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FROM PATHS TO STARS

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ABSTRACT. The number of cycles in the complement T' of a tree T is known to increase with the diameter of the tree. A smimilar question is raised and settled for the number of complete subgraphs in T' for a special class of trees via Fibonacci numbers. A structural characterization of extremal trees is also presented.

KEY WORDS AND PHRASES. Cycles, Complete subgraphs, Trees.
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1. INTRODUCTION.

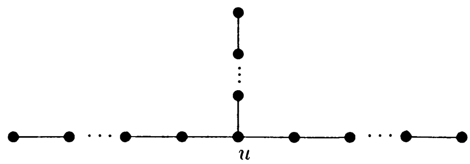
Among all trees T of order n, the number $c(T')$ of all cycles in the complement T' and the structural characterization of those trees which optimize $c(T')$ have been dealt with in [1,2]. The same problem was solved in [3] for the number $i(T')$ of all complete subgraphs in the complement of an arbitrary T. It turns out that among all n-trees T, the path $P_n (n \geq 9)$ has both the maximum number of cycles [1] and the minimum number of complete subgraphs in its complement [3]. The star S_n also maximizes both $c(T')$, $5 \leq n \leq 8$ [1] and $i(T')$ for $n \geq 4$ [2]. It minimizes $c(T')$ for $n \geq 9$.

The problem of characterizing the n-vertex trees T for extremal values with respect to $c(T')$ or $i(T')$ loses some structural significance in the generality of Y. Suppose we consider a class of trees which keep out all paths and stars, for example, the class $\mathcal{F}_3$ of all those trees T_3 having exactly three endvertices. What structural similarities between $c(T_3')$ and $i(T_3')$ are inherited from $c(T')$ and $i(T')$? We recall that the diameter of a graph G is the maximum distance $d(u, v)$ taken over all pairs of vertices u, v in G. The following theorem [1, p.93] relates $c(T')$ with the diameter of T.

THEOREM 1. For each $n \geq 6$ and every tree T of order n and diameter d, $4 \leq d \leq n-2$, there is a tree T_1 of order n and at least diameter $d+1$ such that $c(T') < c(T_1')$.

2. TREES WITH THREE ENDVERTICES.

Utilizing enumerative techniques [2] we conclude that among all trees T_3 of order n, the tree with the smallest number of cycles in its complement is a tree with the smallest diameter as shown in Figure 1. Moreover, the tree T_3



$$n = \begin{cases} 3k-1 \\ 3k \\ 3k+1 \end{cases} \Rightarrow d = \begin{cases} 2k-1 \\ 2k \\ 2k \end{cases}$$

Figure 1. A tree with three endvertices and minimum $c(T_3')$.

with the largest number of cycles in its complement is a tree with the largest diameter as shown in Figure 2.

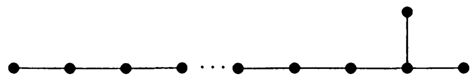


Figure 2. A tree with maximum $c(T_3')$.

So, among all trees with three endvertices, of order n, the direct relationship between $c(T_3')$ and the diameter of T_3 is inherited from the class of all trees, i.e., $\min_{\mathcal{F}_3} c(T_3')$ and $\max_{\mathcal{F}_3} c(T_3')$ are still associated with the smallest and the largest diameters of T_3, respectively. Can we make the same claim about $i(T_3')$, the number of complete subgraphs in the complement of T_3? We recall that when T is arbitrary, $i(T')$ is maximum when the diameter of T is minimum (T is a star) and $i(T')$ is minimum when the diameter is maximum (T is a path). Does this relationship between $i(T')$ and the diameter of T remain true when T is restricted to $\mathcal{F}_3$? To this end, we need the concept of a Fibonacci number $f(G)$ of a graph G.

According to [4, p. 45], the total number of subsets of $\{1,2,3,\ldots,n\}$ such that no two elements are adjacent is F_{n+1}, where F_n is the nth Fibonacci number, which is defined by

$$F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2.$$

The sequence $\{1,2,3,\ldots,n\}$ can be regarded as the vertex set of the path P_n. This definition covers the empty graph also; so, $f(G) = i(G')$. We note that $i(P_0') = 1, i(P_1') = 2, i(P_2'), \ldots, i(P_n') = F_{n+1}$.

