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SOME PROPERTIES OF THE FUNCTIONAL EQUATION

$$\phi(x) = f(x) + \int_0^{\lambda x} g(x,y,\phi(y))\,dy\ .$$

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ABSTRACT. A discussion is given of some of the properties of the functional Volterra Integral equation

$$\phi(x) = f(x) + \int_0^{\lambda x} g(x,y,\phi(y))\,dy\ .$$

and of the corresponding multidimensional equation. Sufficient conditions are given for the uniqueness of the solution, and an iterational process is provided for the construction of the solution, together with error estimates. In addition bounds are provided on the solution. The results obtained are illustrated by means of the pantograph equation.

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1. INTRODUCTION.

A certain amount of attention has been paid to functional differential equations [1] but it appears that very little attention has been paid to functional Volterra Equations such as

$$\phi(x) = f(x) + \int_0^{\lambda x} g(x,y,\phi(y))\,dy\ . \qquad 0 < \lambda\ ,\ 0 < x \tag{1.1}$$

The first remark which may be made is that if $\lambda > 1$, there may not be uniqueness. For consider the equation

$$\phi(x) = 1 + \frac{a}{\lambda}\int_0^{\lambda x} \phi(y)\,dy \qquad \lambda > 1\ . \tag{1.2a}$$

Differentiation shows that this integral equation is equivalent to the differential equation problem

$$\phi'(x) = a\,\phi(\lambda x) \tag{1.2b}$$

with

$$\phi(0) = 1 \tag{1.2c}$$

and it is well known that the problem defined by (1.2b) and (1.2c) does not have a unique solution [2].

In this paper, only values of λ such that $0 < \lambda \le 1$ will be considered. Sufficient conditions will be obtained for the solution of the integral equation (1.1) to be unique, a number of bounds will be found for the solutions of the equation and an iterational process, with error analysis, will be given for the construction of the solution. These results will be valid provided $f(x)$ and $g(x,y,z)$ obey certain conditions as follows. The relevant conditions will be mentioned before each piece of analysis.

A) $$|g(x,y,z_1) - g(x,y,z_2)| \le P(x)\, Q(y)\, |z_1 - z_2| \qquad P(x)\,,\ Q(y) > 0 \tag{1.3}$$

In part of the analysis

$$P(x) = px^{\alpha}\ ,\ Q(x) = y^{\beta}\ ,\ p > 0\ ,\ \alpha \ge 0\ ,\ \beta \ge 0\ . \tag{1.4}$$

(In some places an alternative condition on α , β , $\alpha + \beta + 1 > 0$ is used.)

The equation (1.1) may be rewritten as

$$\phi(x) = f(x) + \int_0^{\lambda x} g(x,y,0)\, dy + \int_0^{\lambda z} [g(x,y,\phi(y)) - g(x,y,0)]\, dy \tag{1.5a}$$

$$= f^*(x) + \int_0^{\lambda x} g^*(x,y,\phi(y))\, dy\ . \tag{1.5b}$$

Clearly it follows from the inequality (1.3) that

B) $$|g^*(x,y,\phi(y))| \le P(x)\, Q(y)\, |\phi(y)|\ . \tag{1.5c}$$

C) $$|k(x)| \le M\lambda^{\gamma}x^{\delta} \qquad \gamma \ge 0\ ,\ \delta \ge 0\ ,\ M \ge 0 \tag{1.6a}$$

where

$$k(x) = \int_0^{\lambda x} g(x,y,f(y))\, dy\ . \tag{1.6b}$$

If $\lambda = 1$, the theory is that of the usual Volterra equation which is well known.

2. UNIQUENESS

It may be shown that, if condition A holds there is uniqueness. For suppose that $\phi^A(x)$, $\phi^B(x)$ are two possible solutions

$$\left|\phi^A(x) - \phi^B(x)\right| = \left| \int_0^{\lambda x} \left\{g(x,y,\phi^A(y) - g(x,y,\phi^B(y)\right\} dy \right| \tag{2.1}$$

$$\le \int_0^{x} \left|g(x,y,\phi^A(y)) - g(x,y,\phi^B(y))\right| dy$$

$$\le \int_0^{x} P(x)\, Q(y) \left|\phi^A(y) - \phi^B(y)\right| dy\ . \tag{2.2}$$

By using the processes similar to those involved in the proof of Gronwall's inequality [3], it follows that $\left|\phi_A(x) - \phi_B(x)\right|$ vanishes and there is thus uniqueness.

3. BOUNDS ON THE SOLUTION

It is possible, using Gronwall's inequaliy to obtain functions which bound the solution of equation (1.1).

a) Suppose that condition A holds.

The equation (1.1) can be rewritten as

$$\phi(x) - f(x) = \int_0^{\lambda x} g(x,y,f(y))\,dy + \int_0^{\lambda x} \{g(x,y,\phi(y)) - g(x,y,f(y))\}\,dy$$

$$= k(x) + \int_0^{\lambda x} \{g(x,y,\phi(y)) - g(x,y,f(y))\}\,dy\ . \qquad (3.1)$$

It follows that

$$|\phi(x) - f(x)| \le |k(x)| + \int_0^{\lambda x} |g(x,y,\phi(y)) - g(x,y,f(y))|\,dy$$

and using condition A it follows that

$$|\phi(x) - f(x)| \le |k(x)| + \int_0^{\lambda x} P(x)\,Q(y)\,|\phi(y) - f(y)|\,dy\ . \qquad (3.2)$$

Let

$$|\phi(x) - f(x)| = P(x)\,\psi(x) \qquad (3.3a)$$

$$|k(x)| = P(x)\,h(x)\ . \qquad (3.3b)$$

Using the relations (3.3a), (3.3b) and (2.3b), the inequality (3.2) may be rewritten as

$$\psi(x) \le h(x) + \int_0^{\lambda x} U(y)\,\psi(y)\,dy \qquad (3.4a)$$

whence

$$\psi(x) \le h(x) + \int_0^{x} U(y)\,\psi(y)\,dy\ . \qquad (3.4b)$$

The inequality (3.4b) is in a form suitable for the application of Gronwall's inequality. Multiplying by $U(x)$, and writing the result as a first order differential relation for

$$\Phi(x) = \int_0^{x} U(y)\,\psi(y)\,dy \qquad (3.4c)$$

it follows that

$$\int_0^{x} U(y)\,\psi(y)\,dy \le \int_0^{x} h(y)\,Y(y)\left[\exp\left\{\int_y^{x} U(z)\,dz\right\}\right]dy\ . \qquad (3.5)$$

