

SUBWEAKLY α - CONTINUOUS FUNCTIONS

DAVID A. ROSE

Department of Mathematics
Francis Marion College
Florence, South Carolina 29501

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ABSTRACT. In a recent paper by T. Noiri [1], a function $f: X \rightarrow Y$ is said to be weakly α -continuous if $f: X^\alpha \rightarrow Y$ is weakly continuous where X^α is the space X endowed with the α -topology. Similarly, we define subweak α -continuity and almost α -continuity and show that almost α -continuity coincides with the almost continuity of T. Husain [2] and H. Blumberg [3]. This implies a functional tridecomposition of continuity using almost continuity and subweak α -continuity.

KEYWORDS AND PHRASES. Weakly continuous, subweakly continuous, almost continuous, semi-continuous, α -continuous, weakly α -continuous, subweakly α -continuous, $w^*.c.$, locally weak* continuous, A-function, α -function, semi-open, almost open, α -open, tri-decomposition of continuity.

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1. INTRODUCTION.

In 1961, Norman Levine showed that a function $f: X \rightarrow Y$ between arbitrary topological spaces X and Y is continuous if and only if it is both weakly continuous and $w^*.c.$ [4], where weak continuity and $w^*.c.$ are independent strict generalizations of continuity. (This weak continuity was called weak θ -continuity by S. Fomin in 1940 [5]). Levine's decomposition of continuity was strengthened in 1978 [6] by replacing $w^*.c.$ with a strictly weaker condition, local weak* continuity. Another important generalization of continuity is the almost continuity of T. Husain [2] and H. Blumberg [3]. For closed graph functions between complete metric spaces almost continuity implies continuity [7] [8]. Recently, M. Ganster and I.L. Reilly [9] have found a strict generalization of continuity which for almost continuous functions between arbitrary topological spaces implies continuity. In this paper, a tridecomposition of continuity is found using almost continuity and based on the improved version of Levine's decomposition of continuity. Along the way, some recent results of T. Noiri [1] on weakly α -continuous functions are extended, a new characterization of semi-open functions is found, and some results of A.S. Mashhour et.al [12] are improved.

2. PRELIMINARIES.

Unless specified, no structure beyond a topology $O(X)$ is assumed for a space X . Relative to this topology, the interior and closure of a subset $A \subset X$ are denoted $\text{Int } A$ and $\text{Cl } A$ respectively. For the boundary of A we write $\text{Bd } A$. The collection of almost open subsets of the space X , denoted $\text{AO}(X)$, consists of those subsets $A \subset X$ such that $A \subset \text{Int } \text{Cl } A$. The collection of semi-open subsets of X , written $\text{SO}(X)$, consists of those subsets A satisfying $A \subset \text{Cl } \text{Int } A$ [10]. A subset $A \subset X$ is called α -open if $A \subset \text{Int } \text{Cl } \text{Int } A$, and the α -open subsets of X are precisely the sets of the form $U - N$ where $U \in O(X)$ is open and $N \subset X$ is nowhere dense. i.e. $\text{Int } \text{Cl } N = \emptyset$. The collection of α -open subsets of X , written $\alpha O(X)$, is a topology for X called the α -topology for X , and it contains $O(X)$ [11]. The underlying set for the space X , endowed with the α -topology for X , will be written X^α so that $\alpha O(X) = O(X^\alpha)$. For a subset $A \subset X^\alpha$, the interior and closure (relative to the α -topology for X) will be denoted $\alpha\text{-Int } A$ and $\alpha\text{-Cl } A$ respectively. It is also useful to note that $\alpha O(X) = O(X^\alpha) = \text{AO}(X) \cap \text{SO}(X)$ [14] [15].

DEFINITION 1. A function $f: X \rightarrow Y$ is

- (a) almost continuous if $f^{-1}(V) \in \text{AO}(X)$ for each $V \in O(Y)$ [2];
- (b) semi-continuous if $f^{-1}(V) \in \text{SO}(X)$ for each $V \in O(Y)$ [10];
- (c) α -continuous if $f: X^\alpha \rightarrow Y$ is continuous [12] [13].

Clearly, a function f is α -continuous if and only if it is both almost continuous and semi-continuous [14] [15].

DEFINITION 2. A function $f: X \rightarrow Y$ is

- (a) weakly continuous if $f^{-1}(V) \subset \text{Int } f^{-1}(\text{Cl } V)$ for each $V \in O(Y)$ [5] [4];
- (b) subweakly continuous if there is an open basis B for $O(Y)$ such that $\text{Cl } f^{-1}(V) \subset f^{-1}(\text{Cl } V)$ for each $V \in B$ [16];
- (c) weakly α -continuous if $f: X^\alpha \rightarrow Y$ is weakly continuous [1];
- (d) $w^*.c.$ if $f^{-1}(\text{Bd } V)$ is closed for each $V \in O(Y)$ [4];
- (e) locally weak* continuous if there is an open basis B for $O(Y)$ such that $f^{-1}(\text{Bd } V)$ is closed for each $V \in B$ [6].

