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MAXIMAL SUBGROUPS OF FINITE GROUPS

by

S. SRINIVASAN

DEPARTMENT OF ISDP, UNIVERSITY OF LOUISVILLE

LOUISVILLE, KY 40292, USA

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ABSTRACT: In finite groups maximal subgroups play a very important role. Results in the literature show that if the maximal subgroup has a very small index in the whole group then it influences the structure of the group itself. In this paper we study the case when the index of the maximal subgroups of the groups have a special type of relation with the Fitting subgroup of the group.

KEY WORDS: Maximal subgroup, Fitting subgroup, supersolvable group.

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1. INTRODUCTION

P.Hall proved that a finite group with the property that its maximal subgroups have index a prime or square of a prime is solvable. J.Kohler studied in detail finite groups with the property of Hall mentioned above (see [5]). B.Huppert [3] proved that if every maximal subgroup has index prime, then the group is supersolvable. O.U.Kramer [6] proved that if a finite solvable group G has the property that for every $M <\cdot G$, $[F(G) : F(G) \cap M] = 1$ or prime, then G is supersolvable. In this paper we consider groups with the following property: For every $M <\cdot G$, $[F(G) : F(G) \cap M] = 1$, a prime or square of a prime. We consider only finite groups.

2. NOTATION AND KNOWN RESULTS

$F(G)$ is the Fitting subgroup of G, $\Phi(G)$ is the Frattini subgroup of G, $\pi(G)$ denotes the set of distinct prime divisors of order of $|G|$. $\Delta(G)$ denotes the intersection of all nonnormal maximal subgroups of G. $M <\cdot G$ means that M is a maximal subgroup of G. Consider the exponents in the orders of the chief factors of a chief series of a solvable group G. For each prime $p \in \pi(G)$, the maximal such exponent is denoted by $r_p(G)$, called the **p-rank** of G. $r(G) = \max \{ r_p(G) ; p \in \pi(G) \}$, is called the **rank** of G.

We mention below the following known results for easy reference.

LEMMA 2.1 (Kohler [5], Lemma 3.3): Let G be an irreducible subgroup of $GL(2, p)$ with $|G|$ odd. Then G is cyclic and $|G|$ divides $(p^2 - 1)$.

THEOREM 2.2 (Huppert [3], Satz 1): Let G be solvable. Let p^n be the highest power of p that occurs in a maximal chain of G. Then $r_p(G) = n$.

THEOREM 2.3 (Gaschütz [1], Satz 13): For a finite group G, $F(G)/\Phi(G)$ is a direct product of minimal abelian normal subgroups of $G/\Phi(G)$.

LEMMA 2.4 (Kohler [5], Lemma, p.440): If G is a subdirect product of primitive solvable groups on a prime or prime square number of letters, then $r_p(G) \leq 2$ for every $p \in \pi(G)$.

THEOREM 2.5 (Gaschütz [1], Satz 15): If $r_p(G/\Delta(G)) \leq 2$, then $r_p(G) \leq 2$.

3. MAIN RESULTS

We prove the following lemma in a general setting.

LEMMA 3.1: Let G be a solvable group with $\Phi(G) = 1$. Let $F = F(G)$. For every $M <\cdot G$, let $[F : F \cap M] = p^i$, p an arbitrary prime and $i \geq 0$. Then every G-chief factor of F has order q^j, q an arbitrary prime and $j \leq e = \max \{ i \ ; [F : F \cap M] = p^i$, p an arbitrary prime $\}$.

PROOF: Since $\Phi(G) = 1$, it follows from Theorem 2.3 that $F(G)$ is the direct product of minimal normal subgroups of G. This means that for each G-chief factor H/K of F there exists a minimal normal subgroup S of G which lies in F with $S \approx H/K$. Then there is a maximal subgroup M of G with $MS = G$ and $M \cap S = 1$. So it follows that $[F : F \cap M] = [G : M] = |S|$. Hence the lemma is proved.

REMARK: The condition $\Phi(G) = 1$ is needed in the hypothesis of Lemma 3.1 as can be easily seen from the example of Huppert [4], Beispiel 2, p.140.

THEOREM 3.2: Let G be a solvable group. Let $F = F(G)$. For every $M <\cdot G$, let $[F : F \cap M] = p^i$, $i = 0$, 1 or 2. If $|G|$ is odd, then $r_p(G) \leq 2$ for all primes $p \in \pi(G)$.

PROOF: If $\Phi(G) \neq 1$, then consider $G / \Phi(G)$. By induction on $|G|$ we can conclude that $r_p(G/\Phi(G)) \leq 2$ and hence $r_p(G) \leq 2$. So assume that $\Phi(G) = 1$. By Lemma 3.1 it suffices to show that chief factors of G/F are of order a prime or prime square. By Theorem 2.3, $F = H_1 \times H_2 \times \dots \times H_r$ where H_i are minimal abelian normal subgroups of G. Since $\Phi(G) = 1$, for every H_i there exists $M_i <\cdot G$ such that $G = M_i H_i$. $M_i \cap H_i = 1$ since H_i is a minimal abelian normal subgroup of G. Since $H_i \leq F$, $G = M_i H_i = M_i F$. Therefore $|H_i| = [G : M_i] = [F : F \cap M_i]$. Hence by hypothesis on $[F : F \cap M_i]$ we conclude that $|H_i| = p^2$ or p for some $p \in \pi(G)$. If $|H_i| = p$, then $G/C_G(H_i)$ is cyclic. If $|H_i|=p^2$, then $G/C_G(H_i)$ is an irreducible subgroup of $GL(2, p)$ as H_i is a minimal normal subgroup of G. By hypothesis $|G|$ is odd. So we can apply Lemma 2.1 to conclude that $G/C_G(H_i)$ is cyclic of order dividing $(p^2 - 1)$. $F = H_1 \times H_2 \ldots H_r$ shows that $C_G(F) = \bigcap_{i=1}^{r} C_G(H_i)$. Since G is solvable, $C_G(F) \le F$. Since F is abelian, $F \le C_G(F)$. Thus $F = C_G(F)$. Therefore $G/F = G/C_G(F) \lesssim \times_{i=1}^{r} G/C_G(H_i)$ implies that G/F is abelian. This means that all chief factors of G/F are of prime order. This completes the proof of the theorem.