It follows from the inequalities (3.4a) and (3.5) that

$$\psi(x) \le h(x) + \int_0^{\lambda x} h(y)\,U(y)\left[\exp\left\{\int_y^{\lambda x} U(z)\,dz\right\}\right]dy \qquad (3.6)$$

whence there follows the bounding inequality

$$|\phi(x) - f(x)| \le |k(x)| + P(x)\int_0^{\lambda x} |k(y)|\,Q(y)\left[\exp\left\{\int_y^{\lambda x} P(z)\,Q(z)\,dz\right\}\right]dy. \qquad (3.7a)$$

In the special case $P(x) = p$, $Q(y) = 1$, and $|k(x)|$ is differentiable the right hand side of (3.7a) becomes

$$|k(x)| - |k(\lambda x)| + |k(0)|e^{p\lambda x} + \int_0^{\lambda x} \frac{d}{dy} |k(y)|e^{p(\lambda x-y)}\, dy\ . \qquad (3.7b)$$

b) Suppose that condition B holds. Then, using the relations (1.5b) and (1.5c), it follows that

$$|\phi(x)| \le |f^*(x)| + \int_0^{\lambda x} P(x)\ Q(y)\ |\phi(y)|\ dy\ . \qquad (3.8)$$

It will be noted that if $f^*(x)$ is zero the inequality (3.8) implies that $\phi(x)$ vanishes.

By exactly the same process as the inequality (3.7a) was deduced from the inequality (3.2), it follows that

$$|\phi(x)| \le |f^*(x)| + P(x) \int_0^{\lambda x} |f^*(y)|\ Q(y) \left[\exp\left\{ \int_y^{\lambda x} P(z)\ Q(z)\ dz \right\}\right] dy \qquad (3.9a)$$

and in the special case mentioned above, the right hand side of this becomes

$$|f^*(x)| - |f^*(\lambda x)| + |f^*(0)|e^{p\lambda x} + \int_0^{\lambda x} \frac{d}{dy} |f^*(y)|e^{p(\lambda x-y)}\, dy\ . \qquad (3.9b)$$

A simplification occurs if $P(x) = 1$, when

$$|\phi(x) \le |f^*(x)| + \int_0^{\lambda x} |f^*(y)|\ Q(y) \left[\exp\left\{ \int_y^{\lambda x} Q(z)\ dz \right\}\right] dy \qquad (3.10a)$$

which can be rewritten as

$$|\phi(x)| \le |f^*(x)| - |f^*(\lambda x)| + |f^*(0)| \exp\left\{ \int_0^{\lambda x} Q(z)\ dz \right\}$$

$$+ \int_0^{\lambda x} \left[\frac{d}{dy} |f^*(y) \exp\left\{ \int_y^{\lambda x} Q(z)\ dz \right\}\right] dy \qquad (3.10b)$$

if $|f^*(x)|$ is differentiable.

4. CONSTRUCTION OF SOLUTIONS BY AN ITERATIVE PROCESS

a) It will now be shown that, if conditions A , B^* and C are satisfied, the sequence of functions $\phi_n(x)$ defined by

$$\phi_{n+1}(x) = f(x) + \int_0^{\lambda x} g(x,y,\phi_n(y))\ dy \qquad n \ge 0 \qquad (4.1a)$$

$$\phi_0(x) = f(x) \qquad (4.1b)$$

converges to the solution $\phi(x)$ of equation (1.1). It is also possible to use this sequence to obtain a further bound for $|\phi(x)|$. (In fact it is possible to start with an arbitrary $\phi_0(x)$ in which case there wil be slightly different results.)

Let

$$\chi_n(x) = |\phi(x) - \phi_n(x)|\ . \qquad (4.2)$$

$\chi_n(x)$ may be regarded as the error involved in stopping at the nth iteration.

Then

$$\chi_{n+1}(x) = \left| \int_0^{\lambda x} \{g(x,y,\phi(y)) - g(x,y,\phi_n(y))\}\ dy \right|\ . \qquad (4.3a)$$

$$\leq \int_0^{\lambda x} \left| g(x,y,\phi(y)) - g(x,y,\phi_n(y)) \right| dy .$$

$$\leq \int_0^{\lambda x} px^\alpha y^\beta \left| \phi(y) - \phi_n(y) \right| dy .$$

$$= \int_0^{\lambda x} px^\alpha y^\beta \chi_n(y) \, dy . \tag{4.3b}$$

Now

$$\chi_0(x) = \left| \phi(x) - \phi_0(x) \right| = \left| \phi(x) - f(x) \right| . \tag{4.3c}$$

Suppose that the range of interest is $0 \leq x \leq c$. Then it follows that, using the inequality (3.7a),

$$\chi_0(x) \leq k_{max} + P_{max} \int_0^{\lambda c} |k(y)| \, Q(y) \left[\exp\left\{ \int_y^{\lambda c} P(z)\, Q(z)\, dz \right\} \right] dy \tag{4.4a}$$

where k_{max} , P_{max} are respectively the maximum values of $|k(x)|$ and $P(x)$ over $0 \leq x \leq c$.

$$= C(c) \quad \text{say} \tag{4.4b}$$

Look for a solution of the recurrence inequality (4.3b) of the form

$$\chi_n(x) \leq C_n \lambda^{s_n} x^{t_n} . \tag{4.5}$$

Substitution of the inequality (4.5) in the inequality (4.3b) gives

$$\chi_{n+1}(x) \leq \int_0^{\lambda x} px^\alpha y^\beta C_n \lambda^{s_n} y^{t_n} dy . \tag{4.6a}$$

$$= pC_n \lambda^{s_n} x^\alpha \int_0^{\lambda x} y^{t_n+\beta} dy$$

$$= pC_n \lambda^{s_n} x^\alpha \frac{(\lambda x)^{t_n+\beta+1}}{t_n + \beta + 1} . \tag{4.6b}$$

