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ON THE SYMMETRY AXIOM FOR VALUES OF NONATOMIC GAMES

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ABSTRACT. In this paper, a weaker version of the Symmetry Axiom on BV, and values on subspaces of BV are discussed. Included are several theorems and examples.

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1. INTRODUCTION AND STATEMENT OF RESULTS.

It has been shown by Aumann and Shapley [1] that there is no value defined on the entire space BV. However, it was shown in Ruckle [2] that there do exist continuous, efficient projections from BV onto FA which satisfy a weaker form of the Symmetry Axiom. In this paper we shall pursue this phenomenon to a greater extent.

Throughout this paper we use the terminology and notation of Aumann and Shapley [1]. Let (I,C) denote a standard measureable space which will remain fixed throughout the discussion. A symmetry of (I,C) is a one to one bi-measureable transformation of (I,C) onto itself. The group of all symmetries of (I,C) is denoted by G. For every v in BV let G(v) be the subgroup of all symmetries π which preserve v, i.e. $v\circ\pi = v$. Let Q be a symmetric linear subspace of BV, and let ϕ: Q $\to$ FA be a value. By the Symmetry Axiom it follows that G(v) is contained in G(ϕv) for every v in BV. This motivates us to define a <u>measure group</u> to be a group of symmetries H for which there is μ in FA such that $H \subset G(\mu)$. A game v will be called a <u>valueable game</u> if G(v) is a measure group. A symmetric linear space of games is called a <u>valueable space</u> if each of its members is a valueable game.

The proof of the result of Aumann and Shapley cited in the first paragraph can be analyzed as follows: First it is shown that G is not a measure group. Then the unanimity game w is defined as the game for which w(I) = 1 but w(S) = 0 for every proper subset S of I. Since G(w) clearly equals G, w cannot be a valueable game so that BV is not a valueable space. From the above considerations we conclude that when looking for values on subspaces of nonatomic games one must either restrict the search to valueable subspaces, necessarily proper subspaces, or weaken the symmetric axiom or else combine these two approaches in an appropriate way.

2. WEAKENING THE AXIOM OF SYMMETRY ON BV.

Let H be a subgroup of G. An H-value on BV is a linear positive efficient projection ρ: BV $\to$ FA which is H-symmetric. If an H-value exists then it follows that H is a valueable group. For let P be an H-value on BV and let w denote the unanimity game. Then we have

$$H = H(w) \subset H(Pw) \subset G(Pw)$$

This shows that H is a measure group. We have just proved:

LEMMA 2.1. Every valueable group is a measure group.

In view of Lemma 2.1 it is natural to pose the following problem to which we have no answer.

PROBLEM 2.1. Is every measure group a valueable group? The first attempt to seek valueable groups was in Ruckle, [2] where it was shown that every locally finite group is a valueable group. The next two theorems strengthen that result.

THEOREM 2.1. Let H be a subgroup of G. Then H is a valueable group if and only if the following condition is satisfied:

The group generated by every finite subset of H is a valueable group.

Next we place a topology on G and define a subgroup H of G to be almost compact if every finitely generated subgroup of H is compact in this topology. Using these notions we obtain the following result.

THEOREM 2.2. Every almost compact group is a valueable group.

If H is a locally finite group then by Lemma 2.1, H is a measure group. This means there is μ in FA' such that $H \subset G(\mu)$. Sometimes it is desired that the value of a game be a countably additive measure and not just a member of FA. Thus the following question was posed by R.J. Aumann (Ruckle [2]): Is every locally finite group a measure group for some countable additive measure? This question is answered in Example 4 below where a locally finite group is constructed which does not belong to $G(\mu)$ for any μ in CA.

