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A NOTE ON QUASI R*-INVARIANT MEASURES ON SEMIGROUPS

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ABSTRACT. A characterization of quasi r*-invariant measures on metric topological semigroups is obtained by showing that their support has a left group structure thus generalizing previously known results for relatively r*-invariant measures and the topo-algebraic structure of their support.

KEY WORDS AND PHRASES. Measures on topological semigroups. Radon, Invariant, absolutely continuous measures; metric semigroups; left groups; Baire space.
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1. INTRODUCTION.

Two interesting characterizations of absolute continuity of a Radon measure μ with respect to a right Haar measure λ on a locally compact group S is the continuity of $x \to \mu(Kx)$ for all compact K and also the condition

$$\mu * \delta_x << \lambda \quad \text{for all } x \in S, \ \delta_x \text{ the point-mass at } x, \tag{1.1}$$

(cf. [1],V, 19.18, 19.27, 20.28, 20.31)).

One is tempted to replace λ with μ in (1.1) to obtain thus a measure μ which turns out to be also absolutely continuous (cf. [2] and [3]). Moreover this condition i.e.,

$$\mu * \delta_x << \mu \quad \text{for all } x \in S \tag{1.2}$$

makes sense even when there is no invariant λ and S is only a semigroup. Following [2] and [3] we define a non-negative, inner regular, locally finite Radon measure on a semitopological semigroup S to be <u>quasi r*-invariant</u> if in addition to (1.2), the companion condition $\mu << \mu * \delta_x$ for all x, also holds. Similarly one defines <u>quasi l*-invariant</u> measures using $\mu << \delta_x * \mu$ for all $x \in S$ and its "companion" condition i.e. the "equivalence of these measures, where $\delta_x * \mu(B) = \mu(x^{-1}B) = \mu\{s;\ xs \in B\}$. In [4] A. Mukherjee considered relatively r*-invariant measures satisfying $\mu(Bx^{-1}) = \beta(x)\mu(B)$ for all Borel B and x, with β being a fixed homomorphism $\beta : S \to (0,\infty)$, and proved that their support is a <u>left group</u> i.e., left simple and right cancellative semigroup. Such measures are clearly quasi r*-invariant but in the non locally compact case there may exist quasi r*-invariant measures although there is no right invariant one (cf. e.g., [5]). In this note we prove similar results (under weaker conditions) for quasi r*-invariant measures and study the topo-algebraic structure of their support $F = \{x \in S;$ every open neighborhood of x has positive μ measure$\}$.

Throughout we will assume that S is (at least) a T_2 semitopological semigroup in which the right (continuous) translations $t_x:s\to sx$ are closed maps. Topologically, in most cases, we will assume (and state it explicitly when we shall do so) that S is either a locally compact paracompact 1° countable or a metrizable space. The measure μ is called purely atomic if every singleton in F has positive measure. Clearly, a purely atomic measure on a left simple support (i.e., $Fx = F$ for all $x\in F$) is trivially quasi r*-invariant. Also, pure atomicity for σ-compact S, forces F to be countable. Although the purely atomic case with the extra condition that $\mu\{xx^{-1}\}<\infty$, may have some interesting pathology, in this note however we consider mostly the non-purely atomic case and the continuous case (i.e. $\mu\{x\}=0$ for all $x\in F$) and show in this case that F is a left group and a right ideal of S. (See Theorem 4).

2. STRUCTURE OF THE SUPPORT.

The effect of the existence of an atom (singleton) on the structure of F is shown by the following.

THEOREM 1. Suppose μ quasi r*-invariant on S and that the right translations t_x, $x\in S$, are closed.

(i) If S is σ-compact and if for some $a\in F$, $\mu\{a\}>0$, then aF is a countable left simple, left cancellable (separable metric) semigroup.

(ii) If for some $b\in F$, $\mu\{b\}=0$, then F is a left group.

REMARK. In this Theorem S is either a locally compact paracompact 1° countable (or a metrizable) topological semigroup and the attributes in parenthesis hold if S is metrizable.

PROOF. First we observe that the condition $\mu<<\mu*\delta_x$ for all x, immediately implies that $F\subset\overline{Fx}$ = closure (Fx); for V open in F implies $\mu(Vx^{-1})>0$ which implies $V\cap Fx \neq \emptyset$. Also the companion condition $\mu*\delta_x<<\mu$ implies that F is a semigroup, in fact a right ideal. For if $\mu(B)>0$, then since $B\subset Bxx^{-1}$, we have $\mu(Bxx^{-1})>0$ and hence $\mu(Bx)>0$. (Of course, we need the semitopological property of S). Hence, we have $\overline{Fx} = F$ for all x and since the t_x are closed we have $Fx = F$. It follows by a standard argument in ([6], p. 144, Lemma 2.1.), that for every $a\in F$, aF is left cancellable. Also aF is left simple, since F itself is so. (i): Since in this part we assume F σ-compact, aF is also σ-compact and all its elements have positive measure, so it must be countable (μ on aF is purely atomic). (ii): Since $\mu\{bb^{-1}\}=0$, Interior $(bb^{-1})=\emptyset$ and so Frontier (bb^{-1}) must be countably compact ([7], p. 254, Exerc. 14) and by paracompactness, it is a compact subsemigroup and hence contains an idempotent element. It follows that F is a left group. (Observe that $bb^{-1}\neq\emptyset$ since F is left simple). The rest follow easily.