The form (4.5) will be preserved if

$$C_{n+1} = \frac{pC_n}{t_n + \beta + 1} \tag{4.7a}$$

$$t_{n+1} = t_n + \alpha + \beta + 1 \tag{4.7b}$$

$$s_{n+1} = s_n + t_n + \beta + 1 . \tag{4.7c}$$

Comparison with the relation (4.4b) gives

$$C_0 = C \tag{4.8a}$$

$$t_0 = 0 \tag{4.8b}$$

$$s_0 = 0 \tag{4.8c}$$

Solution of the recurrence relations (4.7) with the initial conditions (4.8) gives the following results

$$C_n = \frac{p^n C}{\prod_{\ell=0}^{n-1} \left(t_\ell + \beta + 1 \right)} \tag{4.9a}$$

$$t_n = n(\alpha + \beta + 1) \tag{4.9b}$$

$$s_n = \frac{n(n-1)}{2}(\alpha + \beta + 1) + n(\beta + 1) \tag{4.9c}$$

and thus the error in stopping at the nth approximation has been defined by (4.5) and (4.9). These results hold for $x \le c$. In particular they hold for $x = c$ and so it follows that

$$\chi_n(c) \le \frac{p^n \lambda^{s_n} c^{t_n} C(c)}{\prod_{\ell=0}^{n-1}\left(t_\ell + \beta + 1\right)} \tag{4.9d}$$

c however is arbitrary and can be replaced by x and so

$$\chi_n(x) \le \frac{p^n \lambda^{s_n} x^{t_n} C(x)}{\prod_{\ell=0}^{n-1}\left(t_\ell + \beta + 1\right)} \tag{4.9e}$$

where $C(x)$ is defined by the relations (4.4). It can easily be verified that if $\lambda \le 1$ and $\alpha + \beta + 1 > 0$, the sequence $\chi_n(x)$ converges to zero.

The sequence functions can also be used to determine a bound for $|\phi(x)|$. Consider the sequence

$$\phi_{n+1}(x) = \phi_1(x) + \sum_{m=1}^{n} \psi_m(x) \tag{4.10a}$$

where

$$\psi_m(x) = \phi_{m+1}(x) - \phi_m(x)(4.10b) \tag{4.10b}$$

and

$$\phi_1(x) = f(x) + \int_0^{\lambda x} g(x,y,f(y))\, dy \tag{4.10c}$$

$$= f(x) + k(x) . \tag{4.10d}$$

Then

$$\left|\phi_{n+1}(x)\right| \le \left|\phi_1(x)\right| + \sum_{m=1}^{n} \left|\psi_m(x)\right| . \tag{4.10e}$$

If the series

$$\sum_{m=1}^{n} \left|\psi_m(x)\right|$$

converges, it dominates the series

$$\sum_{m=1}^{n} \psi_m(x)$$

which must also converge. Therefore the sequence $\phi_{n+1}(x)$ converges and it follows from the relation (4.1a) that the limit is the solution $\phi(x)$ of equation (1.1). It follows from the inequality (4.10d) that there is a further bound to $|\phi(x)|$ namely

$$\left|\phi_1(x)\right| + \sum_{m=1}^{\infty}\left|\psi_m(x)\right| . \tag{4.11}$$

Following the analysis of equations (4.3) it can easily be shown that

$$\left|\psi_{n+1}(x)\right| \le \int_0^{\lambda x} p x^{\alpha} y^{\beta} \left|\psi_n(y)\right| dy \tag{4.12a}$$

$$\psi_0(x) = \phi_1(x) - \phi_0(x) = \phi_1(x) - f(x)$$

$$= \int_0^{\lambda x} g(x, y, f(y))\, dy = k(x) \tag{4.12b}$$

and so

$$\left|\psi_0(x)\right| \le M \lambda^{\gamma} x^{\delta} . \tag{4.12c}$$

In exactly the same way as previously consider the inequality sequence

$$\left|\psi_n(x)\right| \le M_n \lambda^{u_n} x^{v_n} . \tag{4.13}$$

Using the relation (4.12a) in the same way as the relation (4.6a) was used

$$M_{n+1} = \frac{p M_n}{v_n + \beta + 1} \tag{4.14a}$$

$$v_{n+1} = v_n + \alpha + \beta + 1 \tag{4.14b}$$

$$u_{n+1} = u_n + v_n + \beta + 1 . \tag{4.14c}$$

The initial relations are now

$$M_0 = M \tag{4.15a}$$

$$v_0 = \delta \tag{4.15b}$$

$$u_0 = \gamma \tag{4.15c}$$

Thus

$$M_n = \frac{p^n M}{\prod_{\ell=0}^{n-1} \left[v_\ell + \beta + 1\right]} \tag{4.16a}$$

$$v_n = n(\alpha + \beta + 1) + \delta \tag{4.16b}$$

$$u_n = \frac{n(n-1)}{2} (\alpha + \beta + 1) + n(\delta + \beta + 1) + \gamma . \tag{4.16c}$$

It is not now difficult to see that the series

$$\sum_{m=1}^{\infty} \left|\psi_m(x)\right| \quad \text{converges for} \quad \lambda \le 1 .$$

It is possible to use the results (4.15) to obtain a simpler bound by using further inequalities.

$$M_{n+1} = \frac{p^{n+1} M}{\prod_{\ell=0}^{n} \left[v_\ell + \beta + 1\right]} \le \frac{p^{n+1} M}{(\beta + 1) \prod_{\ell=1}^{n} \{\ell(\alpha + \beta + 1) + \beta + 1\}}$$

$$\le \frac{p^{n+1} M}{(\beta + 1)(\alpha + \beta + 1)^n} \frac{1}{n!} \tag{4.17a}$$

$$u_{n+1} = \frac{(n+1)(n)}{2} (\alpha + \beta + 1) + (n + 1)(\beta + \delta + 1) + \gamma$$

$$\ge n(\alpha + \beta + 1) + (n + 1)(\beta + \delta + 1) + \gamma$$

$$= n(\alpha + 2\beta + \delta + 2) + (\beta + \gamma + \delta + 1) \tag{4.17b}$$

$$v_{n+1} = n(\alpha + \beta + 1) + (\alpha + \beta + \delta + 1) . \tag{4.17c}$$

