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ON A THEOREM OF H. HOPF

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ABSTRACT. A simple proof of a theorem of H. Hopf [1], via Morse theory, is given.

KEY WORDS AND PHRASES. Hypersurface, Morse function, critical point, Gauss map, degree.
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1. INTRODUCTION AND THE THEOREM.

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth map, and let

$$V = \{(x_1, \cdots, x_n) \in \mathbf{R}^n | f(x_1, \cdots, x_n) = 0\}.$$

Suppose V is compact and the gradient, ∇f, of f is nonzero on V. Then V in an $(n-1)$-dimensional real orientable hypersurface in $\mathbf{R}^n$. Let U be the unbounded component of $\mathbf{R}^n - V$. We may suppose that $f > 0$ on U, otherwise consider $-f$. We shall give V the following orientation. Let $v \in V$ and let $v_1, \cdots, v_{n-1}$ be a positively oriented basis for the tangent space TV_v, regarded as a subspace of $T\mathbf{R}^n_v$. We say that V has the positive orientation at v if

$$det \begin{bmatrix} \nabla f(v) \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} > 0.$$

V has the positive orientation, if it has the positive orientation at each of its points. Let S^{n-1} be the unit sphere in $\mathbf{R}^n$, along with its usual orientation. Consider the Gauss map $\eta : V \to S^{n-1}$ which assigns to each point of V, the unit normal vector $\nabla f / \|\nabla f\|$. Let d be the degree of η. For a real compact manifold W, let $\chi(W)$ denote its Euler characteristic. We can now state the theorem which relates d, with the Euler characteristic of certain hypersurfaces arising from f.

THEOREM (HOPF [1]). Let f, V, d be as above. Then

$$d = \begin{cases} \frac{\chi(V)}{2} & \text{if } n \text{ is odd} \\ \chi(f \leq 0) & \text{if } n \text{ is even.} \end{cases}$$

2. PRELIMINARIES.

The main idea of the proof of the theorem is to apply Morse theory on V, using a convenient Morse function. According to a theorem of Sard, the set of critical values of η has measure zero in S^{n-1} [2]. Hence, after rotating the axis if necessary, we may assume that the points $(0,\cdots,0,\pm 1)$ are not critical values of η. Let $\pi(x_1,\cdots,x_n)=x_n$ be the projection onto the last coordinate, and let $h=\pi|_V$ be the height function on V. Let p be a critical point of h. At p we have:

$$f=0,\quad \frac{\partial f}{\partial x_i}=0,\quad i=1,\cdots,n-1,\quad 1=\lambda\frac{\partial f}{\partial x_n},\quad \lambda\in\mathbf{R}.$$

LEMMA 2.1 [3]. With the above considerations, p is not a critical point of η, and p is a nondegenerate critical point of h.

PROOF. We observe that $\eta(p)=(0,\cdots,0,\pm 1)$, since $\frac{\partial f}{\partial x_n}(p)\neq 0$. Hence, p is not a critical point of η. In terms of local coordinates $u_1,\cdots,u_{n-1}$ on V, this means that the matrix $\left[\frac{\partial \eta_i}{\partial u_j}\right]$, $i,j<n$, is nonsingular at p. In fact, near p we can choose local coordinates $u_1,\cdots,u_{n-1}$ so that $x_1=u_1,\cdots,x_{n-1}=u_{n-1},x_n=h(u_1,\cdots,u_{n-1})$. Then,

$$\eta(u_1,\cdots,u_{n-1})=\pm\left(\frac{\partial h}{\partial u_1},\cdots,\frac{\partial h}{\partial u_{n-1}},-1\right)\Big/\sqrt{1+\sum_{j=1}^{n-1}(\tfrac{\partial h}{\partial u_j})^2}.$$

Hence, $\frac{\partial \eta_i}{\partial u_j}=\pm\frac{\partial^2 h}{\partial u_i\partial u_j}$ at p. Therefore, the matrix $\left[\frac{\partial^2 h}{\partial u_i\partial u_j}\right]$, $i,j<n$, is nonsingular, which implies that p is a nondegenerate critical point of h. ∎

Set $S=\eta^{-1}(0,\cdots,-1)$, $N=\eta^{-1}(0,\cdots,1)$. Then the above Lemma shows that h is a Morse function on V with critical set $S\cup N$. For $p\in S\cup N$, we denote by $i(p)$ the Morse index of h at p, which is equal to the number of negative eigenvalues, multiplicities counted, of the real symmetric matrix $\left[\frac{\partial^2 h}{\partial u_i\partial u_j}\right]$ [4].

Also, for $p\in S\cup N$ we define $sgn(p)$ to be

$$sgn(p)=\begin{cases}1 & \text{if near } p,\ \eta \text{ preserves the orientation}\\ -1 & \text{if near } p,\ \eta \text{ reverses the orientation.}\end{cases}$$

In addition, if a is a real number, $a\neq 0$, we will denote its signature by $sign(a)$.

REMARK 2.1. $2d=\sum_{p\in S\cup N} sgn(p)$, $\chi(V)=\sum_{p\in S\cup N}(-1)^{i(p)}$, [4].

