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ON PROBABILISTIC NORMED SPACES UNDER $\tau_{T,L}$

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ABSTRACT. We introduce the operation $\oplus_L$ copulative with $\tau_{T,L}$ to define PN space under $\tau_{T,L}$ and establish some basic properties of probabilistic seminorms and norms under $\tau_{T,L}$. Finally, we discuss so-called L-simple spaces.

KEY WORDS AND PHRASES. Probabilistic normed space, L-simple space.
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1. INTRODUCTION.

In [1-4], Šerstnev introduced the concept of PN space. A triple (V, ν, τ) is called a PN space, if V is a vector space over the field K of real or complex numbers, ν is a function from V into Δ^+, the set of all distance distribution functions, τ is a continuous triangle function, and for any p, q $\in$ V, a $\in$ K with $a \neq 0$, the following conditions hold.

(i) $\nu(0) = \varepsilon_0$, (1.1)

(ii) $\nu(p) \neq \varepsilon_0$ if $p \neq 0$, (1.2)

(iii) $\nu(ap) = |a| \odot \nu(p)$, (1.3)

(iv) $\nu(p+q) \geq \tau(\nu(p), \nu(q))$, (1.4)

where $|a| \odot \nu(p) = \nu(p)(j/|a|)_r$ and j denotes the identity function. Since $\odot$ and τ are not always cooperative as multiplication and addition, there is a certain difficulty in the further development of PN space theory. In fact, for any p, q $\in$ V, a $\in$ K with $a > 0$, we can estimate $\nu(ap+aq)$ in two ways and the two estimates are not always consistent (see Schweizer and Sklar [5], p 238). To overcome this objection, Muštari and Šerstnev [6-7] had to focus their attention on homogeneous triangle functions.

In this paper, we establish the operation $\oplus_L$ copulative with $\tau_{T,L}$ and use it to discuss PN spaces under $\tau_{T,L}$, where $\tau_{T,L}: \Delta^+ \times \Delta^+ \to \Delta^+$,

$\tau_{T,L}(F,G)(x) = \sup\{T(F(u), G(v)) \mid L(u,v) = x\}$, $x \in R^+$, F, G $\in \Delta^+$, T is a continuous t-norm, and $L: R^+ \times R^+ \to R^+$ satisfies:

1) RanL = R;
2) L has 0 as identity;
3) L is a nondecreasing in each place, and if $u_1 < u_2$, $v_1 < v_2$,

then $L(u_1, v_1) < L(u_2,v_2)$;

4) L is continuous on $R^+ \times R^+$, except possibly at the points $(0,\infty)$ and $(\infty,0)$;

5) L is associative;

6) L is Archimedean, i.e., for all $u \in (0,\infty)$, $L(u,u) > u$.

First, we give some simple results which are needed in the sequel. From Theorem 5.7.4 in [5], it is easy to know that there exists an additive generator g of L, i.e., a strictly increasing and continuous function $g: R^+ \to R^+$ with $g(0) = 0$, $g(\infty) = \infty$, such that $L(x,y) = g^{-1}(g(x) + g(y))$, $x, y \in R^+$.

Now, we choose a fixed additive generator g of L, and note that the particular choice of g does not affect the validity of our results.

DEFINITION 1.1. $*_L: R^+ \times R^+ \to R^+$ is defined as

$$\alpha *_L x = g^{-1}(\alpha g(x)), \ \alpha, x \in R^+.$$

Clearly, $\alpha *_{sum} x = \alpha x$, $\alpha, x \in R^+$.

LEMMA 1.1. For any $\alpha, \beta, x, y \in R^+$, the following equalities hold.

(i) $\alpha *_L (\beta *_L x) = (\alpha\beta) *_L x$ (1.5)

(ii) $\alpha *_L L(x,y) = L(\alpha *_L x, \alpha *_L y)$ (1.6)

(iii) $(\alpha+\beta) *_L x = L(\alpha *_L x, \beta *_L x)$ (1.7)

Clearly, if $\alpha \in (0,\infty)$, then $f(x) = \alpha *_L x$, $x \in R^+$ is strictly increasing and continuous. So we may give

DEFINITION 1.2. For any $\alpha \in (0,\infty)$, $x \in R^+$, $x \delta_L \alpha$ is defined as the only solution of the equation $\alpha *_L t = x$.

LEMMA 1.2. For any $\alpha, \beta \in (0,\infty)$, $x,y \in R^+$, the following equalities hold.