Thus

$$\left|\psi_{m+1}(x)\right| \le \frac{p\ M}{\beta + 1} \lambda^{\beta+\gamma+\delta+1} x^{\alpha+\beta+\delta+1} g_n(x) \tag{4.18a}$$

where

$$g_n(x) = \frac{\{p\lambda^{\alpha+2\beta+\delta+2} x^{\alpha+\beta+1} \big/ (\alpha + \beta + 1)\}^n}{n!} . \tag{4.18b}$$

Thus

$$\sum_{m=1}^{\infty} \left|\psi_m(x)\right| \le \frac{p\ M}{\beta + 1} \lambda^{\beta+\gamma+\delta+1} x^{\alpha+\beta+\delta+1} \tag{4.19}$$

$$\exp \{p\lambda^{\alpha+2\beta+\delta+2} x^{\alpha+\beta+1} / (\alpha+\beta+1)\}$$

and from this a bound can be obtained using the expression (4.10d). It may be noted that, even if $f(x)$ is zero and there is no free term in the integral equation (1.1) it is possible if the equation is non-linear to rewrite the equation in the form (3.1) and proceed as indicated. If a relation of form A holds however, the only solution possible, when the free term is zero, is the zero solution.

5. THE SPECIAL CASE OF A LINEAR EQUATION

In this case

$$g(x,y,\phi(y)) = K(x,y)\ \phi(y) \tag{5.1}$$

and equation (1.1) is of the form

$$\phi(x) = f(x) + \int_0^{\lambda x} K(x,y)\ \phi(y)\ dy . \tag{5.2a}$$

The alternative form corresponding to equation (3.1) is

$$\phi(x) - f(x) = k(x) + \int_0^{\lambda x} K(x,y)\ \{\phi(y) - f(y)\}\ dy \tag{5.2b}$$

$$= k(x) + G(x) \quad \text{say} \tag{5.2c}$$

where

$$k(x) = \int_0^{\lambda x} K(x,y)\ f(y)\ dy . \tag{5.2d}$$

If $\lambda \le 1$, and condition A holds, that is

$$\left|K(x,y)\right| \le P(x)\ Q(y) , \tag{5.2f}$$

the solution will be unique, and evidently if $f(x)$ is zero, the solution will be the zero solution. Furthermore the whole of the theory of section 3 and section 4 remains valid.

The relation (3.7a), which is a bounding inequality for $|\phi(x) - f(x)|$ still holds, and, by analogy, it follows from equation (5.2a), on taking moduli and applying the Gronwall inequality that

$$|\phi(x)| \le |f(x)| + P(x) \int_0^{\lambda x} |f(y)|\ Q(y) \left[\exp \left\{\int_y^{\lambda x} P(z)\ Q(z)\ dz\right]\right.\ dy \tag{5.3}$$

The iterative solution is given by

$$\phi_{n+1}(x) = f(x) + \int_0^{\lambda x} K(x,y)\,\phi_n(y)\,dy \qquad n \ge 0 \tag{5.4a}$$

$$\phi_0(x) = f(x) \tag{5.4b}$$

amd it can be shown that

$$\phi_n(x)$$

is of the form

$$\phi_n(x) = f(x) + \sum_{m=1}^{n} \int_0^{\lambda_m x} K_m(x,y)\,f(y)\,dy\ . \tag{5.4c}$$

The values of λ_m and K_m may be obtained by substituting the relation (5.4c) into the Right Hand Side of equation (5.4a).

Let

$$\psi(x) = f(x) + \int_0^{\lambda x} K(x,z)\left[f(z) + \sum_{m=1}^{n} \int_0^{\lambda_m z} K_m(z,y)\,f(y)\,dy\right] dz \tag{5.5a}$$

$$= f(x) + \int_0^{\lambda x} K(x,y)\,f(y)\,dy + \sum_{m=1}^{n} \int_0^{\lambda\lambda_m x} f(y) \left[\int_{y\lambda_m^{-1}}^{\lambda x} K(x,z)\,K_m(z,y)\,dz\right] dy\ . \tag{5.5b}$$

The reversal of the order of integration is in fact valid for fairly wide conditions on K and f. It can be seen that the Right Hand Side of the equation (5.5b) can be rewritten as

$$f(x) + \sum_{m=1}^{n+1} \int_0^{\lambda_m x} K_m(x,y)\,f(y)\,dy \tag{5.5c}$$

provided that

$$\lambda_m = \lambda^m \tag{5.6a}$$

and the K_m are defined by the iterative sequence

$$K_{m+1}(x,y) = \int_{y\lambda_m^{-1}}^{\lambda x} K(x,z)\,K_m(z,y)\,dz \qquad m \ge 1 \tag{5.6b}$$

and

$$K_1(x,y) = K(x,y) \tag{5.6c}$$

and thus the solution of equation (5.1), if the appropriate conditions hold is

$$f(x) + \sum_{m=1}^{\infty} \int_0^{\lambda_m x} K_m(x,y)\,f(y)\,dy\ . \tag{5.6d}$$

The error in stopping at the nth term is given again by the relations (4.5) and (4.9).