We will now compute $sgn(p)$, for $p\in S\cup N$. Let $G:U\to V$ be a local parametrization of V near p, defined by $G(x_1,\cdots,x_{n-1})=(x_1,\cdots,x_{n-1},h(x_1,\cdots,x_{n-1}))$. Set $\bar{p}=(p_1,\cdots,p_{n-1})$. Then,

$$sgn\, dG_{\bar{p}} = sign\, det \begin{bmatrix} \nabla f(p) \\ dG(\frac{\partial}{\partial x_1}) \\ \vdots \\ dG(\frac{\partial}{\partial x_{n-1}}) \end{bmatrix} = (-1)^{n-1} sign \frac{\partial f}{\partial x_n}(p).$$

On the other hand, if $k : U_1 \to S^{n-1}$ is a local parametrization of S^{n-1} near the point $(0, \cdots, 0, \frac{\partial f}{\partial x_n}(p)/|\frac{\partial f}{\partial x_n}(p)|)$, defined by $k(s_1, \cdots, s_{n-1}) = (s_1, \cdots, s_{n-1}, sign\frac{\partial f}{\partial x_n}(p)\sqrt{1 - \sum s_j^2})$, then,

$$sgn\, dk_0 = sign\, det \begin{bmatrix} \nabla f(p) \\ dk(\frac{\partial}{\partial s_1}) \\ \vdots \\ dk(\frac{\partial}{\partial s_{n-1}}) \end{bmatrix} = (-1)^{n-1} sign \frac{\partial f}{\partial x_n}(p) .$$

Also, near p, $\eta = -sign\frac{\partial f}{\partial x_n}(p)\frac{(\nabla h, -1)}{\sqrt{1+\sum(\frac{\partial h}{\partial u_j})^2}}$. Hence,

$$sgn(p) = sgn\, d(k^{-1} \circ \eta \circ G)(\bar{p}) = \left(-sign \frac{\partial f}{\partial x_n}(p)\right)^{n-1} \cdot sign\, det \left[\frac{\partial^2 h}{\partial u_i \partial u_j} \right]. \qquad (2.1)$$

LEMMA 2.2. For $p \in S \cup N$, $sgn(p) = -sign\, detBH(f)(p)$, where $BH(f) = \begin{bmatrix} 0 & \nabla(f) \\ \nabla^t f & H(f) \end{bmatrix}$, is the Bordered Hessian matrix of f.

PROOF. We have $f(u_1, \cdots, u_{n-1}, h(u_1, \cdots, u_{n-1})) = 0$, where $u_1, \cdots, u_{n-1}, h$, are as in Lemma 2.1. By differentiating the above identity twice, and evaluating at p, we get

$$\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_n}(p)\frac{\partial^2 h}{\partial u_i \partial u_j} = 0, \quad 1 \le i, j \le n-1. \qquad (2.2)$$

Using (2. 1) we get $sgn(p) = \left(-sign\frac{\partial f}{\partial x_n}(p)\right)^{n-1} \cdot sign\, det\left[\frac{\partial^2 h}{\partial u_i \partial u_j}\right] = sign\, det\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right] = -sign\, detBH(f)(p)$. ∎

REMARK 2.2. If n is even, then $\chi(V) = 0$.

PROOF. We have $\chi(V) = \sum_{p \in S}(-1)^{i(p)} + \sum_{p \in N}(-1)^{i(p)}$. But if $p \in S$ then $sgn(p) = (-1)^{i(p)}$, while if $p \in N$, $sgn(p) = (-1)(-1)^{i(p)}$. Hence,

$$\chi(V) = \sum_{p \in S} sgn(p) - \sum_{p \in N} sgn(p) = d - d = 0 . \quad ∎$$

3. PROOF OF THE THEOREM . *Case i.* n is odd. We observe from (2. 1), that

$sgn(p) = sign\, det\left[\frac{\partial^2 h}{\partial u_i \partial u_j}\right] = (-1)^{i(p)}$. Hence, by Remark 2. 1,

$$\chi(V) = \sum_{p\in S\cup N} (-1)^{i(p)} = \sum_{p\in S\cup N} sgn(p) = 2d.$$

Case ii. n is even. Then, let us consider $V^- = \{f \leq 0\}$. This is a compact orientable manifold with boundary V. Consider the double covering W of V^-, ramified along V, which is defined by

$$W = \{(x_1, \cdots, x_n, x_{n+1}) \in \mathbf{R}^{n+1} | f(x_1, \cdots, x_n) + x_{n+1}^2 = 0\}.$$

W is a compact n-dimensional nonsingular hypersurface and $\chi(W) = 2\chi(V^-) - \chi(V) =$ $= 2\chi(V^-)$, since n is even. We orient W as we oriented V. On W we consider the height function $\bar{h}$, where $\bar{h} = \bar{\pi}|_W, \bar{\pi}(x_1, \cdots, x_n, x_{n+1}) = x_n$. Let $\bar{\eta} : W \rightarrow S^n$ be the Gauss map, and let $\bar{d}$ be its degree. Regard $\mathbf{R}^n, S, N$ as subsets of $\mathbf{R}^{n+1}$.

As in Lemma 2.1, we have that if $p \in S \cup N$, then p is a nondegenerate critical point of $\bar{h}$. In fact, $S \cup N$ is the critical set of $\bar{h}$, and the points $(0, \cdots, 0, \pm 1, 0)$ are not critical values of $\bar{h}$. Let now $p \in S \cup N$. p is viewed as a critical point of both h and $\bar{h}$, and also as a noncritical point of η and $\bar{\eta}$. Denote by $\overline{sgn}(p)$, the $sgn(p)$ viewed as a noncritical point of $\bar{\eta}$. We have:

$$sgn(p) = (-1) sign\, det \begin{bmatrix} 0 & \nabla f \\ \nabla^t f & H(f) \end{bmatrix} = (-1) sign\, det \begin{bmatrix} 0 & \nabla f & 0 \\ \nabla^t f & H(f) & 0 \\ 0 & 0 & 2 \end{bmatrix} = \overline{sgn}(p).$$

Hence, $d = \bar{d} = \frac{\chi(W)}{2} = \chi(V^-) = \chi(f \leq 0)$. The proof of the theorem is now complete. ∎

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