EXAMPLE (Kohler [5]):

$$x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Let $D = \langle x, y\rangle \approx D_4$, the dihedral group of order 8.
Let $K = A_1 * A_2$ where * denotes the free product. $|A_1| = |A_2| = p^2$, A_i elementary abelian with $A_i = \langle a_i, b_i\rangle$.

Let $H = H_1 \times H_2$ where $H_i \approx D_4$ is a group of automorphisms of A_i,

$$a_i^{x_i} = a_i \ , \quad a_i^{y_i} = b_i^{-1} \ , \quad b_i^{y_i} = a_i \ , \quad b_i^{x_i} = b_i^{-1}$$

for i = 1, 2. Let $W = \langle [a_1, a_2], [a_1, b_2], [a_2, b_1], [a_2, b_2]\rangle$.
W is a normal, elementary abelian p-subgroup of order p^4 in HW.
$H <\!\!\cdot\, HW$. $H \cap W = 1$, $W = F(HW)$. HW has the property that for every $N <\!\!\cdot\, HW$, $[F(HW) : F(HW) \cap N] = 1$, p or p^2 for some $p \in \pi(HW)$.
$[HW : H] = p^4$. $|HW| = |H|\,|W| = 2^6\, p^4$.

Thus, we see that in Theorem 3.2 we require that G is of odd order.

However, for groups of even order we have the following theorem.

THEOREM 3.3: Let G be a solvable group. Let $F = F(G)$. For every $M <\!\!\cdot\, G$, let $[F : F \cap M] = p^i$, i = 0, 1 or 2. If a Sylow 2-subgroup of G centralizes every Sylow q-subgroup of G for all q, q odd, then $r_p(G) \le 2$ for all $p \in \pi(G)$.

PROOF: As in the proof of Theorem 3.2 we can assume that $\Phi(G) = 1$ and write $F = F(G) = H_1 \times \ldots \times H_r$ with $|H_i| = p$ or p^2 for $p \in \pi(G)$. If $|H_i| = p$, then $G/C_G(H_i)$ is cyclic. If $|H_i| = p^2$ with p odd, then also we can conclude as in the proof of Theorem 3.2 that $G/C_G(H_i)$ is cyclic. If $|H_i| = 4$, then $G/C_G(H_i)$ is an irreducible subgroup of $GL(2, 2) \approx S_3$ and hence is cyclic. The rest of the proof now follows as in the proof of Theorem 3.2. This completes the proof of the theorem.

THEOREM 3.4: Let G be a solvable group of odd order. Let $F = F(G)$. For every $M <\!\!\cdot\, G$, let $[F : F \cap M] = p^i$, i = 0, 1 or 2. Then every subgroup of G has the same property.

PROOF: It is enough to show that every maximal subgroup of G has the same property as G. By Theorem 3.2 we have $r_p(G) \leq 2$ for all primes $p \in \pi(G)$. Now applying Theorem 2.2 we see that p^2 is the highest power of p that occurs as the index in a maximal chain of G for some $p \in \pi(G)$ corresponding to $r_p(G) \leq 2$.

Let H be any maximal subgroup of G. Let $K <^{c} H$. Therefore $[H : K] = p$ or p^2 for some $p \in \pi(H)$. If $F(H) \leq K$, then clearly $[F(H) : F(H) \cap K] = 1$. If $F(H) \nleq K$, then $[F(H) : F(H) \cap K] = [H : K]$. Thus H has the same property as the group G. This completes the proof of the theorem.

REMARK: Theorem 3.4 can be modified as in Theorem 3.3 for the even order case.

THEOREM 3.5: Let G be a solvable group of odd order. G has the property that for every $M <\circ G$, $[F(G) : F(G) \cap M] = p^i$ for $i = 0, 1$ or 2 if and only if $G/\Delta(G)$ is isomorphic to a subdirect product of primitive solvable groups on a prime or prime square number of letters.

PROOF: Assume that $G/\Delta(G)$ has the above property. By Lemma 2.4, $r_p(G/\Delta(G)) \leq 2$. By Theorem 2.5 we then have $r_p(G) \leq 2$. Hence G has the required property.

Now suppose that G has the property mentioned in the statement of the theorem. By Theorem 3.2, $r_p(G) \leq 2$. Using Theorem 2.2 we see that every maximal subgroup has index either a prime or square of a prime. Let π be a permutation representation on the conjugacy classes of maximal subgroups of G. Let π^* be the restriction of π to one of these conjugacy classes. Since $M <\circ G$, $N_G(M) = G$, and thus π^* is the identity. If $M = N_G(M)$, then π^* is primitive of degree $[G : M] = p$ or p^2 for some $p \in \pi(G)$. Hence $G/\ker(\pi)$ is a subdirect product of primitive solvable groups on prime or prime square number of letters. $x \in \ker(\pi)$ if and only if $x \in N_G(M)$ for every $M <\circ G$ and M nonnormal in G. Hence $\ker(\pi) = \Delta(G)$. This completes the proof of the theorem.

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