Consider now how the theory of sections 3 and 4 can be applied when $K(x,y)$ and $f(x)$ obey the fairly loose condition of boundedness.

$$|K(x,y)| \le p \tag{5.7a}$$

$$|f(x)| \le \ell \ . \tag{5.7b}$$

It follows immediately that

$$|k(x)| = \left| \int_0^{\lambda x} K(x,y)\ f(y)\ dy \right| \le p\ell\lambda x \tag{5.7c}$$

and the C defined in (4.4b) is given by

$$C \le p\ell\lambda c \left[1 + p \int_0^{\lambda x} e^{p(\lambda c-y)}\ dy \right] = p\ell\lambda c\ e^{\lambda pc}\ . \tag{5.7d}$$

$G(x)$ as defined by equation (5.2c) now obeys the inequality

$$|G(x)| \le p \int_0^{\lambda x} \ell\ e^{p(\lambda x-y)}\ dy = \ell\left(e^{p\lambda x} - 1\right)\ . \tag{5.7e}$$

Thus, the bound given by equation (5.2b) becomes

$$|\phi(x) - f(x)| \le p\ell\lambda x + \ell\left(e^{p\lambda x} - 1\right) \tag{5.8a}$$

and the bound given by the inequality (5.3) becomes

$$|\phi(x)| \le |f(x)| + \ell\left(e^{p\lambda x} - \right)\ . \tag{5.8b}$$

The following results follow for the various quantities defined in equations (1.4) and (1.6)

$$\alpha = 0 \tag{5.9a}$$

$$\beta = 0 \tag{5.9b}$$

$$\gamma = 1 \tag{5.9c}$$

$$\delta = 1 \tag{5.9d}$$

$$M = p\ell\ . \tag{5.9e}$$

The relevant results from equations (4.9) become

$$C_n \le p^n \frac{Cp\ell\lambda e^{\lambda pc}}{\prod_{\ell=0}^{n-1} \left(t_\ell + 1\right)} \tag{5.10a}$$

$$t_n = n \tag{5.10b}$$

$$s_n = \frac{n(n-1)}{2} + n = \frac{n^2 + n}{2} \tag{5.10c}$$

giving

$$C_n \le \frac{p^{n+1}\ C\ell\lambda e^{\lambda pc}}{n!} \tag{5.10d}$$

and the error involved in stopping at the nth member of the sequence in the iterative solution is given by

$$\chi_n(x) \le \frac{p\ell\lambda e^{\lambda pc}}{n!}\ \lambda^{\frac{1}{2}\left(n^2+n\right)}\ C(px)^n\ . \tag{5.10e}$$

The bound given by (4.10) can also be obtained. Equation (4.10c) gives

$$\left|\phi_1(x)\right| = \left|f(x) + \int_0^{\lambda x} g(x,y,f(y))\ dy\right|$$

$$= \left|f(x) + \int_0^{\lambda x} K(x,y)\ f(y)\ dy\right|$$

$$\leq \ell\ (1 + p\lambda x) \tag{5.11a}$$

and equation (4.19) gives

$$\sum_{m=1}^{\infty} \left|\psi_m(x)\right| \leq p^2 \ell \lambda^3 x^2\ e^{p\lambda^3 x} \tag{5.11b}$$

giving a further bound

$$\left|\phi(x)\right| \leq p + p\ell\lambda x + p^2\ell\lambda^3 x^2\ e^{p\lambda^3 x}\ . \tag{5.11c}$$

6. MULTIDIMENSIONAL EQUATION

The theory outlined above may easily be extended to multidimensional systems of equations.

Consider the n dimensional system of equations

$$\underline{\phi}(x) = \underline{f}(x) + \int_0^{\lambda x} g(x,y,\underline{\phi}(y))\ dy \tag{6.1}$$

were

$$\underline{\phi}(x) = (\phi_1, \ldots, \phi_n)$$

g satisfies the conditions

$$\|\underline{g}(x,y,\underline{z}_1) - \underline{g}(x,y,\underline{z}_2)\| \leq P(x)\ Q(y)\|\underline{z}_1 - \underline{z}_2\| \tag{6.2a}$$

$$\|\underline{g}(x,y,\underline{z})\| \leq R(x)\ S(y)\|\underline{z}\| \tag{6.2b}$$

and

$$\|\underline{k}(x)\| \leq M\lambda^{\gamma} x^{\delta} \tag{6.2c}$$

where

$$\underline{k}(x) = \int_0^{\lambda x} \underline{g}(x,y,\underline{f}(y))\ dy\ . \tag{6.2d}$$

$\|\ \|$ denotes an appropriate norm.

In exactly the same way as previously it follows that

$$\|\underline{\phi}(x) - \underline{f}(x)\| \leq \|\underline{k}(x)\| + P(x) \int_0^{\lambda x} \|\underline{k}(y)\|\ Q(y) \left[\exp\left\{\int_y^{\lambda x} P(z)\ Q(z)\ dz\right\}\right] dy \tag{6.3}$$

which corresponds to (3.7a) and

$$\|\underline{\phi}(x)\| \leq \|\underline{f}(x)\| + R(x) \int_0^{\lambda x} \|\underline{f}(y)\|\ S(y) \left[\exp\left\{\int_y^{\lambda x} R(z)\ S(z)\ dz\right\}\right] dy \tag{6.4}$$

which corresponds to (3.9b).

An iterative sequence function vector may be generated

$$\underline{\phi}_{n+1}(x) = \underline{f}(x) + \int_0^{\lambda x} \underline{g}(x,y,\underline{\phi}_n(y)\ dy \qquad n \geq 0 \tag{6.5a}$$

$$\underline{\phi}_0(x) = \underline{f}(x) \ . \tag{6.5b}$$

The error in stopping at the nth iteration

$$\chi_n(x) \quad \text{is defined as} \quad \|\underline{\phi}(x) - \underline{\phi}_n(x)\|$$

and identical formulae to those of (4.4) to (4.9) are obtained, save that $\|\underline{k}(x)\|$ replaces $|k(x)|$.

Similarly

$$\|\underline{\phi}_{n+1}(x)\| \le \|\underline{\phi}_1(x)\| + \sum_{m=1}^{n} \|\underline{\psi}_m(x)\| \tag{6.6a}$$

where

$$\underline{\psi}_m(x) = \underline{\phi}_{m+1}(x) - \underline{\phi}_m(x) \tag{6.6b}$$

giving

$$\|\underline{\phi}(x)\| \le \|\underline{f}(x) + \underline{k}(x)\| + \sum_{m=1}^{\infty} \|\underline{\psi}_m(x)\| \ , \tag{6.6c}$$

a bound for the infinite sum being given by the expression (4.19), when $P(x) = p$.

