Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 199-204

RESEARCH NOTES

NOTES ON CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

SHIGEYOSHI OWA

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
JAPAN

TADAYUKI SEKINE

Department of Mathematics
College of Science and Technology
Nihon University
Kanda-Surugadai, Chiyoda, Tokyo 101
JAPAN

TERUO YAGUCHI

Department of Mathematics
College of Humanities and Sciences
Nihon University
Sakurajosui, Setagaya, Tokyo 156
JAPAN

MAMORU NUNOKAWA

Department of Mathematics
Gunma University
Aramaki, Maebashi 371
JAPAN

and

DONKA PASHKOULEVA

Institute of Mathematics with Computer Center
Bulgarian Academy of Sciences
1090 Sofia
BULGARIA

(Received January 10, 1987)

ABSTRACT. The object of the present paper is to show the extreme points of the class $A(n,B_k)$ consisting of analytic functions with negative coefficients and the support points of the subclass $A^*(n,B_k)$ of $A(n,B_k)$.

KEYS WORDS AND PHRASES. Starlike of order α, convex of order α, extreme point, continuous linear functional, support point.

1980 AMS SUBJECT CLASSIFICATION CODE. 30C45.

1. INTRODUCTION.

Let A(n) be the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geq 0;\ n \in N = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

A function $f(z)$ in $A(n)$ is said to be in the class $A(n,B_k)$ if and only if it satisfies the condition

$$\sum_{k=n+1}^{\infty} B_k a_k \leq 1 \qquad (B_k > 0). \tag{1.2}$$

Note that $A(n,C_k) \subseteq A(n,B_k)$ for $0 < B_k \leq C_k$. The class $A(n,B_k)$ was introduced by Sekine [1].

A function $f(z)$ belonging to $A(n)$ is said to be starlike of order α if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in U$. We denote by $T_n^*(\alpha)$ the subclass of $A(n)$ consisting of functions which are starlike of order α in the unit disk U.

Further, a function $f(z)$ belonging to $A(n)$ is said to be convex of order α if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.4}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in U$. Also we denote by $C_n(\alpha)$ the subclass of $A(n)$ consisting of all convex functions of order α in the unit disk U.

For the above classes $T_n^*(\alpha)$ and $C_n(\alpha)$, Chatterjee [2] has showed the following results.

LEMMA 1. The function $f(z)$ defined by (1.1) is in the class $T_n^*(\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} \left(\frac{k - \alpha}{1 - \alpha}\right) a_k \leq 1. \tag{1.5}$$

LEMMA 2. The function $f(z)$ defined by (1.1) is in the class $C_n(\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} \left(\frac{k(k - \alpha)}{1 - \alpha}\right) a_k \leq 1. \tag{1.6}$$

It follows from Lemma 1 that $A(n,B_k) \subseteq T_n^*(\alpha)$ for $B_k \geq (k - \alpha)/(1 - \alpha)$, and from Lemma 2 that $A(n,B_k) \subseteq C_n(\alpha)$ for $B_k \geq k(k - \alpha)/(1 - \alpha)$. Further, we note that a function $f(z)$ in $A(n,B_k)$ with $B_k \geq k$ is univalent in the unit disk U.

2. EXTREME POINTS.

We begin with the statement and the proof of the following result.

THEOREM 1. $A(n,B_k)$ is the convex subfamily of $A(n)$.

PROOF. Let the functions

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geq 0;\ j = 1,2) \tag{2.1}$$

be in the class $A(n,B_k)$. Defining the function $h(z)$ by

$$\begin{aligned} h(z) &= \lambda f_1(z) + (1 - \lambda) f_2(z) \qquad (0 \leq \lambda \leq 1) \\ &= z - \sum_{k=n+1}^{\infty} \{\lambda a_{k,1} + (1 - \lambda) a_{k,2}\} z^k \\ &= z - \sum_{k=n+1}^{\infty} A_k z^k, \end{aligned} \tag{2.2}$$

we have

$$\sum_{k=n+1}^{\infty} B_k A_k = \sum_{k=n+1}^{\infty} B_k\{\lambda a_{k,1} + (1-\lambda)a_{k,2}\}$$

$$= \lambda \sum_{k=n+1}^{\infty} B_k a_{k,1} + (1-\lambda) \sum_{k=n+1}^{\infty} B_k a_{k,2} \leq 1 \tag{2.3}$$

which implies that $h(z) \in A(n,B_k)$. This completes the proof of Theorem 1.

