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THE DITTERT'S FUNCTION ON A SET OF NONNEGATIVE MATRICES

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ABSTRACT. Let K_n denote the set of all $n \times n$ nonnegative matrices with entry sum n. For $X \in K_n$ with row sum vector $(r_1,\ldots,r_n)$, column sum vector $(c_1,\ldots,c_n)$, Let $\phi(X) = \Pi_i r_i + \Pi_j c_j - \text{per}X$. Dittert's conjecture asserts that $\phi(X) \leq 2 - n!/n^n$ for all $X \in K_n$ with equality iff $X = [1/n]_{n\times n}$. This paper investigates some properties of a certain subclass of K_n related to the function ϕ and the Dittert's conjecture.

KEY WORDS AND PHRASES. Permanent, Dittert's function, Λ-admissable matrix.
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1. INTRODUCTION.

Let K_n denote the set of all $n \times n$ nonnegative matrices whose entries have sum n, and let ϕ denote a real valued function of K_n defined by

$$\phi(X) = \prod_{i=1}^{n} \sum_{j=1}^{n} x_{ij} + \prod_{j=1}^{n} \sum_{i=1}^{n} x_{ij} - \text{per}X$$

for $X = [x_{ij}] \in K_n$ where perX stands for the permanent of X;

$$\text{per}X = \sum_{\sigma \in S_n} x_{1\sigma(1)} \cdots x_{n\sigma(n)}.$$

Let J_n denote the $n \times n$ matrix all of whose entries are $1/n$. For the function ϕ there is a conjecture due to Eric Dittert.

CONJECTURE (Marcus and Merris [1], Conjecture 28]). For $A \in K_n$,

$$\phi(A) \leq 2 - \frac{n!}{n^n}$$

with equality if and only if $A = J_n$.

In this paper, we will call ϕ the Dittert's function. It is proved that the Dittert's conjecture is true for $n \leq 3$ (Marcus and Merris [1], Sinkhorn [2], and Hwang [3]). For a matrix $X \in K_n$ whose row sum vector is $(r_1,\ldots,r_n)$ and whose column sum vector is $(c_1,\ldots,c_n)$,

Let

$$\overline{r}_i = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n \quad (i=1,\ldots,n),$$

$$\overline{c}_j = c_1 \cdots c_{j-1} c_{j+1} \cdots c_n \quad (j=1,\ldots,n)$$

and

$$\phi_{ij}(X) = \overline{r}_j + \overline{c}_j - \operatorname{per}X(i|j) \qquad (i,j = 1,2,\ldots,n)$$

where $X(i|j)$ denotes the matrix obtained from X by deleting the row i and column j. A matrix $A \in K_n$ is called a ϕ- maximizing matrix on K_n if $\phi(A) \geq \phi(X)$ for all $X \in K_n$. In [3], the following results are proved.

THEOREM A. *If $A = [a_{ij}]$ is a ϕ- maximizing matrix on K_n, then*

$$\phi_{ij}(A) \begin{cases} = \phi(A) & \text{if } a_{ij} > 0 \\ \leq \phi(A) & \text{if } a_{ij} = 0 . \end{cases}$$

THEOREM B. *If, for every ϕ- maximizing matrix A on K_n, $\phi_{ij}(A) = \phi(A)$ for all $i,j=1,\ldots,n$, then J_n is the unique ϕ- maximizing matrix on K_n.*

We see that $\phi(A) \geq 0$ for all $A \in K_n$. For $A \in K_n$ with row sum vector $(r_1,\ldots,r_n)$ and column sum vector $(c_1,\ldots,c_n)$, if either $r_1\cdots r_n > 0$ or $c_1\cdots c_n > 0$, then $\phi(A) > 0$. Now, for $A \in K_n$ with $\phi(A) > 0$, Let $A^* = [a^*_{ij}]$ denote the $n \times n$ matrix defined by

$$a^*_{ij} = \frac{\phi_{ij}(A)}{\phi(A)} \qquad (i,j = 1,\ldots,n).$$

For $\Lambda \in K_n$, we say that $A \in K_n$ with $\phi(A) > 0$ is Λ-admissable (or A is admissable by Λ) if $\operatorname{tr}(\Lambda^T A^*) \geq n$ where Λ^T denotes the transpose of Λ and tr denotes the trace function. Let $\mathcal{C}(\Lambda)$ denotes the set of all Λ-admissable matrices.

It follows from Theorem A that every ϕ-maximizing matrix A is self-admissable i.e. $A \in \mathcal{C}(A)$.