For a linear system, the set of equations assume the form

$$\underline{\phi}(x) = \underline{f}(x) + \int_0^{\lambda x} K(x,y)\ \underline{\phi}(y)\ dy \tag{6.7}$$

where $K(x,y)$ is a matrix.

One bound is given by

$$\|\underline{\phi}(x) - \underline{f}(x)\| \le \|\underline{k}(x)\| + G(x) \tag{6.8a}$$

where

$$\underline{k}(x) = \int_0^{\lambda x} K(x,y)\ \underline{f}(y)\ dy \tag{6.8b}$$

and $G(x)$ is defined by (5.2c), $f(y)$ being replaced by $\|\underline{f}(y)\|$ and another bound is given by

$$\|\underline{\phi}(x)\| \le \|\underline{f}(x)\| + G(x) \ . \tag{6.8c}$$

The solution sequence becomes

$$\underline{\phi}_{n+1}(x) = \underline{f}(x) + \int_0^{\lambda x} K(x,y)\ \underline{\phi}_n(y)\ dy \ , \qquad n \ge 0 \tag{6.9a}$$

$$\underline{\phi}_0(x) = \underline{f}(x) \tag{6.9b}$$

and if

$$\|K(x,y)\| \le p \tag{6.10a}$$

$$\|\underline{f}(x)\| \le \ell \ , \tag{6.10b}$$

the results of (5.7) hold, giving

$$\|\underline{\phi}(\underline{x}) - \underline{f}(x)\| \le p\ell\lambda x + \ell\{e^{p\lambda x} - 1\} \tag{6.11a}$$

$$\|\underline{\phi}(x)\| \le \|\underline{f}(x)\| + \ell\{e^{p\lambda x} - 1\} \tag{6.11b}$$

$$\|\underline{\phi}(x)\| \le p + p\ell\lambda x + p^2\ell\lambda^3 x^2 e^{p\lambda^3 x} \ . \tag{6.11c}$$

It can easily be shown that for small x (6.11b) gives the tighter bound for $\|\underline{\phi}(x)\|$, but for large x (6.11c) gives the tighter bound.

7. APPLICATION TO EXAMPLES

A) Consider now the generalised pantograph equation

$$\theta'(x) = a\,\theta(\lambda x) + b\,\theta(x)\ . \tag{7.1a}$$

This equation has been discussed extensively [2], [4], [5].
Let

$$\theta(x) = e^{bx}\,\phi(x)\ . \tag{7.1b}$$

Equation (7.1a) then assumes the form

$$\phi'(x) = a\exp[b(\lambda - 1)]\,\phi(\lambda x) \tag{7.1c}$$

and if $\theta(0) = \phi(0) = 1$, this assumes the integral equation form

$$\phi(x) = 1 + a\lambda^{-1}\int_0^{\lambda x} \exp\,(b^*y)\,\phi(y)\,dy \tag{7.1d}$$

where

$$b^* = b(\lambda - 1) \tag{7.1e}$$

$$b = c + i\omega\quad,\quad b^* = c^* + i\omega^*\ . \tag{7.1f}$$

The alternative form for (7.1d) becomes

$$\phi(x) - 1 = k(x) + a\lambda^{-1}\int_0^{\lambda x} \exp\,(b^*y)[\phi(y) - 1]\,dy \tag{7.1g}$$

where

$$k(x) = a\lambda^{-1}\int_0^{\lambda x} \exp\,(b^*y)\,dy\ . \tag{7.1h}$$

Taking moduli, equations (7.1d) , (7.1g) , (7.1h) take the respective forms

$$|\phi(x)| \le 1 + |a|\lambda^{-1}\int_0^{\lambda x} \exp\,(c^*y)\,|\phi(y)|\,dy \tag{7.2a}$$

$$|\phi(x) - 1| \le |k(x)| + |a|\lambda^{-1}\int_0^{\lambda x} \exp\,(c^*y)\,|\phi(y) - 1|\,dy \tag{7.2b}$$

$$|k(x)| \le |a|\lambda^{-1}\int_0^{\lambda x} \exp\,(c^*y)\,dy = |a|\,(\lambda c^*)^{-1}\,[\exp\,(c^*\lambda x - 1)] \tag{7.2c}$$

In the special case of c zero, it can easily be seen from the inequality (7.2c) that

$$|k(x)| \le |a|\,x \tag{7.2d}$$

and the inequality (7.2b) becomes

$$|\phi(x) - 1| \le |a|x + |a|\lambda^{-1}\int_0^{\lambda x} |\phi(y) - 1|\,dy \tag{7.2e}$$

and application of the Gronwald-Bellman-Reid inequality gives

$$\begin{aligned}|\phi(x) - 1| &\le |a|x + |a|\lambda^{-1}\int_0^{\lambda x} |a|y\,[\exp\,|a|\lambda^{-1}\,(\lambda - y)]\,dy\\ &= |a|x(1 - \lambda) + \int_0^{\lambda x} |a|\,\exp\,\{|a|\lambda^{-1}\,(\lambda x - y)\}\,dy\\ &= |a|x(1 - \lambda) + \lambda(e^{|a|x} - 1)\ .\end{aligned} \tag{7.2f}$$

Clearly, because of linearity, the results for $\phi(0)$ arbitrary can easily be obtained.

The bound given by (3.10a) becomes, on using the inequality (7.2a),

$$|\phi(x)| \le \exp\left[\int_0^{\lambda x} |a|\lambda^{-1} \exp(c^* z)\, dz\right] \tag{7.3a}$$

or in the special case of c zero

$$|\phi(x)| \le \exp[|a|x] . \tag{7.3b}$$

Using the fact that

$$|\theta(x)| = e^{cx} |\phi(x)| , \tag{7.3c}$$

a bound for $\theta(x)$ can easily be obtained.