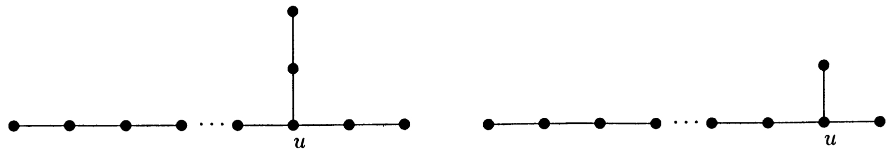
3. MAIN RESULTS

If T_3 is a tree with three endvertices, then it has a unique vertex u of degree three. We count $i(T_3')$ [5] by considering two disjoint sets of complete subgraphs of T_3', say S_1 and S_2, where S_1 is the set of those complete subgraphs not containing the vertex u, and S_2 consists of those that do contain u. Let v_1, v_2 and v_3 be the three vertices adjacent to u in T_3. We have

$$i(T_3') = |S_1| + |S_2| = i(T_3 - u)' + i(T_3 - v_1 - v_2 - v_3)'.$$

If $n = 3k+1, T_3 - u$ is a union of three disjoint paths on $3k$ vertices, where $T_3 - v_1 - v_2 - v_3$

is also a union of three disjoint paths on $3k-3$ vertices together with the isolated vertex u (see Figure 1). The following theorem on Fibonacci numbers shows that $i(T_3')$ is minimized and maximized by the trees in Figures 3a and 3b, respectively. This shows



(a) $\min i(T_3')$ (b) $\max i(T_3')$

Figure 3. Extremal trees in $\mathcal{F}_3$.

that the inverse relationship between $i(T')$ and the diameter of T is not inherited in the class of trees $\mathcal{F}_3$ with exactly three endvertices.

THEOREM 2. Let n be an integer ≥ 7. Then among all summands r_1, s_1, t_1 and r_2, s_2, t_2 satisfying (i) $r_1+s_1+t_1=n+2$, (ii) $r_1=r_2+1, s_1=s_2+1, t_1=t_2+1$, and $r_1, s_1, t_1 \geq 2$ we have

the sum of the two products $F_{r_1}F_{s_1}F_{t_1}+F_{r_2}F_{s_2}F_{t_2}$ of the Fibonacci numbers $F_{r_1}, F_{s_1}, F_{t_1}$ and $F_{r_2}, F_{s_2}, F_{t_2}$ is

(a) minimum if $r_1=s_1=3$ and $t_1=n-4$ and

(b) maximum if $r_1=s_1=2$ and $t_1=n-2$.

PROOF. The order of growth of F_n is governed by the golden ratio $\tau=(1+\sqrt{5})/2$. Moreover, $F_n \approx c\tau^n$ where $c=\tau/\sqrt{5}$ and $F_{n+1}\approx \tau F_n$. We have $r_1+s_1+t_1=n+2$ and $r_2+s_2+t_2=n-1$, and for large n,

$$\begin{aligned} F_{r_1}F_{s_1}F_{t_1} &+ F_{r_2}F_{s_2}F_{t_2} \approx (\tau F_{r_2})(\tau F_{s_2})(\tau F_{t_2}) + F_{r_2}F_{s_2}F_{t_2} = (1+\tau^3)F_{r_2}F_{s_2}F_{t_2} \\ &\approx (5.236068\cdots)(c^3\tau^{r_2+s_2+t_2}) \approx 13.59764677\cdots\tau^{n-5} > 13.43181071\cdots\tau^{n-5} \\ &\approx 9F_{n-4}+4F_{n-5} = F_3F_3F_{n-4}+F_2F_2F_{n-5} \\ &= i(P_2')i(P_2')i(P_{n-5}') + i(P_1')i(P_1')i(P_{n-6}'). \end{aligned}$$

That is, $\min_{\mathcal{F}_3} i(T_3')$ is realized in Figure 3a.

To prove (b), we note that for the tree in Figure 3b, we have $i(T_3')=4F_{n-2}+F_{n-3}\approx 12.09016992\cdots F_{n-4}$. On the other hand, for an arbitrary T_3 with n large enough, we have $F_{r_1}F_{s_1}F_{t_1}+F_{r_2}F_{s_2}F_{t_2}\approx 13.59764677\cdots\tau^{n-5}\approx 11.61377685\cdots F_{n-4} < 12.09016992\cdots F_{n-4}\approx 4F_{n-2}+F_{n-3}$. That is, $\max_{\mathcal{F}_3} i(T_3')$ is realized in Figure 3b.

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