(i) $(x \delta_L \alpha) \delta_L \beta = x \delta_L (\alpha\beta)$, (1.8)

(ii) $L(x,y) \delta_L \alpha = L(x\delta_L \alpha, y\delta_L \alpha)$. (1.9)

DEFINITION 1.3. $\circledast_L: (0,\infty) \times \Delta^+ \to \Delta^+$ is defined as

$$\alpha \circledast_L F = F(j \delta_L \alpha), \ \alpha \in (0,\infty), \ F \in \Delta^+.$$

In particular, $\alpha \circledast_{sum} F = \alpha \odot F$, $\alpha \in (0,\infty)$, $F \in \Delta^+$.

LEMMA 1.3. For any $\alpha, \beta \in (0,\infty)$, $x \in R^+$, $F, G \in \Delta^+$, the following equalities hold.

(i) $\alpha \circledast_L \varepsilon_x = \varepsilon_{\alpha *_L x}$, (1.10)

(ii) $\alpha \circledast_L (\beta \circledast_L F) = (\alpha\beta) \circledast_L F$, (1.11)

(iii) $\alpha \circledast_L \tau_{T,L}(F,G) = \tau_{T,L}(\alpha \circledast_L F, \alpha \circledast_L G)$. (1.12)

COROLLARY 1.1. (cf. Lemma 15.1.3 in [5]) For any $\alpha \in (0,\infty)$, $F, G \in \Delta^+$,

$$\tau_{T,L}(F,G) = \tau_{T,L}(F(a *_L j), G(a *_L j))(j \delta_L a), \tag{1.13}$$

i.e., $\tau_{T,L}$ is homogenous in the sense of (1.13).

DEFINITION 1.4. For any $x, y \in [0,\infty)$ with $y < x$, $x \sim_L y$ is defined as the only solution of the equation $L(y,t) = x$.

DEFINITION 1.5. For any $a, b \in I$,

$$a \alpha_T b = \operatorname{Sup}\{x \mid T(b,x) \le a\}.$$

LEMMA 1.4. For any $a, b, c, a_\lambda, b_\lambda (\lambda \in \Lambda) \in I$,

(i) $T(a,b)\alpha_T a \geq b$, (1.14)

(ii) If $a \leq b$, then $a\alpha_T c \leq b\alpha_T c$, $c\alpha_T b \leq c\alpha_T a$, (1.15)

(iii) $a\alpha_T \inf_{\lambda\in\Lambda} b_\lambda \geq \operatorname{Sup}_{\lambda\in\Lambda}(a\alpha_T b_\lambda)$, (1.16)

$a\alpha_T \operatorname{Sup}_{\lambda\in\Lambda} b_\lambda = \inf_{\lambda\in\Lambda}(a\alpha_T b_\lambda)$, (1.17)

$\inf_{\lambda\in\Lambda} a_\lambda \alpha_T b = \inf_{\lambda\in\Lambda}(a_\lambda \alpha_T b)$, (1.18)

$\operatorname{Sup}_{\lambda\in\Lambda} a_\lambda \alpha_T b \geq \operatorname{Sup}_{\lambda\in\Lambda}(a_\lambda \alpha_T b)$. (1.19)

DEFINITION 1.6. $\eta_{T,L}: \Delta^+ x\ \Delta^+ \to I^{R^+}$ is defined as: for all $F, G \in G$,

$$(F\eta_{T,L}G)(x) = \begin{cases} \inf\{F(u)\alpha_T G(v) \mid u \hat{_L}\ v = x,\ v \leq u,\ u,\ v \in [0,\infty)\}, & x \in [0,\infty); \\ 1, & x = \infty. \end{cases}$$

It is easy to check that for any $F, G \in \Delta^+$, $F\eta_{T,L}G$ is left-continuous and increasing, but it is possible that $(F\eta_{T,1}G)(0) > 0$. In addition, from Lemma 2.4. (ii), we know that $\eta_{T,L}$ is increasing in the first place and decreasing in the second place.

LEMMA 1.5. For any $F, G \in \Delta^+$,

$\tau_{T,L}(F,G)\eta_{T,L}, G \geq F$. (1.20)

2. PROBABILISTIC SEMINORMS AND NORMS UNDER $\tau_{T,L}$

DEFINITION 2.1. Let V be a vector space over the field K of real or complex numbers, $\nu : V \to \Delta^+$. Then (V,ν) is called a PSN space under $\tau_{T,L}$ if for all $p, q \in V$, $\alpha \in K$ with $\alpha \neq 0$, the following conditions hold.