V. Popa [17] showed that every almost continuous and semi-continuous function is weakly continuous. So α -continuity implies weak continuity which implies individually both subweak continuity and weak α -continuity. None of these implications individually is reversible and subweak continuity and weak α -continuity are independent [16] [1]. It is known [4] ([6]) that a function f is continuous if and only if it is both weakly continuous and $w^*.c.$ (locally weak* continuous).

DEFINITION 3. A function $f: X \rightarrow Y$ is

- (a) subweakly α -continuous if $f: X^\alpha \rightarrow Y$ is subweakly continuous;
- (b) almost α -continuous if $f: X^\alpha \rightarrow Y$ is almost continuous;
- (c) semi- α -continuous if $f: X^\alpha \rightarrow Y$ is semi-continuous.

O. Njastad [11] showed that $\text{SO}(X^\alpha) = \text{SO}(X)$ so that semi- α -continuity is simply semi-continuity. It is also true that $\alpha O(X^\alpha) = \alpha O(X) = O(X^\alpha)$ so that $(X^\alpha)^\alpha = X^\alpha$ [11]. Therefore, $f: X^\alpha \rightarrow Y$ is α -continuous if and only if $f: (X^\alpha)^\alpha \rightarrow Y$ is continuous which occurs if and only if $f: X \rightarrow Y$ is α -continuous. In the next section we will show that $\text{AO}(X^\alpha) = \text{AO}(X)$ and consequently almost α -continuity

coincides with almost continuity. We conclude this section with two more definitions.

DEFINITION 4. A function $f: X \rightarrow Y$ is

- (a) semi-open if $f(U) \in SO(Y)$ for each $U \in O(X)$ [10];
- (b) almost open if $f(U) \in AO(Y)$ for each $U \in O(X)$;
- (c) α -open if $f: X \rightarrow Y^\alpha$ is open [12].

Almost openness in the sense of A. Wilansky [18] is equivalent to almost openness here [16].

DEFINITION 5. A function $f: X \rightarrow Y$ is an

- (a) A-function if $f^{-1}(B) \in AO(X)$ whenever $B \in AO(Y)$;
- (b) α -function if $f: X^\alpha \rightarrow Y^\alpha$ is continuous.

In the literature, A-functions have been called pre-irresolute functions [19], and α -functions were introduced as α -irresolute functions by S.N. Maheshwari and S.S. Thakur [23].

3. ALMOST α -CONTINUITY IS ALMOST CONTINUITY.

It is known [16] that functions which are both almost continuous and subweakly continuous are weakly continuous. Thus, functions which are both almost α -continuous and subweakly α -continuous are weakly α -continuous. It will be shown that such functions are actually weakly continuous by proving the title result of this section. We begin with a lemma giving a new characterization of semi-open functions.

LEMMA 1. A function $f: X \rightarrow Y$ is semi-open if and only if

$$f^{-1}(\text{Int Cl } B) \subset \text{Cl } f^{-1}(B) \text{ for each subset } B \subset Y.$$

PROOF. (sufficiency) For any subset $B \subset Y$,

$$f^{-1}(Y - \text{Int Cl } B) = X - f^{-1}(\text{Int Cl } B) \supset X - \text{Cl } f^{-1}(B) = \text{Int}(X - f^{-1}(B)) = \text{Int } f^{-1}(Y - B).$$

Thus, $f(\text{Int } f^{-1}(Y - B)) \subset Y - \text{Int Cl } B = \text{Cl Int}(Y - B)$ for all $B \subset Y$. Now for any subset $A \subset X$; by setting $B = Y - f(A)$, we have $Y - B = f(A)$ and $A \subset f^{-1}(Y - B)$.

Thus, $f(\text{Int } A) \subset f(\text{Int } f^{-1}(Y - B)) \subset \text{Cl Int } f(A)$.

Therefore, if $A \in O(X)$, $f(A) \in SO(Y)$ showing that f is semi-open.

(necessity) For any subset $B \subset Y$, $f(X - \text{Cl } f^{-1}(B)) = f(\text{Int } f^{-1}(Y - B)) \subset$

(by semi-openness) $\text{Cl Int } f(\text{Int } f^{-1}(Y - B)) \subset \text{Cl Int}(Y - B) = Y - \text{Int Cl } B$.