The bound given by (3.7a) becomes, on using the inequalities (7.2b) and (7.2c)

$$|\phi(x) - 1| \le |a|(\lambda c^*)^{-1} [\exp(c^*\lambda x) - 1]$$
$$+ |a|\lambda^{-1} \int_0^{\lambda x} |a|(\lambda c^*)^{-1} [\exp(c^*\lambda y) - 1]\exp(c^* y)\exp\left[|a|\lambda^{-1} \int_y^{\lambda x} \exp(c^* z)dz\right]dy . \tag{7.4a}$$

In the special case of c zero, the formula (3.7b) may be used, giving

$$|\phi(x) - 1| = \lambda\left(e^{|a|x} - 1\right) . \tag{7.4b}$$

$$\le |a|x(1 - \lambda) + \int_0^{\lambda x} |a| \exp[|a|\lambda^{-1} (\lambda x - y)]\, dy$$

$$\le |a|x(1 - \lambda) + \lambda\{e^{|a|x} - 1\} . \tag{7.4c}$$

Slightly more complicated formulae will follow for the corresponding bound for $\theta(x)$.

B) Consider now the many dimensional generalised pantograph equation

$$\underline{\theta}'(x) = A\underline{\theta}(\lambda x) + B\underline{\theta}(x) \tag{7.5a}$$

where $\underline{\theta}$ is an n dimensional vector and A and B are constant complex square matrices of order n . An analytical discussion of this has been given in [6]

Bounds associated with this equation may be obtained by extensions of the methods discussed previously.

Suppose that, first of all a suitable linear (possibly complex) transformation has been made so that B is diagonal. If B is degenerate, all that happens is that some of the diagonal terms will be zero.

Then equation (7.5a) may be rewritten as

$$\theta_r'(x) = \sum_{s=1}^{n} a_{rs}\, \theta_s(\lambda x) + b_r\, \theta_r(x) , \qquad 1 \le r \le n \tag{7.5b}$$

with an obvious notation.

Let

$$\theta_r(x) = \exp(b_r x)\, \phi_r(x) . \tag{7.5c}$$

The set of equations (7.5b) then assumes the form

$$\exp(b_r x)\, \phi_r'(x) = \sum_{s=1}^{n} a_{rs} \exp(b_s \lambda x)\, \phi_s(\lambda x)$$

or

$$\phi_r'(x) = \sum_{s=1}^{n} a_{rs} \exp(\beta_{rs}\lambda x)\, \phi_s(\lambda x) \tag{7.5d}$$

where

$$\beta_{rs} = b_s - b_r/\lambda \, . \tag{7.5e}$$

If

$$\phi_r(0) = c_r \, , \tag{7.5f}$$

equation (7.5d) takes the form

$$\phi_r(x) = c_r + \sum_{s=1}^{n} a_{rs}\, \lambda^{-1} \int_0^{\lambda x} \exp(\beta_{rs}y)\, \phi_s(y)\, dy \, . \tag{7.5g}$$

The alternative form for equation (7.5g) is

$$\phi_r(x) - c_r = k_r(x) + \sum_{s=1}^{n} a_{rs}\, \lambda^{-1} \int_0^{\lambda x} \exp(\beta_{rs}y)\, \{\phi_s(y) - c_s\}\, dy \tag{7.5h}$$

where

$$k_r(x) = \sum_{s=1}^{n} a_{rs}\, \lambda^{-1} \int_0^{\lambda x} \exp(\beta_{rs}y)\, c_s\, dy \, . \tag{7.5i}$$

Let

$$\max_{r,s} \operatorname{Re} \beta_{rs} = \beta \, . \tag{7.6a}$$

Then equation (7.5g) becomes

$$|\phi_r(x)| \le |c_r| + \sum_{s=1}^{n} |a_{rs}|\, \lambda^{-1} \int_0^{\lambda x} \exp(\beta y)\, |\phi_s(y)|\, dy \, . \tag{7.6b}$$

Let

$$\max_{s} |a_{rs}| = a_r \, . \tag{7.6c}$$

Then the inequality (7.6b) becomes

$$|\phi_r(x)| \le |c_r| + a_r\, \lambda^{-1} \int_0^{\lambda x} \exp(\beta y) \sum_{s=1}^{n} |\phi_s(y)|\, dy \, . \tag{7.6d}$$

Let

$$\sum_{r=1}^{n} a_r = a \, , \quad \sum_{r=1}^{n} |c_r| = c \, , \quad \sum_{r=1}^{n} |\phi_r(x)| = \phi(x) \, . \tag{7.6e}$$

Then it follows from the inequality (7.6d) that

$$\phi(x) \le c + a\lambda^{-1} \int_0^{\lambda x} \exp(\beta y)\, \phi(y)\, dy \, . \tag{7.6f}$$

It follows immediately that in the same way as previously

$$\phi(x) \le c \exp\left[\int_0^{\lambda x} a\lambda^{-1} \exp(\beta z)\, dz\right] \, . \tag{7.6g}$$

A second bound follows from equation (7.5h).

Let

$$\phi_r(x) - c_r = \psi_r(x) \, . \tag{7.7a}$$

Then equation (7.5h) assumes the form

$$\psi_r(x) = k_r(x) + \sum_{s=1}^{n} a_{rs}\ \lambda^{-1} \int_0^{\lambda x} \exp\ (\beta_{rs} y)\ \psi_s(y)\ sy\ . \tag{7.7b}$$

In the same way as before

$$|\psi_r(x)| \le |k_r(x)| + a_r \lambda^{-1} \int_0^{\lambda x} \exp\ (\beta y) \sum_{s=1}^{n} |\psi_s(y)|\ dy\ . \tag{7.7c}$$

With an obvious notation, it follows that

$$\psi(x) \le k(x) + a\lambda^{-1} \int_0^{\lambda x} \exp\ (\beta y)\ \psi(y)\ dy\ . \tag{7.7d}$$

Thus, using the inequality (3.6), it follows that

$$\psi(x) \le k(x) + \int_0^{\lambda x} k(y)\ a\lambda^{-1} \exp\ (\beta y) \left[\exp\left\{\int_y^{\lambda x} a\lambda^{-1} \exp\ (\beta z)\ dz\right\}\right] dy \tag{7.7e}$$

and, if $k(x)$ is differentiable, the relation (7.7e) can be written as

$$\psi(x) \le \int_0^{\lambda x} k'(y) \left[\exp\left\{\int_0^{\lambda x} a\lambda^{-1} \exp\ (\beta z)\ dz\right\}\right] dy\ . \tag{7.7f}$$

Clearly, if $\beta = 0$, these formulae simplify.

