P'(1-\alpha,0)$, subordination.
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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ belonging to A is said to be in the class $P'(\alpha)$ (according to Goodman [4]) if and only if it satisfies the condition

$$\text{Re}\{f'(z)\} > \alpha \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$ and for all $z \in U$. Note that $P'(\alpha)$ the subclass of close-to-convex functions of order α in the unit disk U. Further, let $P'(1-\alpha,0)$ (according to Goodman [4]) be the subclass of A consisting of all functions which satisfy the condition

$$|f'(z) - 1| < 1 - \alpha \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$ and for all $z \in U$.

It is clear that $P'(1-\alpha,0)$ is the subclass of $P'(\alpha)$ for $0 \leq \alpha < 1$. Nunokawa, Fukui, Owa, Saitoh and Sekine [1] showed that functions in $P'(1-\alpha,0)$ are starlike in $|z| < r_1$, where r_1 is the root of the equation

$$\log \left\{ \frac{1 - (2/(3-\alpha))^2 \, (r - (1-\alpha)r^2/2)^2}{1 - r^2} \right\} + \mathrm{Sin}^{-1}((1-\alpha)r) = \pi.$$

Also, Fukui, Owa, Ogawa and Nunokawa [2] proved that functions in $P'(\alpha)$ are starlike in $|z| < r_2$, where r_2 is the smallest root in $[0,1)$ of the equation

$$\mathrm{Sin}^{-1} \frac{2(1-\alpha)r}{1 - (2\alpha-1)r^2} + \log \frac{1}{1 - r^2} = \pi.$$

For the functions $f(z)$ and $g(z)$ belonging to A, we say that $f(z)$ is subordinate to $g(z)$ in U if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1$ for $z \in U$ and $f(z) = g(w(z))$. We denote by $f(z) \prec g(z)$ this subordination. In particular, if $g(z)$ is univalent in U the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (cf. [3]).

2. MAIN RESULT

In order to prove our main result, we have to recall here the following lemma due to Miller and Mocanu [5].

LEMMA. Let $q(z)$ be an injective mapping of $\bar{U}$ onto $\bar{Q}$, with $q(0) = 1$, such that $q(z)$ is regular on $\bar{U}$ except for at most one pole on ∂U. Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in U with $p(z) \not\equiv 1$. If there exists a point $z_0 \in U$ such that $p(z_0) \in \partial U$ and $p(|z| < |z_0|) \subset Q$, then $z_0 p'(z_0) = m w_0 q'(w_0)$, where $m \geq 1$ and $w_0 = e^{i\theta_0} = q^{-1}(p(z_0))$.

Applying the above lemma, we derive

THEOREM. Let the function $f(z)$ defined by (1) be in the class $P'(1-\alpha,0)$. Then

$$\frac{f(z)}{z} \prec 1 + \frac{(1 - \alpha)z}{2}. \tag{1.4}$$

PROOF. Let $q(z) = 1 + (1 - \alpha)z/2$ and $p(z) = f(z)/z$. It is clear that the result holds true if $p(z) \equiv 1$ for $z \in U$.

Assume that $p(z) \not\equiv 1$ for $z \in U$ and the subordination $p(z) \prec q(z)$ does not hold in U. Then there exists a point $z_0 \in U$ such that $p(z_0) \in \partial q(U)$ and $p(|z| < |z_0|) \subset q(U)$. Therefore, applying the lemma, we get

$$\begin{aligned} f'(z_0) &= z_0 p'(z_0) + p(z_0) \\ &= n w_0 q'(w_0) + q(w_0) \\ &= \frac{m(1-\alpha)w_0}{2} + \frac{(1-\alpha)w_0}{2} + 1 \\ &= 1 + \frac{(m + 1)\,(1-\alpha)w_0}{2}, \end{aligned} \tag{1.5}$$

where $m \geq 1$ and $|w_0| = 1$. Thus

$$|f'(z_0) - 1| = \frac{(m + 1)(1 - \alpha)}{2} \geq 1 - \alpha, \tag{1.6}$$

which contradicts the hypothesis that $f(z) \in P'(1-\alpha,0)$. So we must have $p(z) \prec q(z)$ in U. This completes the proof of Theorem.

Finally, we have

CORALLARY 1. Let the function $f(z)$ defined by (1.1) be in the class $P'(1-\alpha,0)$. Then

$$\text{Re}\left\{e^{i\beta}\frac{f(z)}{z}\right\} > 0,$$

where $|\beta| \leq \pi/2 - \text{Sin}^{-1}(1 - \alpha)/2$.

CORALLARY 2. Let the function $f(z)$ defined by (1.1) be in the class $P'(1-\alpha,0)$. Then

$$\text{Re}\left\{\frac{f(z)}{z}\right\} > 0.$$

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