Therefore, $X - \text{Cl } f^{-1}(B) \subset X - f^{-1}(\text{Int Cl } B)$ and $f^{-1}(\text{Int Cl } B) \subset \text{Cl } f^{-1}(B)$

for every subset $B \subset Y$.

The next theorem was obtained by D.S. Jankovic [24, Proposition 3.4].

THEOREM 1. If $f: X \rightarrow Y$ is an almost continuous and semi-open function, then f is an A-function.

PROOF. For $B \in AO(Y)$, $B \subset \text{Int Cl } B$, so that $f^{-1}(B) \subset f^{-1}(\text{Int Cl } B) \subset$ (by almost continuity) $\text{Int Cl } f^{-1}(\text{Int Cl } B) \subset$ (by Lemma 1) $\text{Int Cl } f^{-1}(B)$.

Therefore, $f^{-1}(B) \in AO(X)$ showing that f is an A-function.

COROLLARY 1. For any space X , $AO(X^\alpha) = AO(X)$.

PROOF. If $f: X \rightarrow X^\alpha$ is the identity function then f and f^{-1} are each both almost continuous and semi-open.

As an immediate consequence of Corollary 1 we have the title result of this section.

COROLLARY 2. A function $f: X \rightarrow Y$ is almost α -continuous if and only if it is almost continuous.

4. A TRIDECOMPOSITION OF CONTINUITY.

In the literature, a decomposition (or more accurately, a bidecomposition) of continuity consists of two independent function properties each strictly weaker than continuity which jointly imply continuity [4] [20]. Such a decomposition is improved or strengthened by weakening one or both of the function properties while maintaining continuity jointly [6] [9]. In this section we find a tridecomposition of continuity into three independent function properties each strictly weaker than continuity. By three independent properties we mean that not one of them is implied jointly by the other two properties. Our tridecomposition is based on the improved version of Levine's decomposition [6], and the fact that subweak continuity and almost continuity jointly imply weak continuity [16]. The following theorem is an almost immediate consequence of this fact.

THEOREM 2. If $f: X \rightarrow Y$ is almost continuous and subweakly α -continuous then it is weakly continuous.

PROOF. By the remarks above and corollary 2 of theorem 1, for a function $f: X \rightarrow Y$ satisfying the hypotheses of this theorem, $f: X^\alpha \rightarrow Y$ is almost continuous and subweakly continuous. Thus, $f: X^\alpha \rightarrow Y$ is weakly continuous. So, $f: X \rightarrow Y$ is almost continuous and weakly α -continuous and T. Noiri showed [1] that such functions are weakly continuous.

Example 1 below shows that subweak α -continuity is strictly weaker than weak α -continuity so that theorem 2 is a strict improvement of the supporting result of Noiri [1, theorem 4.9].

The next theorem gives three properties each strictly weaker than continuity which jointly are equivalent to continuity. A discussion of existing examples will show that these three properties are independent and thus comprise a tridecomposition of continuity.

THEOREM 3. A function $f: X \rightarrow Y$ is continuous if and only if it is almost continuous, subweakly α -continuous, and locally weak* continuous.

PROOF. The necessity is obvious and the sufficiency follows from theorem 2 and the improved version of Levine's decomposition of continuity [6].

To see that none of the weaker than continuity properties of theorem 3 is implied by the other two jointly, recall first that a function was found [6, Example 3] which was almost continuous, weakly continuous, and hence also weakly α -continuous and therefore subweakly α -continuous, but not locally weak* continuous. Next, the identity function from a set with at least two elements having the indiscrete topology onto the same set with the discrete topology [16, Example 1], is almost continuous since the domain is indiscrete, and w*.c. and hence also locally weak* continuous since the range is discrete. But by theorem 3, this function cannot be subweakly α -continuous and therefore not weakly α -continuous since it is not continuous.

Finally, consider the identity function from a set with non-discrete T_1 topology onto the same set with the discrete topology [16, Example 4]. Because the domain is T_1 , the function is subweakly continuous and hence subweakly α -continuous using the basis of singleton subsets of the range. Also, this function is $w^*.c.$ and thus locally weak* continuous since the range is discrete. Again, by theorem 3, the function cannot be almost continuous since it is not continuous. These examples not only show the independence of the three weaker than continuity properties of theorem 3, but also of the triplets {almost continuity, subweak continuity, $w^*.c.$ } and {almost continuity, subweak continuity, local weak* continuity}. Hence, each of these triplets is a tridecomposition of continuity with the latter being strictly stronger than the preceding one since local weak* continuity is strictly weaker than $w^*.c.$ [6]. The following example shows that subweak α -continuity is strictly weaker than subweak continuity so that the tridecomposition of continuity presented in theorem 3 is strictly the strongest of these three tridecompositions.