3. VALUES ON SUBSPACES OF BV.

Most effort on values of nonatomic games has been dedicated to constructing values on subspaces of BV. The main existence and uniqueness results are the existence of a unique value on bv'NA (viz. Aumann and Shapley [1], the existence of a value on ASIMP (viz. Aumann and Shapley [1] weak ASIMP (viz. Neyman [3]) and the Mertens space (viz. Mertens [4]). Other than for a few results which are quite basic, there are no uniqueness theorems besides that for bv'NA. Thus the most pressing need in this area is for uniqueness results. Even the existence theory is far from satisfactory. The question of existence is in doubt for several "nice" spaces such as AC, pNA' $\cap$ AC and AC_∞ (viz. (Monderer [5]). We have already mentioned that a necessary condition for the existence of a value on Q is that Q be a valueable space. This leads to the following problem:

PROLBEM 3.1. Does there exist a value on every valueable space?

Hoping for a positive answer to this problem we prove the following result:

THEOREM 3.1. The space AC is a valueable space. Moreover, for every v in AC we can find a μ in NA such that $\mu(I) = v(I)$, $||\mu|| \leq ||v||$, $G(v) \quad G(\mu)$ and $\mu << \lambda$ for every λ in NA for which $v << \lambda$. Moreover, if $v \in BV^+$ we can choose the corresponding μ to be in NA^+.

4. PROOFS OF RESULTS.

We need the following Lemma.

LEMMA 4.1. Suppose P is an efficient linear projection from BV onto FA. Then $||P|| = 1$ if and only if P is a positive operator.

PROOF. Assume P is a positive operator. For every v in BV we have

$$||P(v^+) - P(v^-)|| \leq ||P(v^+)|| + ||P(v^-)||$$

$$= (P(v^+)\ (I) + P(v^-)\ (I)$$

since $P(v^+)$ and $P(v^-)$ are in FA^+ by the positivity of P. Since P is efficient we further note that

$$P(v^+)\ (I) + P(v^-)\ (I) = v^+(I) + v^-(I) = ||v||.$$

Therefore, we conclude that $||P|| \leq 1$. As with all projections we know $||P|| \geq 1$ (since $||P|| = ||P^2|| \leq ||P||^2$).

To establish the converse let P denote an efficient projection with $||P|| = 1$. Suppose P is a monotone game. Then Pv is equal to $\mu_1 - \mu_2$ where $\mu_1 = (Pv^+)$ and $\mu_2 = (Pv)^-$. We shall verify that $\mu_2(I) = 0$ which implies $\mu_2 = 0$. Indeed, we have

$$\mu_1(I) - \mu_2(I) = PV(I) = V(I) = ||v|| \geq ||Pv|| = \mu_1(I) + \mu_2(I)$$

from which we conclude that $\mu_2(I) \leq 0$ so that $\mu_2(I) = 0$.

The set of all efficient, positive, linear projections P from BV onto FA will be denoted by Γ. For every subgroup H of G, $\Gamma(H)$ denotes the set of all P in Γ which are H-values, i.e. $\pi^{-1} o P o \pi = P$ for all $\pi \in H$. By Ruckle [2] Γ and $\Gamma(H)$ are compact in the w^*-topology of operators in L(BV) - the space of all continuous linear operators from BV into itself. Moreover, Γ is nonempty by Ruckle [2].

PROOF OF THEOREM 2.1. Let H be a subgroup of G which satisfies the condition state in the theorem. For every π in H let $\Gamma(\pi)$ be $\Gamma(D)$, where D is the group generated by π. We shall prove that $\bigcap_{\tau \in H} F(\pi)$ is nonempty. Since each $\Gamma(\pi)$ is compact, it suffices to prove that each finite intersection is nonempty. Indeed, if E is the group generated by the finite subset $\{\pi_1, \pi_2, \ldots, \pi_n\}$ of H we have $\cap_{i=1}^{n} \Gamma(\pi_i) = \Gamma(E)$, and $\Gamma(E) \neq \phi$ by the hypothesis of the theorem.

The converse of the theorem is obvious.

A subgroup H of G will be called a compact group if there is a topology $\tau(H)$ on H for which (i) H is a compact topological group and (ii) the mapping from H X BV into BV defined by $(\pi,v) \to \pi(v)$ is continuous with respect to the product topology on H X BV and the Banach space topology on BV. Thus H acts on BV as a group of continuous operators in the sense of Rudin [6]. For example, if H is finite the discrete topology of H satisifes these conditions. A subgroup H of G is called almost compact if every finite subset F of H is contained in some compact group H_1.

