Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 113-118

BOUNDED SPIRAL-LIKE FUNCTIONS WITH FIXED SECOND COEFFICIENT

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(Received July 10, 1986)

ABSTRACT. Let $F_p(\alpha,\beta,M)$ $(o \leq p \leq 1,\ |\alpha| < \frac{\pi}{2},\ o \leq \beta < 1$ and $M > 1/2)$, denote the class of functions $f(z)$ which are regular in $U = \{z:|z| < 1\}$ and of the form $f(z) = z + |a_2|\, e^{-i\alpha} z^2 + \ldots$, where $|a_2| = p(1+\sigma)(1-\beta)\cos\alpha$, which satisfy for fixed M, $z = re^{i\theta} \in U$ and

$$\left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)} - \beta\cos\alpha - i\sin\alpha}{(1-\beta)\cos\alpha} - M\right| < M.$$

In this paper we have found the sharp radius of γ-spiralness of the functions belonging to the class $F_p(\alpha,\beta,M)$.

KEY WORDS AND PHRASES. Spirallike, bounded functions, radius of γ-spiralness. .
1980 AMS CLASSIFICATION CODES. 30A32, 30A36.

1. INTRODUCTION. Let A denote the class of functions which are regular and univalent in the unit disc $U = \{z:|z| < 1\}$ and satisfy the conditions $f(0) = 0 = f'(0)-1$.

Let $F(\alpha,\beta,M)$ $(|\alpha| < \frac{\pi}{2},\ 0 \leq \beta < 1$ and $M > 1/2)$ denote the class of bounded α-spirallike functions of order β, that is $f \in F(\alpha,\beta,M)$ if and only if for fixed M,

$$\left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)} - \beta\cos\alpha - i\sin\alpha}{(1-\beta)\cos\alpha} - M\right| < M,\ z \in U. \tag{1.1}$$

The class $F(\alpha,\beta,M)$ introduced by Aouf [1], he proved that if $f(z) = z + a_2 z^2 + \ldots \in F(\alpha,\beta,M)$ then,

$$|a_2| \leq (1+\sigma)(1-\beta)\cos\alpha,\ \sigma = 1 - \frac{1}{M}. \tag{1.2}$$

If $\varepsilon = \exp(-i \arg a_2 - i\alpha)$, then $\frac{f(\varepsilon z)}{\varepsilon} = z + |a_2| e^{-i\alpha} z^2 + \ldots \in F(\alpha,\beta,M)$, whenever $f(z) \in F(\alpha,\beta,M)$. Thus without loss of generality we can replace the second coefficient a_2 of $f(z) \in F(\alpha,\beta,M)$ by $|a_2| e^{-i\alpha}$.

Let $F_p(\alpha,\beta,M)$ denote the class of functions $f(z) = z + |a_2| e^{-i\alpha} z^2 + \ldots.,$

which satisfy (1.1), where $|a_2| = p(1 + \sigma)(1 - \beta) \cos \alpha$. In view of (1.2) it follows that $0 < p < 1$.

Let $G_p(\alpha,\beta,M)$ denote the class of functions $g(z) = z + |b_2| e^{-i\alpha} z^2 + \ldots$, regular in U and satisfy the condition

$$\left| \frac{e^{i\alpha}\left(1+ \frac{zg''(z)}{g'(z)}\right) - \beta\cos\alpha - i \sin \alpha}{(1-\beta) \cos \alpha} - M \right| < M, \quad z \in U, \tag{1.3}$$

where $|b_2| = \frac{1}{2} p(1+\sigma)(1-\beta) \cos \alpha$.

It follows from (1.1) and (1.3) that

$$g(z) \in G_p(\alpha,\beta,M), \text{ if and only if } zg'(z) \in F_p(\alpha,\beta,M). \tag{1.4}$$

We note that by giving specific values to p,α,β and M, we obtain the following important subclasses studied by various authors in earlier papers:

(i) $F_1(\alpha,\beta,M) = F_M(\alpha,\beta)$ and $G_1(\alpha,\beta,M) = G_M(\alpha,\beta)$, are respectively the class of bounded spirallike functions of order β and the class of bounded Robertson functions of order β investigated by Aouf [1] and $F_1(\alpha,0,M) = F_{\alpha,M}$ and $G_1(\alpha,0,M) = G_{\alpha,M}$, are respectively the class of bounded spirallike functions and the class of bounded Robertson functions investigated by Kulshrestha [2].