(i) $\nu(0) = \varepsilon_0$, (2.1)

(ii) $\nu(\alpha p) = |\alpha| \circledast_L \nu(p)$, (2.2)

(iii) $\nu(p+q) \geq \tau_{T,L}(\nu(p), \nu(q))$. (2.3)

If (V,ν) is a PSN space and satisfies: for all $p \in V$,

(iv) $\nu(p) \neq \varepsilon_0$ if $p \neq 0$, (2.4)

then (V,ν) is called a PN space.

THEOREM 2.1. If (V,ν) is a PSN space under $\tau_{T,L}$, then for all $p, q \in V$,

$\nu(p-q) \leq M(\nu(p)\eta_{T,L}\nu(q), \nu(q)\eta_{T,L}\nu(p))$, (2.5)

where M denotes the minimum function.

PROOF. From Lemma 1.5, we have

$$\nu(p)\ \eta_{T,L}\ \nu(q) \geq \tau_{T,L}(\nu(p-q), \nu(q))\ \eta_{T,L}\ \nu(q) \geq \nu(p-q)$$

because $\nu(p) \geq \tau_{T,L}(\nu(p-q), \nu(q))$.

In addition,

$$\begin{aligned}\nu(p-q) &= 1 \circledast_L \nu(q-p) \\ &= \nu(q-p) \\ &\leq \nu(q)\ \eta_{T,L}\ \nu(p).\end{aligned}$$

THEOREM 2.2. In a PSN space (V,ν) under $\tau_{T,L}$, for all $\varepsilon \in R^+$, $\lambda \in I$, $p \in V$, the ball with center p and radius ε of level λ $B_p(\varepsilon,\lambda) = \{q \mid T(\nu_{q-p}(\varepsilon), \lambda) = \lambda\}$ is convex.

PROOF. If $q_1, q_2 \in B_p(q,\lambda)$, $t \in [0,1]$, then

$$\nu_{[tq_1+(1-t)q_2]-p}(\varepsilon) = \nu_{t(q_1-p)+(1-t)(q_2-p)}(\varepsilon)$$

$$\geq \tau_{T,L}\left(\nu_{t(q-p)}, \nu_{(1-t)(q-p)}\right)(\varepsilon)$$

$$= \operatorname{Sup}\left\{T\left(\nu_{t(q_1-p)}(\varepsilon_1), \nu_{(1-t)(q_2-p)}(\varepsilon_2)\right) \mid L(\varepsilon_1,\varepsilon_2) = \varepsilon\right\}$$

$$\geq T(\nu_{t(q_1-p)}(t *_L \varepsilon), \nu_{(1-t)(q_2-p)}((1-t) *_L \varepsilon))$$

$$= T\left(\nu_{q_1-p}(\varepsilon), \nu_{q_2-p}(\varepsilon)\right).$$

$$T\left(\nu_{[tq_1+(1-t)q_2]-p}(\varepsilon), \lambda\right) \geq T\left(T(\nu_{q_1-p}(\varepsilon), \nu_{q_2-p}(\varepsilon)), \lambda\right)$$

$$= T\left(\nu_{q_1-p}(\varepsilon), T(\nu_{q_2-p}(\varepsilon), \lambda\right)$$

$$= T(\nu_{q_1-p}(\varepsilon),\lambda)$$

$$= \lambda.$$

THEOREM 2.3. In a PN space (V,ν) under $\tau_{T,L}$, let
$U_p(\varepsilon,\lambda) = \{q \mid \nu_{q-p}(\varepsilon) > 1-\lambda\}$, $\varepsilon, \lambda > 0$, $p \in V$.
Then the family $\mathcal{U} = \{\nu_p(\varepsilon,\lambda) \mid \varepsilon > 0, \lambda > 0, p \in V\}$ generates a Hausdorff topology $\mathcal{T}$ which is called the strong topology of V. Moreover,

(1) $+: V \times V \to V$, $(p,q) \to p+q$, $p, q \in V$ is continuous;

(2) If Ran $\nu \subseteq \mathcal{D}^+$, then $.: k \times V \to V$, $(\alpha,p) \to \alpha p$, $\alpha \in R$, $p \in V$ is continuous, where $\mathcal{D}^+ = \{F \in \Delta^+ \mid \sup_{x<\infty} F(x) = 1\}$;

(3) $\nu: V \to \Delta^+$, $p \to \nu(p)$, $p \in V$ is continuous.

PROOF. Straightforward.

