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ON THE BOUNDS OF MULTIVALENTLY STARLIKENESS AND CONVEXITY

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ABSTRACT. The object of the present paper is to prove some interesting results for the bounds of starlikeness and convexity of certain multivalent functions.

KEY WORDS AND PHRASES. p-valently starlike, p-valently convex, typically real.

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1. INTRODUCTION.

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\})$$

which are analytic in the unit disk $\mathbb{U} = \{z: |z| < 1\}$. A function $f(z)$ in the class $\mathcal{A}(1)$ is said to be a member of the class $\mathcal{P}$ if and only if it satisfies

$$\text{Re}\{f'(z)\} > 0 \qquad (z \in \mathbb{U}).$$

It is well known that if f(z) belongs to the class $\mathcal{P}$, then f(z) is univalent in $\mathcal{U}$ (cf. [1], [2]).

Many results for this class were obtained, but the radius of starlikeness for the class $\mathcal{P}$ is not known.

Lewandowski [3] has proved that if f(z) belongs to the class $\mathcal{P}$, then f(z) is starlike in $|z| < 4\sqrt{2} - 5 \fallingdotseq 0.6568$. This result has been improved in ([4], [5]) as follows:

If f(z) belongs to the class $\mathcal{P}$, then f(z) is univalently starlike in $|z| < \rho$, where ρ is the smallest positive root of the equation

$$\log \frac{1}{1-r^2} + \mathrm{Sin}^{-1}\frac{2r}{1+r^2} = \pi$$

and $0.901 < \rho < 0.902$.

2. PRELIMINARIES.

DEFINITION 1. Let $f(z) \in \mathcal{A}(p)$ and

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (|z| < r \leqq 1).$$

Then we shall call a function f(z) p-valently starlike in $|z| < r$. We denote by $\mathcal{S}(p)$ the subclass of $\mathcal{A}(p)$ consisting of functions which are p-valently starlike in $\mathcal{U}$.

DEFINITION 2. Let $f(z) \in \mathcal{A}(p)$ and

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (|z| < r \leqq 1).$$

Then we shall call a function f(z) p-valently convex in $|z| < r$. Also we denote by $\mathcal{C}(p)$ the subclass of $\mathcal{A}(p)$ consisting of all p-valently convex functions in the unit disk $\mathcal{U}$.

LEMMA 1. (Ruscheweyh [6]) Let f(z) be in the class $\mathcal{P}$, and assume that f'(z) is typically real in $\mathcal{U}$. Then f(z) is univalently starlike in the unit disk $\mathcal{U}$.

LEMMA 2. (Nunokawa [7]) Let f(z) be in the class $\mathcal{A}(p)$, and suppose that

$$\mathrm{Re}\left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \qquad (|z| < r \leqq 1).$$

Then we have

$$\mathrm{Re}\left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0 \qquad (|z| < r)$$

or $f^{(p-k)}(z) \in S(k)$ in $|z| < r$ for $k = 1,2,3,\cdots,p$.

LEMMA 3. (Nunokawa [7]) Let $f(z)$ be in the class $A(p)$, and suppose

$$p + \mathrm{Re}\left\{ \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right\} > 0 \qquad (|z| < r \leqq 1).$$

Then we have

$$k + \mathrm{Re}\left\{ \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} \right\} > 0 \qquad (|z| < r)$$

for $k = 0,1,2,\cdots,p-1$. This shows that $f(z) \in C(p)$ and $f(z) \in S(p)$.

3. BOUNDS OF STARLIKENESS AND CONVEXITY.

We begin with the statement and the proof of the following result.

THEOREM 1. Let the function $f(z)$ belong to the class $A(p)$ with $p \geqq 2$, $(f^{(p-1)}(z)/p!) \in P$, and $(f^{(p)}(z)/p!)$ be typically real in U. Then $f(z)$ is p-valently convex in U and p-valently starlike in U.