(ii) $F_p(\alpha,\beta,\infty) = F_p(\alpha,\beta)$ and $G_p(\alpha,\beta,\infty) = G_p(\alpha,\beta,)$, are considered by Umarani [3].

In this paper we determine the sharp radius of γ-spiralness of the functions belonging to the class $F_p(\alpha,\beta,M)$, generalizing an earlier result due to Kulshrestha [2], Libera [4], Umarani [5,3].

The technique employed to obtain this result is similar to that used by McCarty [6] and Umarani [3].

2. THE SHARP RADIUS OF γ-SPIRALNESS OF THE CLASS $F_p(\alpha,\beta,M)$, $M > 1$.

LEMMA 1. If $f(z) \in F_p(\alpha,\beta,M)$ $M > 1$, then $\left|\frac{zf'(z)}{f(z)} - w_o\right| < \rho_o$, $\quad$ (2.1) where

$$w_o = \frac{(1+pr)^2 + \{[(1-\beta)(\frac{1+\sigma}{\sigma}) -1] \cos \alpha - i \sin \alpha\} e^{-i\alpha} r^2 (r+p)^2}{(1-r^2)(1+2pr + r^2)} \tag{2.2}$$

and

$$\rho_o = \frac{(1+\sigma)(1-\beta)\cos\alpha \; r(1+pr)(r+p)}{(1-r^2)(1+2pr+r^2)} . \tag{2.3}$$

This result is sharp.

PROOF. Let $f(z) \in F_p(\alpha,\beta,M)$, $M > 1.$, then there exists a function $w(z)$ analytic in U and $|w(z)| < 1$ in U such that

$$e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos \alpha \left\{\frac{1 + [(1-\beta)(\frac{1+\sigma}{\sigma}) -1] \sigma w(z)}{1 - \sigma w(z)}\right\} + i \sin \alpha, \; \sigma = 1 - \frac{1}{M}$$

or

$$\frac{zf'(z)}{f(z)} = \frac{1 + \{[(1-\beta)(\frac{1+\sigma}{\sigma}) -1] \cos \alpha - i \sin \alpha\} e^{-i\alpha} \sigma w(z)}{1 - \sigma w(z)} .$$

Solving for $w(z)$,

$$w(z) = \frac{\frac{zf'(z)}{f(z)} - 1}{\sigma[\frac{zf'(z)}{f(z)} + \{[1-\beta)(\frac{1+\sigma}{\sigma}) -1] \cos \alpha - i \sin \alpha\} e^{-i\alpha}]} .$$

Since $f(z) = z + |a_2| e^{-i\alpha} z^2 + \ldots$, we obtain $w(z)=pz + \ldots = z\phi(z)$, where $\phi(z)$ is analytic in U, $\phi(0) = p$ and $|\phi(z)| < 1$ in U. Now $\frac{\phi(z)-p}{1-p\phi(z)}$ z. Therefore $\phi(z)$ $\frac{z + p}{1 + pz}$. Also $|w(z)| = |z\phi(z)| < \frac{|z| + p}{1 + |z|p} |z|$. Let $g(z) = \frac{|z| + p}{1+p|z|} z$

and

$$h(z) = \frac{1 + \{[(1-\beta)(\frac{1+\sigma}{\sigma})-1] \cos \alpha - i \sin \alpha\} e^{-i\alpha} \sigma z}{1 - \sigma z} .$$

Since the image of $|z| < r$ under $g(z)$ is a disc and $h(z)$ is a bilinear transformation, then $\frac{zf'(z)}{f(z)}$ is subordinate to $(h \circ g)(z)$. That is, the image of $|z| < r$ under $\frac{zf'(z)}{f(z)}$ is contained in the image of $|z| < r$ under $(h \circ g)(z)$.