To illustrate that the condition in Theorem 2.3. (2) is necessary, we give

EXAMPLE 2.2. Let $\nu_o: R \to \Delta^+$

$$\nu_o(x) = \begin{cases} \varepsilon_o, & \text{if } x = 0, \\ \varepsilon_\infty, & \text{if } x \neq 0 \,. \end{cases}$$

Then (R,ν_o) is a PN space under $\tau_{T,L}$. However, $1/n \to 0$, but $1/n \xrightarrow{\mathcal{F}(R,\bar{\nu})} 0$ does not hold.

THEOREM 2.4. If (V,ν) is a PSN (or PN) space under $\tau_{T,L}$, $\mathcal{F}: V \times V \to \Delta^+$ is defined as

$$F(p,q) = \nu\,(p-q),\ p,\ q \in V, \tag{2.6}$$

then (V,F) is a PPM (resp, PM) space under $\tau_{T,L}$ which has the following properties: for all $p, q, r \in V$, $\alpha \in K$ with $\alpha \neq 0$,

(i) $F(\alpha p, \alpha q) = |\alpha| \circledast_L F(p,q)$, (2.7)

(ii) $F(p+r, q+1) = F(p,q)$. (2.8)

Conversely, if (V,F) is a PPM (or PM) space under $\tau_{T,L}$ with (2.7), (2.8), then there exists a PSN (resp. PN) space under $\tau_{T,L}$ such that (2.6) holds.

PROOF. Immediate.

3. L-SIMPLE SPACES.

DEFINITION 3.1. Let $(V,||.||)$ be a normed space, and $G \in \Delta^+ \setminus \{\varepsilon_o, \varepsilon_\infty\}$. Then $(V,||.||,G)$, the L-simple space generated by $(V,||.||)$ and G, is the pair

(V,ν) in which $\nu: V \to \Delta^{+}$ is defined by

$$\nu(p) = \|p\| \circledast_L G, \quad p \in V. \tag{3.1}$$

In Particular, Sum-simple spaces are also simple spaces.

DEFINITION 3.2. Let $\mathfrak{C}$ be a class of pairs (V,ν) in which V is a vector space, and $\nu: V \to \Delta^{+}$ satisfies (2.1), (2.4), and τ is a triangle function. If for any $(V,\nu) \in \mathfrak{C}$ it holds that

$$\nu(p+q) > \tau(\nu(p), \ \nu(q)), p, q \in V, \tag{3.2}$$

then τ is said to be universal for $\mathfrak{C}$.

DEFINITION 3.3. If $F, G \in \Delta^{+}$, and there exists $\alpha \in (0,\infty)$ such that $G = \alpha \circledast_L F$, then F and G are said to be L-comparable. We write

$$\mathcal{C}_L(\Delta^{+}) = \{(F,G) \in \Delta^{+} \times \Delta^{+} \mid F, G \text{ are L-comparable}\}.$$

THEREOM 3.1. (cf. Theorem 8.4.2, 8.4.4 and Problem 8.8.1 in [5]). Triangle function τ is universal for the class $\mathcal{S}_L$ of all L-simple spaces if and only if $\tau|_{C_{L(\Delta^{+})}} < \tau_{M,L}|_{C_{L(\Delta^{+})}}$.

PROOF. ($\Longleftarrow$) First, we show that $\tau_{M,L}$ is universal for $\mathcal{S}_L$. In fact, if (V,ν) is a L-simple space, then for all $p \in V$, $y \in I$,

$$\nu(p)^{\wedge}(y) = \operatorname{Sup}\{x \mid \nu(p)(x) < y\}$$

$$= \operatorname{Sup}\{x \mid G(x \,\delta_L \|p\|) < y\}$$

$$= \operatorname{Sup}\{\|p\| *_L \zeta \mid G(\zeta) < y\}$$

$$= \| p\| *_L \operatorname{Sup}\{\zeta \mid G(\zeta) < y\}$$

$$\| p\| *_L G(y).$$

Therefore, from Lemma 1.1. (iii), we obtain that for all $p, q \in V$, $y \in I$,

$$\nu(p+q)^{\wedge}(y) = \|p+q\| *_L G(y)$$

$$< (\|p\| + \|q\|) *_L G(y)$$

$$= L(\|p\| *_L G^{\wedge}(y), \ \|q\| *_L G^{\wedge}(y))$$

$$= L(\nu(p)^{\wedge}(y), \ \nu(q)^{\wedge}(y)).$$

and from (7.7.10) in [5], we have

$$\nu(p+q) > [L(\nu(p)^{\wedge}, \ \nu(q)^{\wedge})] = \tau_{M,L}(\nu(p), \ \nu(q)).$$

