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CHANGES IN SIGNATURE INDUCED BY THE LYAPUNOV MAPPING $\mathcal{L}_A : X \to AX + XA^*$

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ABSTRACT. The Lyapunov mapping on n x n matrices over C is defined by $\mathcal{L}_A(X) = AX + XA^*$; a matrix is stable iff all its characteristic values have negative real parts; and the inertia of a matrix X is the ordered triple $In(X) = (\pi,\nu,\delta)$ where π is the number of eigenvalues of X whose real parts are positive, ν the number whose real parts are negative, and δ the number whose real parts are 0. It is proven that for any normal, stable matrix A and any hermitian matrix H, if $In(H) = (\pi,\nu,\delta)$ then $In(\mathcal{L}_A(H)) = (\nu,\pi,\delta)$. Further, if stable matrix A has only simple elementary divisors, then the image under $\mathcal{L}_A$ of a positive-definite hermitian matrix is negative-definite hermitian, and the image of a negative-definite hermitian matrix is positive-definite hermitian.

KEY WORDS AND PHRASES. Lyapunov, stable matrix, matrix inertia, positive-definite matrix
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For many years stable matrices have interested applied mathematicians because, for a system of linear homogeneous differential equations whose coefficients are constant, a stable matrix of coefficients is a necessary and sufficient condition that the solution be asymptotically stable. Recently, algebraists too have become interested in stable matrices.

Definition: A square matrix is stable $\Leftrightarrow$ all its characteristic values have negative real parts.

(In this article, the entries of all matrices are complex numbers unless stated otherwise.)

A classical test for stability of matrices is Lyapunov's theorem, whose statement is facilitated by some notation:

$\mathcal{S}$ = set of all nxn stable matrices
$\mathcal{H}$ = set of all nxn hermitian matrices
$i\mathcal{H}$ = set of all nxn skew-hermitian matrices
Π = set of all nxn positive-definite hermitian matrices
$\mathcal{N}$ = set of all nxn negative-definite hermitian matrices
$\mathcal{L}_A(X)$ = $AX+XA^*$, where A and X are nxn matrices and A^* is the conjugate transpose of A.

(It is trivial to verify that $\mathcal{L}_A(\bullet)$, the Lyapunov mapping, is a linear transformation on the linear space M_n of nxn matrices.)

Lyapunov's theorem is usually expressed as statement a) of

Theorem 1: The following three statements are equivalent:

a) $A \in \mathcal{S} \Leftrightarrow$ there exists $G \in \Pi$ such that $\mathcal{L}_A(G) = -I$;

b) $A \in \mathcal{S} \Leftrightarrow$ for every $G_1 \in \mathcal{N}$, there exists $G \in \Pi$ such that $\mathcal{L}_A(G) = G_1 \Leftrightarrow$ there exists $G_1 \in \mathcal{N}$ and there exists $G \in \Pi$ such that $\mathcal{L}_A(G) = G_1$ [Taussky, 1964; p. 6, thms 2-3];

c) Let $C = aI+S$ (a real and < 0, $S \in i\,\mathcal{H}$) and $D = \text{diag}(d_1,\ldots,d_n)$ with d_i real ($i=1,\ldots,n$). Then $CD \in \mathcal{S} \Leftrightarrow d_i > 0$ for all i. [Taussky, 1961, J. Math Anal. & App.].

The equivalences are proven (essentially) in Taussky's articles. An analytic proof a) is in Bellman, pp. 242-245, and a topological proof in Ostrowski & Schneider.

Theorem 1 suggests that the operator $\mathcal{L}_A(\bullet)$ might give rise to other tests for stability; such usefulness is limited, however, by the following

Theorem 2: The range of $\mathcal{L}_A(H)$ as a function of $H \in \Pi$ and $A \in \mathcal{S}$ is that subset of $\mathcal{H}$ with $\nu \neq 0$ (where ν denotes the number of characteristic vectors with negative real parts). [Stein, p. 352, thm 2].

Some useful theorems result if further restrictions are imposed on A besides stability. These theorems are obtained via a topological route and require additional concepts.

Definition: The inertia of an nxn matrix X is the ordered triple of integers $(\pi(X), \nu(X), \delta(X)) = \text{In}(X)$ where $\pi(X)$ is the number of characteristic values of X whose real parts are positive, $\nu(X)$ the number whose real parts are negative, and $\delta(X)$ the number whose real parts are 0. If nxn matrices M and N possess the same inertia, this will be denoted by $M \overset{i}{\sim} N$.

Let M and N be nxn hermitian matrices. M and N are congruent (denoted $M \overset{c}{\sim} N$) $\Leftrightarrow \exists$ P non-singular such that $M = P^*NP$.

Recall that all norms in the set of all nxn matrices M_n induce the same topology. In M_n so topologized, matrices M and N are connected $\Leftrightarrow$ there exists a connected set containing both M and N. The relationship of being connected is an equivalence relation, which will be denoted by $\overset{u}{\sim}$. M and N are arc-wise connected $\Leftrightarrow$ there exists a continuous function f from the real interval [0,1] into M_n such that $f(0) = M$ and $f(1) = N$. This, too, is an equivalence relation in M_n and will be denoted by $\overset{a}{\sim}$.

The preceding concepts are brought together by the following theorem:

Theorem 3: In the set N_n of all non-singular nxn matrices with the relative topology induced by any norm, $A \overset{u}{\sim} B$ and $A \overset{a}{\sim} B$ ($\forall A, B \in N_n$). [Schneider; pp. 818-819, lemmata 1 & 2]. Let $\mathcal{H}_r^n$ denote the set of all nxn hermitian matrices of rank r. In $\mathcal{H}_r^n$ with the relative topology induced by any norm the four equivalence relations $\overset{u}{\sim}$, $\overset{a}{\sim}$, $\overset{i}{\sim}$, $\overset{c}{\sim}$ coincide. [Schneider; p. 820].