Next, we show

THEOREM 2. Let

$$f_1(z) = z \tag{2.4}$$

and

$$f_k(z) = z - \frac{1}{B_k} z^k \qquad (k \geq n+1). \tag{2.5}$$

Then $f(z) \in A(n,B_k)$ iff $f(z)$ can be expressed as

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z), \tag{2.6}$$

where $\lambda_1 \geq 0$, $\lambda_k \geq 0$ $(k \geq n+1)$, and

$$\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1. \tag{2.7}$$

PROOF. Suppose that $f(z)$ can be expressed as (2.6). Then we have

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)$$

$$= z - \sum_{k=n+1}^{\infty} \frac{\lambda_k}{B_k} z^k$$

$$= z - \sum_{k=n+1}^{\infty} A_k z^k. \tag{2.8}$$

It follows from (2.8) that

$$\sum_{k=n+1}^{\infty} B_k A_k = \sum_{k=n+1}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1 \tag{2.9}$$

which shows $f(z) \in A(n,B_k)$.

Conversely, suppose that $f(z)$ defined by (1.1) belongs to the class $A(n,B_k)$. Since $B_k a_k \leq 1$ for $k \geq n+1$, we may put $\lambda_k = B_k a_k$ $(k \geq n+1)$ and

$$\lambda_1 = 1 - \sum_{k=n+1}^{\infty} \lambda_k.$$

Therefore, we have

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$$

$$= \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k (z - \frac{1}{B_k} z^k)$$

$$= \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z) \tag{2.10}$$

which completes the assertion of Theorem 2.

Now, we have

COROLLARY 1. The extreme points of $A(n,B_k)$ are the functions $f_1(z)$ and $f_k(z)$ given in Theorem 2.

With the aid of Lemma 1, we have

COROLLARY 2. The extreme points of $T_n^*(\alpha)$ are $f_1(z) = z$ and

$$f_k(z) = z - \left(\frac{1-\alpha}{k-\alpha}\right) z^k \qquad (k \geq n+1).$$

Furthermore, by Lemma 2, we have

COROLLARY 3. The extreme points of $C_n(\alpha)$ are $f_1(z) = z$ and

$$f_k(z) = z - \left(\frac{1-\alpha}{k(k-\alpha)}\right) z^k \qquad (k \geq n+1).$$

3. SUPPORT POINTS.

Let $A^*(n,B_k)$ be the subclass of $A(n,B_k)$ such that $B_k \geq k$. Then a function $f(z)$ belonging to $A^*(n,B_k)$ is univalent in the unit disk U, and $A^*(n,B_k)$ is a convex subfamily of univalent functions with negative coefficients.

A function $f(z)$ in the class $A^*(n,B_k)$ is said to be a support point of $A^*(n,B_k)$ if there exists a continuous linear functional J on $A(n)$ such that $\mathrm{Re}\{J(f)\} \geq \mathrm{Re}\{J(g)\}$ for all $g(z) \in A^*(n,B_k)$, and $\mathrm{Re}\{J\}$ is nonconstant on $A^*(n,B_k)$. We denote by $\mathrm{Supp}\{A^*(n,B_k)\}$ the set of support points of $A^*(n,B_k)$, and also the set of extreme points of $A^*(n,B_k)$ is denoted by $\mathrm{Ext}\{A^*(n,B_k)\}$.

Let F be a subfamily of univalent functions in the unit disk U whose set of extreme points is countable, suppose f_0 is a support point of F, and let J be a corresponding continuous linear functional. Defining G_J by

$$G_J = \{f \in F: \ \mathrm{Re}\{J(f)\} = \mathrm{Re}\{J(f_0)\}\}, \tag{3.1}$$

Deeb [3] has proved the following result.

LEMMA 3. Let G_J be defined by (3.1). Then G_J is convex, $\mathrm{Ext}\{G_J\} \subset \mathrm{Ext}\{F\}$, and

$$G_J = \{f \in F: \ f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z),\ \lambda_i \geq 0,\ \sum_{i=1}^{\infty} \lambda_i = 1,\ f_i(z) \in \mathrm{Ext}\{G_J\}\}. \tag{3.2}$$

Let A denote the class of functions of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{3.3}$$

which are analytic in the unit disk U. Then Brickman, MacGregor and Wilken [4] have shown the following lemma.

LEMMA 4. Let $\{b_k\}$ be a sequence of complex numbers such that

$$\lim_{k\to\infty} \sup |b_k|^{1/k} < 1,$$

and set

$$J(f) = \sum_{k=0}^{\infty} a_k b_k \tag{3.4}$$

for $f(z) \in A$ given by (3.3). Then J is a continuous linear functional on A. Conversely, any continuous linear functional on A is given by such a sequence $\{b_k\}$.