If for each ϕ-maximizing matrix A there exists a positive matrix $\Lambda \in K_n$ such that $A \in \mathcal{C}(\Lambda)$, then the Dittert's conjecture is true (See section 2).

In such a point of view, it would be interesting to study the classes (Λ) for some particular matrices $\Lambda \in K_n$. Such a matrix Λ should be one which is most likely to posess the property that all ϕ-maximizing matrices on K_n are Λ-admissable.

In this paper we find some matrices in $\mathcal{C}(\Lambda)$ for certain Λ's and investigate some properties of the Dittert's function related to the class $\mathcal{C}(\Lambda)$.

2. THE CLASS $\mathcal{C}(\Lambda)$ AND ϕ- MAXIMIZING MATRICES.

From now on let $\operatorname{Max}(K_n)$ denote the set of all ϕ-maximizing matrices on K_n.

THEOREM 2.1. *If each $A \in Max(K_n)$ is admissable by a positive matrix in K_n, then $Max(K_n) = \{J_n\}$, i.e. the Dittert's conjecture holds.*

PROOF. Let $A \in Max(K_n)$ and let $\Lambda = [\lambda_{ij}] \in K_n$ be a positive matrix such that $A \in \mathcal{C}(\Lambda)$. Then

$$n \leq tr(\Lambda^T A^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} \frac{\phi_{ij}(A)}{\phi(A)} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} = n \tag{2.1}$$

by Theorem A. Therefore the inequalitites in (2.1) are all equalities and hence $\phi_{ij}(A) = \phi(A)$ for all $i,j = 1,2,\dots,n$ since Λ is a positive matrix. Now the assertion of the theorem follows from Theorem B.

For $A \in K_n$ with row sum vector $(r_1,\dots,r_n)$ and column sum vector $(c_1,\dots,c_n)$, Let $\hat{A} = [\hat{a}_{ij}]$ denote the $n \times n$ matrix defined by

$$\hat{a}_{ij} = \frac{r_i c_j}{n} \ (i,j = 1,\dots,n).$$

Since $\sum_{i=1}^{n} \sum_{j=1}^{n} r_i c_j = n^2$ we see that $\hat{A} \in K_n$. In particular if $A \in Max(K_n)$, then $\hat{A}$ is a positive matrix since $r_i > 0$, $c_j > 0$ for all $i,j = 1,\dots,n$ because $perA > 0$ [2].

We believe that every $A \in Max(K_n)$ is $\hat{A}$-admissable, which we can not prove yet. We may ask which matrices $A \in K_n$ are $\hat{A}$-admisable and which are not. We have an answer to this question.

THEOREM 2.2. *If A is positive semidefinite symmetric matrix in K_n, then A is $\hat{A}$-admissable.*

PROOF. Let A be a p.s.d. symmetric matrix in K_n and let r_i be the i-th row sum of $A(i=1,\dots,n)$. Then the condition that A is $\hat{A}$-admissable is equivalent to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \phi_{ij}(A) \geq n^2 \phi(A). \tag{2.2}$$

Let $r = r_1 \cdots r_n$ and let $\bar{r}_i = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n$ $(i=1,\dots,n)$. Then

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \phi_{ij}(A) &= \sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j [\bar{r}_i + \bar{r}_j - perA(I|j)] \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} [(r_i + r_j)r - r_i r_j \, perA(i|j)] \\ &= 2n^2 r - \sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \, perA(i|j).\end{aligned}$$

Since

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \, perA(i|j) \leq n^2 \, perA$$

by a theorem of Marcus and Merris [4], we have

$$\sum_{i=1}^{n} \sum_{i=1}^{n} r_i r_j \, \phi_{ij}(A) > 2n^2 r - n^2 \operatorname{per} A = n^2 \phi(A)$$

and the proof is complete.

Note that not every matrix $A \in K_n$ is $\hat{A}$-admissable. For n=2, the matrix

$$A_x = \begin{bmatrix} 2-2x & x \\ x & 0 \end{bmatrix}$$

in K_2 is not $\hat{A}_x$-admissable if $0 < x < 1/2$. For $n > 3$, we have an

EXAMPLE 2.1. Let T_n denote the following $n \times n$ matrix.

$$T_n = \begin{bmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $T_n \in K_n$ and $(r_1,\ldots,r_n) = (1,\ldots,1)$, $(c_1,\ldots,c_n) = (2, \frac{n-2}{n-1},\ldots, \frac{n-2}{n-1})$. We have

$$n^2\phi(T_n) - \sum_{i=1}^{n} \sum_{j=1}^{n} r_i c_j \, \phi_{ij}(T_n) = 2 \frac{(n-1)!}{(n-1)^{n-2}} > 0$$

so that $T_n \in \mathcal{C}(T_n)$ and hence that T_n is not $\hat{T}_n$-admissable.