Equality in (2.1) can be attained by a function

$$f(z) = z(1-2p\sigma z+\sigma z^2)^{-(\frac{1+\sigma}{2\sigma})(1-\beta)\cos\alpha e^{-i\alpha}} \tag{2.4}$$

$$= z + p(1+\sigma)(1-\beta)\cos\alpha\, e^{-i\alpha} z^2 + \ldots \quad ;$$

hence

$$\frac{zf'(z)}{f(z)} = \frac{1-2p\sigma z+\sigma z^2-(1+\sigma)(1-\beta)\cos\alpha\, e^{-i\alpha} z(z-p)}{1-2p\sigma z + \sigma z^2}$$

$$= \frac{1 + \sigma\psi-(1+\sigma)(1-\beta)\cos\alpha\, e^{-i\alpha}\psi}{1 + \sigma\psi}, \qquad (2.5)$$

where $\psi = \frac{z(z-p)}{1-p\sigma z}$.

Since $p \leq 1$, $0 \leq \sigma < 1$, $|\psi| < 1$ for $z \in U$.

This shows that

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = \cos\alpha \left\{\frac{1 + [1- (\frac{1+\sigma}{\sigma})(1-\beta)]\ \sigma\psi(z)}{1 + \sigma\psi(z)}\right\} + i \sin\alpha$$

and

$$\frac{e^{i\alpha}\frac{zf'(z)}{f(z)} - i \sin\alpha-\beta\cos\alpha}{(1-\beta)\cos\alpha} = \frac{1 -\psi(z)}{1+\sigma\psi(z)}.$$

Then it is easy to show that $\left|\frac{1 - \psi(z)}{1 + \sigma\psi(z)} - M\right| < M$, $\sigma = 1 - \frac{1}{M}$. Thus $f \in F_p(\alpha,\beta,M)$.

Substituting $\psi = -\frac{\delta(\delta - \sigma e^{i\alpha})}{\sigma(1-\sigma\delta e^{i\alpha})}$, where $\delta = \frac{r(r+p)}{1+rp}$ in (2.5), we find that

$\left|\frac{zf'(z)}{f(z)} - w_o\right| = \rho_o$, where w_o and ρ_o are given by (2.2) and (2.3).

This completes the proof of the lemma.

REMARK 1.

(i) If $p=1$ and $\beta=0$ in Lemma 1, we obtain a result of Kulshrestha [2].

(ii) If $M = \infty(\sigma=1)$ in Lemma 1, we obtain a result of Umarani [3].

(iii) If $\alpha=0$ and $M=\infty(\sigma=1)$ in Lemma 1, we obtain a result of McCarty [6].

THEOREM 1. If $f(z) \in F_p(\alpha,\beta,M)$, > 1, then $f(z)$ is γ-spiral $|z| < r_\gamma$, where r_γ is the smallest positive root of the equation

$$\cos\gamma + p\,[2\cos\gamma - (1+\sigma)(1-\beta)\cos\alpha]r +$$

$$[p^2\cos\gamma + cp^2 -(1 + \sigma)(1-\beta)\cos\alpha(1+ p^2)]\, r^2$$

$$+p\,[2c-(1+\sigma)(1-\beta)\cos\alpha]r^3 + cr^4=0, \qquad (2.6)$$

where $c = \cos(\gamma-2\alpha) + [(1-\beta)(\frac{1+\sigma}{\sigma})-2] \cos\alpha \cos(\gamma-\alpha)$. The result is sharp.

PROOF. Let $f(z) \in F_p(\alpha,\beta,M)$, $M > 1$, then by the above Lemma, we have

$$\left|\frac{zf'(z)}{f(z)} - w_o\right| < \rho_o.$$

Hence $\mathrm{Re}\, e^{i\gamma} \frac{zf'(z)}{f(z)} > \mathrm{Re}\, e^{i\gamma} \cdot w_o - \rho_o$

$$\cos\gamma\,(1+pr)^2 + \mathrm{Re}\,\{[(1-\beta)(\tfrac{1+\sigma}{\sigma}) -1] \cos\alpha - i \sin\alpha\}\, e^{i(\gamma-\alpha)} r^2 (r+p)^2$$
$$= [\frac{-(1+\sigma)(1-\beta)\cos\alpha\, r(1+pr)(r+p)}{(1-r^2)(1+2pr+r^2)}]$$

$$\cos\gamma(1+pr)^2 + \{\cos(\gamma-2\alpha) + [(1-\beta)(\tfrac{1+\sigma}{\sigma})-2] \cos\alpha \cos(\gamma-\alpha)\} r^2 (r+p)^2$$
$$= [\frac{-(1+\sigma)(1-\beta)\cos\alpha\; r(1+pr)(r+p)}{(1-r^2)(1+2pr+r^2)}] \,. \tag{2.7}$$