REMARK. We remark that we could have assumed S to be only semitopological (multiplication separably continuous) and again F would turn out to be a topological left group in (ii) and aF a topological group by a theorem of Ellis (cf. [8], p. 60). Of cource we would have to impose completeness in the metric case. (cf. e.g. N. Bourbaki, General Topology, Part II, p. 258).

We shall need the following topo-algebraic result which is also of independent interest.

LEMMA 1. Suppose S is a locally compact or complete separable metrizable semitopological semigroup with the t_x (right translations) closed. Suppose further that F is a closed subset of S such that $\overline{Fx} = \overline{xF} = F$ for all $x \in F$. Then F is a topological subgroup.

PROOF. Clearly, $\overline{FF}$ = closure(FF) = F and hence F is a subsemigroup. Since $\overline{xF}=F$ for all $x \in F$, it follows that Fa=F is right cancellable (see proof of Theorem 1), so F is a left group. Since every aF contains now some eF, with ee = e, and since eF is closed, we have aF=F for all $a \in F$, that F is also right simple. Hence, F is a group and being locally compact (or complete separable metric) is a topological group.

THEOREM 2. Suppose S is a topologically complete topological semigroup with the t_x closed and μ is a quasi r*-invariant and also quasi l*-invariant measure on F. Then F ia a locally compact group and μ is absolutely continuous with respect to the right Haar measure on F.

PROOF. From the above Lemma we obtain easily F to be a Baire topological group. By the main result in [3], F is locally compact. Then it follows from the work of ([2], p. 229) that μ is the indefinite integral with respect to the right Haar measure of an almost everywhere (Haar) positive function.

REMARK. Actually half of the definition of l*-invariance is required in the above Theorem, i.e., we only need $\mu \ll \delta_x * \mu$ for all x. Of course we need (fully) quasi r*-invariance.

For continuous measures we have the following theorem.

THEOREM 3. Suppose S is a topologically complete semigroup with the t_x's closed and μ is a quasi r*-invariant continuous (i.e., $\mu\{x\} = 0$ for all $x \in F$) measure on S. Then F is a locally compact left group.

PROOF. Since the t_x are closed, the Proof of Theorem 1 (ii) shows that F is a left group. Now every left group satisfies condition (R) of [9] i.e., that $K^{-1}(Kx)$ is compact for compact K. Also since F is a Baire space, every point of F is a point of condensation (each neighborhood contains uncountably many points, since F has no isolated points by the "continuity" of μ.). Now the proof given in ([9], Lemma 1, p. 255) of producing a relatively compact neighborhood V of a point x, goes through since the process of producing disjoint translates of K (K of positive measure) by points of V (V a Baire space itself) must terminate after a countable enumeration (local finiteness here is needed), and this leads to contradiction.

Question: Can we say anything as regards to absolute continuity of such μ with respect to some "standard" r*-invariant product measure on the left group of the form $\mu_1 \times \lambda$, λ the right Haar measure on the group component (factor) of F?).

The following Theorem summarizes the various conditions on μ (i.e. "fibers" must not be "too big" in measure) and on the t_x, in order that the support of a quasi r*-invariant measure to be a left group.

THEOREM 4. Suppose μ is a quasi r*-invariant measure on S (always the t_x are assumed closed) and further suppose that any one of the following three conditions obtains

(a) μ is not purely atomic (this holds in particular if μ is continuous).

(b) The t_x's are proper maps i.e., in addition to closedness we also have yx^{-1} compact for all $x,y \in F$.

(c) In the purely atomic case, for each $a \in F$ there is a homomorphism $\beta: S \to (0,\infty)$ (possibly depending on $a \in F$) such that $\mu(ab^{-1}) \leq \beta(b)\mu\{a\}$ for all $a,b \in F$.

Then F is a left group.

PROOF. Assume (b). Then aa^{-1} is a compact semigroup (non-empty by left simplicity) and hence contains an idempotent element. Next, assume (c). Trivially one checks that β is a constant (equal to unity) on each aa^{-1}. Now take $z \in aa^{-1}$. Defining the support of a closed set in the natural way, it follows that Support $(aa^{-1}) = aa^{-1}$ and that aa^{-1} is also left simple. For $u \in aa^{-1}$, consider

$$L = (zu)(zu)^{-1} \cap z(aa^{-1})$$

with is non-empty by left simplicity of $z(aa^{-1})$ and every element in it has measure greater or equal to the measure of z, since $z \in zuu^{-1}$ and β is 1 on aa^{-1}. It turns out that L must be a finite simigroup and hence must contain an idempotent. The part using Condition (a) follows from Theorem 1. (cf. the Remark to that Theorem).

We conjecture that one can dispense with the rather unnatural condition (c). Observe that in (b) and (c) we only need S to be T_2 semitopogical (with the t_x's closed).

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