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RESEARCH NOTES

QUASI-PROJECTIVE MODULES AND THE FINITE EXCHANGE PROPERTY

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ABSTRACT. We define a module M to be directly refinable if whenever $M = A + B$, there exists $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$ such that $M = \bar{A} \oplus \bar{B}$. Theorem. Let M be a quasi-projective module. Then M is directly refinable if and only if M has the finite exchange property.

KEY WORDS AND PHRASES. Quasi-projective, finite exchange property, co-continuous.
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Examples of directly refinable modules are semisimple modules, hollow modules [1], dual continuous modules [2], and strongly supplemented modules [6]. In [3, Proposition 2.9], Nicholson has shown that a projective module has the finite exchange property [5] if and only if it is directly refinable. In this note we will generalize this result to quasi-projective modules. We also observe that from Nicholson's result and [4, Corollary 4] every projective module over a regular ring is directly refinable. Our first lemma presents some of the basic properties of directly refinable modules.

LEMMA 1. Let M be a directly refinable module.

(i) If X is a direct summand of M, then X is directly refinable.

(ii) If $M = \Sigma_{i=1}^{n} A_i$, then $M = \oplus_{i=1}^{n} X_i$ where $X_i \subseteq A_i$ for $i = 1, 2, \ldots, n$. In particular, if A_i is cyclic, then X_i is cyclic. Hence, if M is finitely generated, then it is a direct sum of cyclic submodules.

(iii) If M has no infinite direct sum of direct summands of M, then M is strongly supplemented.

PROOF. The proof is routine.

From [6] a module M is said to be <u>co-continuous</u> if whenever $M = A + B$, there exists $f \in \text{End}(M)$ such that $f(M) \subseteq A$ and $(1 - f)(M) \subseteq B$. Zöschinger observes [6, p. 242] that quasi-projective modules are co-continuous.

LEMMA 2. If M is a co-continuous module which has the finite exchange property, then M is directly refinable.

PROOF. Assume $M = A + B$. There exists $f \in \text{End}(M)$ such that $f(M) \subseteq A$ and $(1 - f)(M) \subseteq B$. From [3, Proposition 1.1 and Theorem 2.1], there exists $e, h, k \in \text{End}(M)$ such that $fh = e$ and $(1 - f)k = 1 - e$ where $e = e^2$. Therefore, $M = e(M) \oplus (1 - e)(M)$ where $e(M) \subseteq A$ and $(1 - e)(M) \subseteq B$.

THEOREM 3. Let M be a quasi-projective module. Then M is directly refinable if and only if M has the finite exchange property.

PROOF. If M is directly refinable, Nicholson's proof of Proposition 2.9 [3] can be used. The converse follows from Lemma 2.

REFERENCES

1. FLEURY, P. A Note on Dualizing Goldie Dimension, Canad. Math. Bull. 17 (1974), 511-517.
2. MOHAMED, S. and SINGH, S., Generalizations of Decomposition Theorems Known Over Perfect Rings, J. Austral. Math. Soc. 24 (Series A), (1977), 496-510.
3. NICHOLSON, W. K., Lifting Idempotents and Exchange Rings, Trans. Amer. Math. Soc. 229 (1977), 269-278.
4. OSHIRO, K. Projective Modules Over von Neumann Regular Rings Have the Finite Exchange Property, Osaka J. Math. 20 (1983), 695-699.
5. WARFIELD, R. B., Exchange Rings and Decomposition of Modules, Math. Ann. 199 (1972), 31-36.
6. ZÖSCHINGER, H., Komplemente als Direkte Summanden, Arch. Math. 25 (1974), 241-253.