PROOF. Let $F(z) = f^{(p-1)}(z)/p!$. Then it is clear that $F(0) = 0$ and $F'(0) = 1$. Also, since $F(z) \in P$, $\mathrm{Re}\{F'(z)\} > 0$ $(z \in U)$, and $F'(z)$ is typically real in U. An application of Lemma 1 to the function $F(z)$ gives that

$$\mathrm{Re}\left\{ \frac{zF'(z)}{F(z)} \right\} = \mathrm{Re}\left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \qquad (z \in U).$$

Therefore, with the aid of Lemma 2, we have

$$\mathrm{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \qquad (z \in U)$$

and

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \qquad (z \in \mathcal{U}).$$

The above inequalities imply that $f(z) \in \mathcal{C}(p)$ and $f(z) \in \mathcal{S}(p)$, respectively. Thus we complete the proof of Theorem 1.

Next, we prove

THEOREM 2. Let the function $f(z)$ belong to the class $\mathcal{A}(p)$, and $(f^{(p-1)}(z)/p!) \in \mathcal{P}$. Then $f(z)$ is p-valently convex in $|z| < r(p)$, where

$$r(p) = \frac{\sqrt{p^2+1} - 1}{p}.$$

PROOF. Defining the function $F(z)$ as in the proof of Theorem 1, that is, $F(z) = (f^{(p-1)}(z)/p!)$, we have $F(0) = 0$, $F'(0) = 1$ and $\operatorname{Re}\{F'(z)\} > 0$ $(z \in \mathcal{U})$. Then it is well known that

$$\left| \frac{zF''(z)}{F'(z)} \right| = \left| \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right| \leqq \frac{2|z|}{1-|z|^2} \qquad (z \in \mathcal{U}).$$

Thus, it follows from the above that

$$p + \operatorname{Re}\left\{ \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right\} \geqq p - \frac{2|z|}{1-|z|^2} > 0$$

for $|z| < r(p)$. Making use of Lemma 3 leads to

$$\operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0$$

for $|z| < r(p)$ which completes the proof of Theorem 2.

REMARK. We can not find out an extremal function of Theorem 2.

Applying the same method as in the proof of [5], and using Lemma 2, we have the following result.

THEOREM 3. Let the function $f(z)$ belong to the class $\mathcal{A}(p)$, and $(f^{(p-1)}(z)/p!) \in \mathcal{P}$. Then $f(z)$ is p-valently starlike in $|z| < \rho_1$, where ρ_1 is the smallest positive root of the equation

$$\log \frac{1}{1 - r^2} + \mathrm{Sin}^{-1} \frac{2r}{1 + r^2} = \pi$$

and $0.901 < \rho_1 < 0.902$.

Further, spending the same manner as in the proof of ([8], [9]), and using Lemma 2, we get the following theorem.

THEOREM 4. Let the function $f(z)$ belong to the class $A(p)$, and

$$|f^{(p)}(z) - p!| < p! \qquad (z \in U).$$

Then $f(z)$ is p-valently starlike in $|z| < \rho_2$, where ρ_2 is the smallest positive root of the equation

$$\log(9 - 4r^2 + 4r^3 - r^4) - \log 9(1 - r^2) + \mathrm{Sin}^{-1} r = \pi$$

and $0.933 < \rho_2 < 0.934$.

Letting $p = 1$ in Theorem 4, we have

COROLLARY. Let the function $f(z)$ belong to the class $A(1)$, and

$$|f'(z) - 1| < 1 \qquad (z \in U).$$

Then $f(z)$ is univalently starlike in $|z| < \rho_2$, where ρ_2 is given as in Theorem 4.

REMARK. The above corollary is an improvement of the result in [10, Theorem 6].

Finally, we derive

THEOREM 5. Let the function $f(z)$ belong to the class $A(p)$, and

$$|f^{(p+1)}(z)| \leqq k|z|^{k-1}p! \qquad (z \in U),$$

where k is a positive real number. Then $f(z)$ is p-valent in U.

PROOF. From the assumption of Theorem 5, we see that

$$|f^{(p)}(z) - p!| = \left| \int_0^z f^{(p+1)}(t)\,dt \right|$$

$$\leq \int_0^{|z|} |f^{(p+1)}(t)|\,|dt|$$

$$\leq \int_0^{|z|} k|t|^{k-1}|dt| = |z|^k p! < p!$$

for $z \in U$. This implies that $\mathrm{Re}\{f^{(p)}(z)\} > 0$ ($z \in U$). By applying Ozaki's theorem [11] to the function f(z), we conclude that f(z) is p-valent in U.

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