$f(z)$ is γ-spiral if the R.H.S. of (2.7) is positive. Hence $f(z)$ is γ-spiral for $|z| < r_\gamma$ where r_γ is the smallest positive root of the equation

$$\cos\gamma\,(1+pr)^2 + \{\cos(\gamma-2\alpha) + [(1-\beta)(\tfrac{1+\sigma}{\sigma})-2] \cos\alpha \cos(\gamma-\alpha)\}.\ r^2(r+p)^2$$

$$-(1+\sigma)(1-\beta)\cos\alpha\; r(1+pr)(r+p) = 0.$$

Simplifying the above equation, we obtain (2.6).

If $\gamma=0$ in the above theorem, we obtain the radius of starlikeness of the class $F_p(\alpha,\beta,M)$.

COROLLARY 1. $f(z) \in F_p(\alpha,\beta,M)$, $M > 1$, is starlike for $|z| < r_o$, where r_o is the least positive root of the equation

$$1+p\,[2-(1+\sigma)(1-\beta)\cos\alpha]r +$$
$$(\tfrac{1+\sigma}{\sigma})(1-\beta)\cos\alpha\,[\cos\alpha p^2 - \sigma(1+p^2)]r^2 +$$
$$p[2c-(1+\sigma)(1-\beta)\cos\alpha]\, r^3 + cr^4 = 0, \tag{2.8}$$

where $c = (\frac{1+\sigma}{\sigma})(1-\beta)\cos^2\alpha - 1$.

If $p=1$, $\gamma=0$ and $\beta=0$ in Theorem 1, we obtain a result of Kulshrestha [2].

COROLLARY 2. $f(z) \in F_{\alpha,M}$, $M > 1$, is starlike for $|z| < r_o$, where r_o is the least positive root of the equation

$$1-(1+\sigma)\cos\alpha\; r + [(\tfrac{1+\sigma}{\sigma})\cos^2\alpha - 1]\, r^2 = 0.$$

REMARK 2.

(i) If $M=\infty$ $(\sigma=1)$ in Theorem 1, we obtain a result of Umarani [3].

(ii) If $p=1$ and $M=\infty$ $(\sigma=1)$ in Theorem 1, we obtain a result of Libera [4] and Umarani [5].

(iii) If $p=1$, $\beta=0$, $\gamma=0$ and $M=\infty$ $(\sigma=1)$ in Theorem 1, we obtain a result of Robertson [7].

Since $g(z) \in G_p(\alpha,\beta,M)$ if and only if $zg'(z) \in F_p(\alpha,\beta,M)$ we obtain from Theorem 1,

THEOREM 2. If $g(z) \in G_p(\alpha,\beta,M)$, $M > 1$, then $\operatorname{Re} e^{i\gamma}(1 + \frac{zg''(z)}{g'(z)}) > 0$ for $|z| < r_\gamma$, where r_γ is the least positive root of equation (2.6). The result is sharp.

If $\gamma=0$ in Theorem 2, we obtain the radius of convexity of the class $G_p(\alpha,\beta,M)$.

COROLLARY 3. If $g(z) \in G_p(\alpha,\beta,M)$, $M > 1$, then the radius of convexity of $g(z)$ is the least positive root of equation (2.8).

REMARK 3.

(i) For $M=\infty$ $(\sigma=1)$ in Theorem 2, and Corollary 3, we obtain a results of Umarani [3].

(ii) If $p=1$ and $\beta=0$ in Corollary 3, we obtain a result of Kulshrestha [2].

(iii) For $p=1$ and $M=\infty$ $(\sigma=1)$, Theorem 2, generalizes the result of Umarani [5].

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