Applying the above lemmas, we prove

THEOREM 3. The set Supp $A^*(n,B_k)$ of support points of $A^*(n,B_k)$ is given by

$$\mathrm{Supp}\{A^*(n,B_k)\} = \{f \in A^*(n,B_k):\ f(z) = z - \sum_{k=n+1}^{\infty}\left(\frac{\lambda_k}{B_k}\right)z^k,\ \lambda_k \geqq 0,$$

$$\sum_{k=n+1}^{\infty}\lambda_k \leqq 1,\ \lambda_j = 0 \text{ for some } j\}. \tag{3.5}$$

PROOF. Let the function $f_0(z)$ be in the class $A^*(n,B_k)$, and let

$$f_0(z) = z - \sum_{k=n+1}^{\infty}\left(\frac{\lambda_k}{B_k}\right)z^k, \tag{3.6}$$

where $\lambda_k \geqq 0$, $\sum_{k=n+1}^{\infty}\lambda_k \leqq 1$, and $\lambda_j = 0$ for some $j \geqq n+1$. If $b_k = 0$ for $k \geqq n+1$, $k \neq j$, and $b_1 = b_j = 1$, then $\lim_{k\to\infty}\sup|b_k|^{1/k} < 1$. Then, with the aid of Lemma 4, we can define the continuous linear functional J given by the sequence $\{b_k\}$. It follows the above that $J(f_0) = 1$, and that $J(f) = 1 - a_j \leqq 1$ for $f(z) \in A^*(n,B_k)$ given by (1.1). Thus we have $\mathrm{Re}\{J(f_0)\} \geqq \mathrm{Re}\{J(f)\}$ for all $f(z)$ in $A^*(n,B_k)$. This shows that $f_0(z)$ is a support point of $A^*(n,B_k)$.

Conversely, suppose that $f_0(z)$ is a support point of $A^*(n,B_k)$ and that its continuous linear functional J is given by the sequence $\{b_k\}$. Note that Re{J} is also continuous and linear on $A^*(n,B_k)$. Consequently, by the Krein-Milman theorem, there exists an extreme point $f_k(z)$ of the class $A^*(n,B_k)$ such that

$$\mathrm{Re}\{J(f_0)\} = \mathrm{Max}\{\mathrm{Re}\{J(f)\}:\ f(z) \in A^*(n,B_k)\} = \mathrm{Re}\{J(f_k)\}.$$

Let .

$$G_J = \{f_k:\ \mathrm{Re}\{J(f_0)\} = \mathrm{Re}\{J(f_k)\},\ f_k(z) \in \mathrm{Ext}\{A^*(n,B_k)\}\}.$$

If $G_J = \mathrm{Ext}\{A^*(n,B_k)\}$, then Re{J} must be constant on $A^*(n,B_k)$. This contradicts that $f_0(z)$ is a support point of $A^*(n,B_k)$. Therefore, there exists a j such that $\mathrm{Re}\{J(f_0)\} > \mathrm{Re}\{J(f_j)\}$. It follows that

$$\mathrm{Ext}\{G_J\} \subset \{f_k:\ f_k(z) \in \mathrm{Ext}\{A^*(n,B_k)\},\ k \geqq n+1,\ k \neq j\}.$$

Hence, by Lemma 3, we have

$$f_0(z) = \sum_{k=n+1}^{\infty}\lambda_k f_k(z), \tag{3.7}$$

where $\lambda_k \geqq 0$, $\sum_{k=n+1}^{\infty}\lambda_k = 1$, and $f_k(z) \in \mathrm{Ext}\{G_J\}$, $k \geqq n+1$, $k \neq j$. It follows from this and Corollary 1 that

$$f_0(z) = z - \sum_{\substack{k=n+1\\ k\neq j}}^{\infty}\left(\frac{\lambda_k}{B_k}\right)z^k \tag{3.8}$$

which completes the proof of Theorem 3.

COROLLARY 4. The set of support points of $T_n^*(\alpha)$ is given by

$$\mathrm{Supp}\{T_n^*(\alpha)\} = \{f \in T_n^*(\alpha):\ f(z) = z - \sum_{k=n+1}^{\infty} \left(\frac{(1-\alpha)\lambda_k}{k-\alpha} \right) z^k, \quad \lambda_k \geq 0,$$

$$\sum_{k=n+1}^{\infty} \lambda_k \leq 1,\ \lambda_j = 0 \text{ for some } j\}.$$

Finally, we have

COROLLARY 5. The set of support points of $C_n(\alpha)$ is given by

$$\mathrm{Supp}\{C_n(\alpha)\} = \{f \in C_n(\alpha):\ f(z) = z - \sum_{k=n+1}^{\infty} \left(\frac{(1-\alpha)\lambda_k}{k(k-\alpha)} \right) z^k, \quad \lambda_k \geq 0,$$

$$\sum_{k=n+1}^{\infty} \lambda_k \leq 1,\ \lambda_j = 0 \text{ for some } j\}.$$

REFERENCES

1. SEKINE, T. On new generalized class of analytic functions with negative coefficients, to appear.

2. CHATTERJEA, S.K. On starlike functions, J. Pure Math. 1 (1981), 23-26.

3. DEEB, W. Extreme and support points of families of univalent functions with real coefficients, Math. Rep. Toyama Univ. 8 (1985), 103-111.

4. BRICKMAN, L., MACGREGOR, T.H. and WILKEN, D.R. Convex hulls of some classical families of univalent functions, Trans. Amer. Math. Soc. 156 (1971), 91-107.