The relationship between algebraic features of hermitian matrices and topological features expressed by theorem 3 makes it possible to discover the variation in signature induced by the Lyapunov mapping $\mathcal{L}_A(\bullet)$ whenever $A \in \mathcal{S}$ is normal and $H \in \mathcal{H}$.

Theorem 4: If $A \in \mathcal{S}$ is normal, then for any $H \in \mathcal{H}$ with $\text{In}(H) = (\pi,\nu,\delta)$, $\text{In}(\mathcal{L}_A(H)) = (\nu,\pi,\delta)$.

Proof: Let $A \in \mathcal{S}$ be normal, $\{a_i\}_1^n$ be its characteristic values, $H \in \mathcal{H}$, $\text{In}(H) = (\pi,\nu,\delta)$, and $\mathcal{L}_A(H) = AH+HA^* = C$.

Since A is normal, it is unitarily similar to a diagonal matrix: $VAV^* = \text{diag}(a_1,\ldots,a_n)$, V unitary. Also a basis for n-dimensional space can be formed from the characteristic vectors of A, $\{\alpha_i\}_1^n$.

For any i, $\alpha_i C = \alpha_i(AH+HA^*) = \alpha_i a_i H + \alpha_i HA^* = \alpha_i H(a_i I + A^*)$. The number of independent $\alpha_i C$ is the rank of C; it is also the rank of $H(a_i I + A^*)$ = rank of H (since $a_i I + A^*$ is non-singular, for the characteristic values of $-A^*$ are $\{-\bar{a}_i\}_1^n$ and $\{a_i\}_1^n \cap \{-a_i\}_1^n = \emptyset$ because real part of $\bar{a}_i$ = real part of $a_i < 0$ (i=1,...,n).) Therefore, rank (H) = rank ($\mathcal{L}_A$(H)).

Because $\mathcal{L}_A$ is a linear transformation of M_n onto itself, it is continuous. If $\mathcal{L}_A$ is restricted to $\mathcal{H} \subseteq M_n$ it is continuous and onto $\mathcal{H}$. Therefore, $\mathcal{L}_A$ maps topologically connected components of $\mathcal{H}_r^n$ onto components of $\mathcal{H}_r^n$ since rank is preserved by $\mathcal{L}_A$. But by theorem 3 topologically connected components coincide with inertial components. Therefore, $\mathcal{L}_A$ maps In(H) on In(C).

$H \in \mathcal{H}$ and since VHV* is congruent to H, In(VHV*) = In(H). Hence, In($\mathcal{L}_A$(VHV*)) = In($\mathcal{L}_A$(H)) = In(C).

Let D = $\mathcal{L}_A$(VHV*) = A(VHV*) + (VHV*)A*. Then V*DV = (V*AV)H + H(V*A*V). Because $D \in \mathcal{H}$, In($\mathcal{L}_{VAV^*}$(H)) = In(V*DV) = In(D) = In(C).

H is congruent to $K = I_\pi \oplus -I_\nu \oplus 0_\delta$, so In(K) = In(H), whence In($\mathcal{L}_{VAV^*}$(K)) = In($\mathcal{L}_{VAV^*}$(H)) = In(C). $\mathcal{L}_{VAV^*}$(K) is of the form

$$\text{diag}(a_1,\ldots,a_n)(I_\pi \oplus -I_\nu \oplus 0_\delta) + (I_\pi \oplus -I_\nu \oplus 0_\delta)\,\text{diag}(\bar{a}_1,\ldots,\bar{a}_n)$$
$$= 2\,\text{diag}(R(a_1),\ldots, R(a_\pi), -R(a_{\pi+1}),\ldots, -R(a_{\pi+\nu}), 0,\ldots,0),$$

where R(a) denotes the real part of complex number a, I_m the mxm identity matrix, and 0_m the mxm zero matrix. Since $R(a_i) < 0$ (i=1,...,n), In($\mathcal{L}_{VAV^*}$(K)) = (ν,π,δ). Therefore, In(C) = (ν,π,δ). QED

The preceding theorem was based on the unitary similarity of A to a diagonal matrix; this property was used first to show the invariance of rank and then to display the inertia when both A and H were expressed in canonical form. The next theorem generalizes the last in that A need be similar (not unitarily similar) to a diagonal matrix, but it is more restrictive of the inertia of H.

Theorem 5: If $A \in \mathcal{S}$ has only simple elementary divisors, then $\mathcal{L}_A(\Pi) = \mathcal{N}$ and $\mathcal{L}_A(\mathcal{N}) = \Pi$.

Proof: Since A has only simple elementary divisors, it is similar to a diagonal matrix. As in the proof of the preceding theorem, rank (H) = rank ($\mathcal{L}_A$(H)). Likewise, $\mathcal{L}_A$ maps In(H) on In($\mathcal{L}_A$(H)). By Lyapunov's theorem (1a), $\exists H \in \Pi : \mathcal{L}_A(H) = -I \in \mathcal{N}$. Therefore, $\mathcal{L}_A(\Pi) \subseteq \mathcal{N}$. But by the alternative version (1b) of Lyapunov's theorem, $\mathcal{N} \subseteq \mathcal{L}_A(\Pi)$.

The second equation follows from $-\mathcal{L}_A(H) = \mathcal{L}_A(-H) = I$. QED

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