3. THE CLASS $\mathcal{C}(J_n)$ AND THE MONOTONICITY OF THE DITTERT'S FUNCTION.

Another candidate for positive $A \in K_n$ with "good" $\mathcal{C}(A)$ is the matrix J_n. A nonnegative square matrix is called a <u>doubly stochastic</u> matrix if all the row sums and column sums are equal to 1. It is conjectured that every $n \times n$ doubly stochastic matrix is J_n-admissable (Dokovic [5] and Minc [6]) but this still remains open. Here we have to notice that A is J_n-admissable (i.e. $A \in \mathcal{C}(J_n)$) if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(A) > n^2\phi(A).$$

We can show that $\mathcal{C}(J_n) \neq K_n$ for $n > 3$ (see Example 3.1). However it seems that $\operatorname{Max}(K_n) \not\subset \mathcal{C}(J_n)$. It is clear that J_n and the $n \times n$ identity matrix I_n are J_n-admissable. We can show that all diagonal matrices in K_n are also J_n-admissable.

THEOREM 3.1. <u>Every diagonal matrix in K_n is J_n-admissable.</u>

PROOF. Let $A = \operatorname{diag}(a_1,\ldots,a_n) \in K_n$, $a = a_1 \cdots a_n$ and $\bar{a}_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$ $(i=1,\ldots,n)$. If a=0, there is nothing to prove. Suppose $a > 0$. Then $\phi(A)=a$ and

$$\sum_{i=1}^{n} \sum_{i=1}^{n} \phi_{ij}(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\bar{a}_i + \bar{a}_j) - \sum_{i=1}^{n} \bar{a}_i$$

$$= (2n-1) \sum_{i=1}^{n} \bar{a}_i$$

$$= (2n-1)a \sum_{i=1}^{n} \frac{1}{a_i} \geq n(2n-1)a.$$

Therefore,

$$\phi(A) < \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(A)$$

if $n \geq 2$, and the proof is complete.

The Dittert's function ϕ has some nice behavior on the set $\mathcal{P}(J_n)$ namely that ϕ is monotone on the straight line segment joining J_n and $A \in \mathcal{P}(J_n)$ whenever the line segment lies in $\mathcal{P}(J_n)$. To show this, let Δ be a function define by

$$\Delta(X) = \phi(X) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(X), \quad X \in K_n.$$

Let $A=[a_{ij}] \in K_n$ have row sum vector $(r_1,\dots,r_n)$ and column sum vector $(c_1,\dots,c_n)$. For a real number t, $0 \leq t \leq 1$, let $A_t = (1-t)J_n + tA := [a_{ij}(t)]$ and let the row sum vector and the column sum vector of A_t be $(r_1(t),\dots,r_n(t))$ and $(c_1(t),\dots,c_n(t))$ respectively.

Letting

$$r(t) = r_1(t) \cdots r_n(t),$$

$$c(t) = c_1(t) \cdots c_n(t),$$

$$\bar{r}_i(t) = r_1(t) \cdots r_{i-1}(t) r_{i+1}(t) \cdots r_n(t), \quad (i=1,\dots,n),$$

$$\bar{c}_j(t) = c_1(t) \cdots c_{j-1}(t) c_{j+1}(t) \cdots c_n(t), \quad (j=1,\dots,n),$$

we compute, for $t > 0$, that

$$\frac{d}{dt} r(t) = \frac{1}{t} \sum_{i=1}^{n} \{r(t) - \bar{r}_i(t)\}$$

$$= \frac{n}{t} \{r(t) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{r}_i(t)\},$$

$$\frac{d}{dt} c(t) = \frac{1}{t} \sum_{j=1}^{n} \{c(t) - \bar{c}_j(t)\}$$

$$= \frac{n}{t} \{c(t) - \frac{1}{n^2} \sum_{j=1}^{n} \bar{c}_j(t)\},$$

$$\frac{d}{dt} \operatorname{per}A_t = \frac{n}{t} \{\operatorname{per}A_t - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \operatorname{per}A_t(i|j)\}$$

so that

$$\frac{d}{dt} \phi(A_t) = \frac{n}{t} \{r(t) + c(t) - \operatorname{per}A_t - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\bar{r}_i(t) + \bar{c}_j(t) - \operatorname{per}A_t(i|j)]\}$$

$$= \frac{n}{t} \{\phi(A_t) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(A_t)\},$$

which is

$$\frac{d}{dt} \phi(A_t) = \frac{n}{t} \Delta(A_t).$$

Thus we have the following

THEOREM 3.2. Let $A \varepsilon K_n$. If $A_t \ \varepsilon \ C(J_n)$ for all t, $0 < t < 1$, then the Dirrert's function is monotone decreasing on the straight line segment from J_n to A.