In general, if $\tau|_{C_{L(\Delta^{+})}} < \tau_{M,L}|_{C_{L(\Delta^{+})}}$, then for any L-simple space (V,ν) and for any $p, q \in V$,

$$\tau(\nu(p), \ \nu(q)) < \tau_{M,L}(\nu(p), \ \nu(q)).$$

In fact, if $p = 0$, or $q = 0$, then $\tau(\nu(p), \nu(q)) = \nu(q)$ or $\nu(p)$, and if $p \neq 0$ and $q \neq 0$, then $\|q\|/\|p\| \in (0,\infty)$ and $\nu(q) = \|q\|/\|p\| \circledast_L \nu(p)$, i.e., $(\nu(p), \ \nu(q)) \in C_L(\Delta^{+})$. Consequently, τ is universal for S_L because so is $\tau_{M,L}$.

($\Longrightarrow$). If $\tau|_{C_{L(\Delta^{+})}} < \tau_{M,L}|_{C_{L(\Delta^{+})}}$ does not hold, then there exists a pair $(G,F) \in C_L(\Delta^{+})$ such that $\tau(G,F) \nless \tau_{M,L}(G,F)$. Because $\tau(\varepsilon_0,F) = \tau_{M,L}(\varepsilon_0,F) = F$, we have and $\tau(\varepsilon_\infty,F) = \varepsilon_\infty$, that $G \notin \{\varepsilon_0, \varepsilon_\infty\}$. Now, we consider the L-simple space $(R,\nu) = (R,|\cdot|, G)$ generated by the real line with the usual norm and G. Since

$(G,F) \in \mathcal{C}_L(\Delta^+)$, there exists $\alpha \in (0,\infty)$ such that $F = \alpha *_L G$. In addition, from $\tau(G,F) < \tau_{M,L}(G,F)$, we know that there exists $x_o \in (0,\infty)$ such that

$$\tau(G,F)(x_o) > \tau_{M,L}(G,F)(x_o), \text{ and furthermore}$$

$$\tau(\nu(1),\ \nu(\alpha))\ (x_o) = \tau(\nu(1),\ \alpha *_L \nu(1))(x_o)$$

$$= \tau(G,F)(x_o)$$

$$> \tau_{M,L}(G,F)(x_o)$$

$$= \mathrm{Sup}\,\{M(G(u),\ F(\nu)) \mid L(u,v) = x_o\}$$

$$> M(G(x_o\,\delta_L(1+\alpha)),\ F(\alpha *_L\ (x_o\,\delta_L(1+\alpha)))$$

$$= G(x_o\,\delta_L\ (1+\alpha))$$

$$= \nu(1+\alpha)(x_o)$$

because

$$L(x_o\,\delta_L(1+\alpha)),\ \alpha *_L(x_o\,\delta_L(1+\alpha)) = (1+\alpha)\ *_L(x_o\,\delta_L(1+\alpha)) = x_o$$

and

$$F(\alpha *_L(x_o\,\delta_L(1+\alpha)) = G\ [(\alpha *_L(x_o\,\delta(1+\alpha))\,\delta_L\,\alpha]$$

$$= G\ (x_o\,\delta_L(1+\alpha)).$$

This contradicts (3.2).

COROLLARY 3.1. Any L-simple space is a PN space under $\tau_{T,L}$.

Let $(V,||\cdot||)$ be a normed space, $\alpha \in (0,\infty)$ and $G \in \Delta^+ \setminus \{\varepsilon_o, \varepsilon_\infty\}$. By the α-simple space generated by $(V,||\cdot||)$ and G, $(V,||\cdot||,\ G,\alpha)$, we mean the pair (V,ν) in which $\nu\colon V \to \Delta^+$ is defined by $\nu(p) = G(\delta/||\ p||^\alpha)$, $p \in V$.

The following corollary characterizes α-simple spaces.

COROLLARY 3.2. (cf. Problem 8.8.2 in [5]) For $\alpha \in (0,\infty)$, any α-simple space is a PN space under τ_M, $K_{1/\alpha}$.

PROOF. It is sufficient to note that $x\,\delta_{K_{1/\alpha}}\ a = x/a^\alpha$, $x,\ a \in (0,\infty)$, and so α-simple spaces are also $K_{1/\alpha}$-simple spaces.

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