PROOF OF THEOREM 2.2. Suppose H is an almost compact subgroup of G. Every finitely generated subgroup of H is contained in a compact group H_1. In order to apply Theorem 1.1. it suffices to prove that H_1 is valueable. This is a direct application of Theorem 5.18 of Rudin [6].

EXAMPLE 4.1. Let I be the half open unit interval [0,1). Let I_n, n = 1,2,... be an infinite partition of I into nonempty subintervals. For each $n \geq 1$ let H_n be the group of all symmetries π of I which satisfy the following conditions:

(1) π is the identity on $\cup_{k=n+1}^{\infty} I_k$.

(2) π is linear on each I_k, $k \geq 1$.

(3) π permutes the set $\{I_1, I_2, \ldots, I_n\}$.

Let H be the union of all H_n, $n \geq 1$. Obviously H is a locally finite group and if H preserves some non-zero μ in FA, then $\mu(I_j) = \mu(I_k)$ for all j, k. Therefore μ cannot be in CA.

One can prove that the group H above preserves an FA' measure μ if and only if $\mu(I_n) = 0$ for all n. For example, let λ_i denote the normalized Lebesgue measure on I_i. Define μ by

$$\mu(S) = \text{LIM } n^{-1}\{\lambda_1(S) + \lambda_2(S) + \ldots + \lambda_n(S)\}$$

where LIM is any Banach limit.

PROOF OF THEOREM 3.1. Because of the standardness assumption we can, without loss of generality, assume I is the unit interval [0,1). For each μ in NA, let NA(μ) be the subspace of NA consisting of all measures which are absolutely continuous with respect to μ. It is known that NA(μ) is isometric to $L^1(\mu)$ via the isometry $\xi \to \frac{d\xi}{d\mu}$. Let P: AC $\to$ NA be the operator for which we have $Pv[0,s) = v[0,s)$ for every s. Thus P is precisely the operator ϕ^R defined in (12.2) of Aumann and Shapley, [1] where R is the natural order on I. For each π in G let P^π be ϕ^R, where R is the order on I defined by: $s < t$ if and only if $\pi^{-1}(s) < \pi^{-1}(t)$. Obviously we have, $P^\pi = \pi^{-1} o P o \pi$. Let $v \in$ AC; then by Proposition 12.8 of Aumann and Shapley [1], every $p^\pi v$ satisfies the following conditions: $P^{\pi}(v)(I) = v(I)$, $||P^{\pi}(v)|| < ||v||$ and $P^{\pi}(v) << \mu$ for every μ such that $v << \mu$. Moreover, if v is in NA then $P^{\pi}(v) = v$ and if $v \in AC^{+}$ then $P^{\pi}(v) \in NA^{+}$. Also it was proved in Aumann and Shapley [1] that for every μ in NA for which $v << \mu$, the measures in $Mv = \{P^{\pi}v: \pi \in G\}$ are uniformly absolutely continuous with respect to μ.

Let Kv be the closed (in the weak topology of NA) convex hull of MV. Every ν in Kv satisfies the properties described above. Moreover, if $v << \mu$ then the members of Kv are uniformly absolutely continuous with respect to μ. Therefore, by Theorem IV. 8.9 of Dunford and Schwartz [7], Kv is weakly compact in NA(μ) for every such μ. Now suppose that v is in AC. Every π in G(v) maps Kv into itself since

$\tilde{\pi}P^{\pi}(v) = P^{\tilde{\pi}\circ\pi^{-1}}(v)$. We now use the following fixed point theorem attributed to Ryll-Nardzewski, see Glasner [8]. Let K be a weakly compact convex subset of a Banach space X, and let G be a group of continuous linear operators π with $\pi(k) = K$ for each π in G. If for every $x \neq y$ in K we have

$$\inf_{\pi\in G}||\pi x - \pi y|| > 0.$$

then there exists x in K such that $\pi x = x$ for all π in G.

Condition (2.1) is easily verified in the present case because each π in G is an isometry. Obviously every common fixed point of G(v) is in Kv satisfies the conditions of the theorem.

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