It is not hard to show that, for any $A \varepsilon K_2$,

$$\frac{1}{2^2} \sum_{i=1}^{2} \sum_{j=1}^{2} \phi_{ij}(A) = \frac{3}{2}.$$

On the other hand, the validity of Dittert's conjecture for n=2 gives us that

$$\frac{3}{2} = \phi(J_n) > \phi(A).$$

Therefore it follows that $K_2 = C(J_2)$. However it does not hold in general that $K_n = C(J_n)$.

EXAMPLE 3.1. Let

$$U_n = \begin{bmatrix} & & & & \frac{n}{n+1} & \frac{n}{n+1} \\ & \bigcirc & & \frac{n}{n+1} & 0 & 0 \\ & & \cdot^{\cdot^{\cdot}} & 0 & & \\ & \frac{n}{n+1} & 0 & & & \\ \frac{n}{n+1} & 0 & & & \bigcirc & \\ \frac{n}{n+1} & 0 & & & & \end{bmatrix}_{n \times n}, \quad n \geqq 4$$

and let

$$U_3 = \begin{bmatrix} 0 & \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & 0 \\ \frac{3}{4} & 0 & 0 \end{bmatrix}.$$

Then

$$\phi(U_n) = 4\left(\frac{n}{n+1}\right)^n$$

and

$$U_n^* = \frac{n+1}{4n}\begin{bmatrix} 2 & 3 & \cdots & 3 & 3 & 3 \\ 3 & 4 & \cdots & 4 & 3 & 3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 3 & 3 & \cdots & 3 & 3 & 3 \\ 3 & 3 & & 3 & 3 & 3 \end{bmatrix}.$$

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{ij}(U_n)$$

$$= \phi(U_n) \times (\text{sum of entries of } U_n^*)$$

$$= 4\left(\frac{n}{n+1}\right)^n \frac{n+1}{4n} [2 + 4(n-3)^2 + 3(6n-10)]$$

$$= \left(\frac{n}{n+1}\right)^{n-1} (4n^2 - 6n + 8).$$

Thus we have

$$n^2\phi(U_n) - \sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{ij}(U_n)$$

$$= \left(\frac{n}{n+1}\right)^{n-1} \left(\frac{4n^3}{n+1} - 4n^2 + 6n - 8\right)$$

$$= \frac{n^{n-1}}{(n+1)^n} (2n^2 - 2n - 8),$$

which is positive for all $n > 3$, telling us that U_n is not J_n-admissable.

4. CONCLUDING REMARKS.

If, for every $A \in \text{Max}(K_n)$, we could find a positive matrix $\Lambda \in K_n$ such that A is admissable by Λ, it would prove the Dittert's conjecture by Theorem 2.1. It seems to us that the matrices $\hat{A}$ or J_n are two of the strongest candidates for such matrices. However we may not expect to have a positive matrix $\Lambda \in K_n$ such that all the matrices in K_n are Λ-admissable.

We shall close our discussion here by giving some further research problems.

PROBLEM 4.1. Determine whether there exists a positive matrix $\Lambda \in K_n$ admitting all matrices in K_n.

We conjecture that such a matrix does not exist.

It is proved that every p.s.d. symmetric doubly stochastic matrix is J_n-admissable [4], from which it follows that the permanent function is monotone increasing on the straight line sequment from J_n to any p.s.d. symmetric doubly stochastic matrix (Hwang [7]).

PROBLEM 4.1. Determine whether every p.s.d. symmetric matrix in K_n is J_n-admissable.

If every p.s.d. symmetric matrix in K_n is J_n-admissable, then it follows from Theorem 3.2 that the Dittert's function is monotone decreasing on the straight line segment from J_n to any p.s.d. symmetric matrix in K_n. We conjecture that the Problem 4.1 will have an affirmative answer.

PROBLEM 4.3. Is every ϕ-maximizing matrix A on K_n $\hat{A}$-admissable or J_n-admissable?

If Problem 4.3 has an a affirmative answer, it would prove the Dittert's conjecture as we stated earlier.

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