EXAMPLE 1. Let $N = \{1, 2, \dots\}$ have the cofinite topology $O(N)$ and let $X = \{0, 1, 2, \dots\}$ have the topology $O(X) = O(N) \cup \{X\}$. I.L. Reilly and M.K. Vamanamurthy have shown [21, Example 2] that X is not T_1 but X^α is T_1 . Therefore, if $Y = \{0, 1, 2, \dots\}$ has the discrete topology $O(Y)$, the identity function $f: X \rightarrow Y$ cannot be subweakly continuous since every open basis for $O(Y)$ contains the singleton subsets of Y and for some $n \in Y$, $Cl f^{-1}(\{n\}) \neq \{n\} = f^{-1}(Cl\{n\})$ since X is not T_1 . However, $f: X^\alpha \rightarrow Y$ is subweakly continuous using the basis of singleton sets for $O(Y)$ since X^α is T_1 . Therefore, $f: X \rightarrow Y$ is subweakly α -continuous. Further, because Reilly and Vamanamurthy have shown [21, Proposition 1]) that a space X is discrete if and only if X^α is discrete, X^α is not discrete and $f: X^\alpha \rightarrow Y$ is not continuous. Therefore, since Y is regular, and weakly continuous functions into regular spaces are continuous [4, theorem 2], $f: X^\alpha \rightarrow Y$ cannot be weakly continuous, and hence $f: X \rightarrow Y$ is not weakly α -continuous.

5. PRODUCT THEOREMS FOR SUBWEAK α -CONTINUITY .

In this section it is shown that the class of subweakly α -continuous functions is closed under arbitrary products.

THEOREM 4. If the function $f: X \rightarrow \prod Y_a$ is such that each Q_a of: $X \rightarrow Y_a$ is subweakly α -continuous where $Q_a: \prod Y_a \rightarrow Y_a$ is the a^{th} projection, then f is subweakly α -continuous .

PROOF. If each Q_a of: $X^\alpha \rightarrow Y_a$ is subweakly continuous, then it is known [22, theorem 4] that $f: X^\alpha \rightarrow \prod Y_a$ is subweakly continuous and therefore, $f: X \rightarrow \prod Y_a$ is subweakly α -continuous.

COROLLARY 3. If each $f_a: X \rightarrow Y_a$ is a subweakly α -continuous function and if $f: X \rightarrow \prod Y_a$ is the function defined by $f(x) = \{f_a(x)\}$, then f is subweakly α -continuous .

PROOF. If $Q_a: \prod Y_a \rightarrow Y_a$ is the a^{th} projection, then $f_a = Q_a \circ f$.

LEMMA 2. If $f: X \rightarrow Y$ is a continuous function and $g: Y \rightarrow Z$ is a subweakly continuous function then $g \circ f: X \rightarrow Z$ is subweakly continuous.

PROOF. Let B be an open basis for $O(Z)$ for which g is subweakly continuous. For each $V \in B$, $Cl (gof)^{-1}(V) = Cl f^{-1}(g^{-1}(V)) \subset Cl f^{-1}(Cl g^{-1}(V)) = f^{-1}(Cl g^{-1}(V)) \subset f^{-1}(g^{-1}(Cl V)) = (gof)^{-1}(Cl V)$.

COROLLARY 4. If $f: X \rightarrow Y$ is an α -function and $g: Y \rightarrow Z$ is a subweakly α -continuous function, then $gof: X \rightarrow Z$ is subweakly α -continuous.

PROOF. If $f: X^\alpha \rightarrow Y^\alpha$ is continuous and $g: Y^\alpha \rightarrow Z$ is subweakly continuous, then by lemma 2, $gof: X^\alpha \rightarrow Z$ is subweakly continuous so that $gof: X \rightarrow Z$ is subweakly α -continuous.

THEOREM 5. If each $f_a: X_a \rightarrow Y_a$ is a subweakly α -continuous function, then $\Pi f_a: \Pi X_a \rightarrow \Pi Y_a$ is subweakly α -continuous.

PROOF. Let $X = \Pi X_a$ and $f = \Pi f_a$, and let $Q_a: \Pi Y_a \rightarrow Y_a$ and $P_a: X \rightarrow X_a$ be the a^{th} projections. It was shown by A.S. Mashhour et.al. [12, Theorem 3.2] that functions which are both α -continuous and almost open are α -functions. Thus, each Q_a is an α -function being continuous and open. Therefore, if each $f_a: X_a \rightarrow Y_a$ is subweakly α -continuous, by corollary 5, each $f_a \circ Q_a$ is subweakly α -continuous. Thus, $f: X \rightarrow \Pi Y_a$ is subweakly α -continuous by theorem 4 since each $P_a \circ f = f_a \circ Q_a$.

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