Alternative bounds, based on the results of section 6 may also be obtained. Let

$$\underline{\theta}(x) = \exp\ \{Bx\}\ \underline{\phi}(x) \tag{7.8a}$$

where $\exp\ \{Bx\}$ is interpreted as

$$I + \sum_{s=1}^{\infty} \frac{B^s x^s}{s!} \tag{7.8b}$$

I being the unit matrix of order n .

It is not difficult to see that equation (7.5a) assumes the form

$$\underline{\phi}'(x) = \exp\ \{B(\lambda - 1)x\}\ \underline{\phi}\ (\lambda x)\ . \tag{7.8c}$$

This can be rewritten, using the initial condition as

$$\underline{\phi}(x) = \underline{\phi}(0) + \int_0^{\lambda x} A\lambda^{-1} \exp\ (B^* y)\ \underline{\phi}(y)\ dy \tag{7.8d}$$

where

$$B^* = B(1 - \lambda^{-1})\ . \tag{7.8e}$$

It follows from (6.8) that

$$\underline{k}(x) = \int_0^{\lambda x} A\lambda^{-1} \exp\ (B^* y)\ \underline{\phi}(0)\ dy \tag{7.8f}$$

$$= \int_0^{\lambda x} A\lambda^{-1} \sum_{s=0}^{\infty} \frac{B^{*s} y^s}{s!}\ \underline{\phi}(0)\ dy$$

$$= A\lambda^{-1} \sum_{s=0}^{\infty} \frac{B^{*s} (\lambda x)^{s+1}}{(s+1)!}\ \underline{\phi}(0)\ . \tag{7.8g}$$

Thus

$$\|k(x)\| \le \|A\| \lambda^{-1} \sum_{s=0}^{\infty} \frac{\|B^*\|^s (\lambda x)^{s+1}}{(s+1)!} = \|A\| \, \|\lambda B^*\|^{-1} \, \|\underline{\phi}\|(0)\| \, [\exp\{\|B^*\|\lambda x\} - 1 \; . \tag{7.8h}$$

If $x \le c$,

$$\|k(x)\| \le \int_0^{\lambda x} \|A\| \lambda^{-1} \exp \{\|B^*\|c\} \|\underline{\phi}(0)\| \, dy \; . \tag{7.8i}$$

Thus the constants for the inequality (6.2c) are given by

$$M = \|A\| \lambda^{-1} \exp \{\|B^*\|c\} \|\underline{\phi}(0)\| \; , \; \gamma = 1 \; , \; \delta = 1 \; . \tag{7.8j}$$

Comparing equation (7.8d) and equation (6.7) it can be seen that

$$\underline{f}(x) = \underline{\phi}(0) \tag{7.9a}$$

and

$$K(x,y) = A \lambda^{-1} \exp (B^* y) \; . \tag{7.9b}$$

Now

$$\|A \lambda^{-1} \exp (B^* y)\| \le \|A\| \lambda^{-1} \exp \{\|B^*\|y\} \tag{7.9c}$$

and as

$$y \le \lambda x \le \lambda c \tag{7.9d}$$

it follows that

$$\|K(x,y)\| \le \|A\| \lambda^{-1} \exp \{\|B^*\| \lambda c\} \; . \tag{7.9e}$$

Thus, the quantities p and ℓ defined in (6.10) are given by

$$p = \|A\| \lambda^{-1} \exp \{\|B^*\| \lambda c\} \tag{7.9f}$$

and

$$\ell = \|\underline{\phi}(0)\| \; . \tag{7.9g}$$

Now the relations (6.11) will hold for $x \le c$. In particular, they hold for $x = c$.

The inequality (6.11a) becomes

$$\|\underline{\phi}(c) - \underline{f}(c)\| \le p\ell\lambda c + \ell\left(e^{p\lambda c} - 1\right) \; . \tag{7.10a}$$

The inequality (6.11b) becomes

$$\|\underline{\phi}(c)\| \le \|\underline{f}(c)\| + \ell\left(e^{p\lambda c} - 1\right) \tag{7.10b}$$

and the inequality (6.11c) becomes

$$\|\underline{\phi}(c)\| \le p + p\ell\lambda c + p^2\ell\lambda^3 c^2 e^{p\lambda^3 c} \tag{7.10c}$$

where p is in fact a function of c defined by (7.9f). c however is arbitrary and so the inequalities (7.10) give bounds for all positive c .

8. DISCUSSION

Because of the way in which the bounds discussed in this paper have been derived, namely by means of a generalisation of Gronwall's inequality, they involve exponentials with positive coefficients, associated with an increasing divergence from the initial values of the dependent variable as the independent variable increases. Consequently, they would not be suitable, except near the initial value of the independent variable for discussing problems such as that defined by

$$\frac{dy}{dx} = - y(\lambda x) \quad , \quad y(0) = 1 \tag{8.1a}$$

which is equivalent to the integral equation

$$y(x) = 1 - \lambda^{-1} \int_0^{\lambda x} y(u)\, du \ . \tag{8.1b}$$

Generally, where solutions are asymptotically stable, and converge to some limit for large x, the bounds discussed here will become irrelevant for large enough x. This would be equally true of multidimensional equations for which the solutions are asymptotically stable.

If, however, the equations are such that solutions are unstable - as would, for example, be the case when all the elements of the A and B matrices of (7.5a) are positive - the bounds here will always be relevant. It may be noted that sometimes one bound is better, sometimes another. For example, it can be seen, without much difficulty that if c is near zero, the bound given by the inequality (7.10b) is tighter than that given by the inequality (7.10c) whereas if c is large, the reverse situation holds.

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