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CHARACTERIZATION OF HANKEL TRANSFORMABLE GENERALIZED FUNCTIONS

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ABSTRACT. In this paper we prove a characterization theorem for the elements of the space H'_μ of generalized functions defined by A.H. Zemanian.

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1. INTRODUCTION.

The Hankel transformation defined by

$$h_\mu\{f(x)\}(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x)dx$$

where J_μ denotes the Bessel function of the first kind and order μ, has been extensively studied in recent years.

A classical result concerning the Hankel transformation is the following inversion theorem (see [1]).

THEOREM 1. Let $f(x) \in L_1(0,\infty)$ be of bounded variation in a neighborhood of the point $x = x_0$. If $\mu > -\frac{1}{2}$ and $F(y) = h_\mu\{f(x)\}(y)$, then

$$h_\mu^{-1}\{F(y)\}(x_0) = \int_0^\infty F(y)(x_0y)^{1/2} J_\mu(x_0y)dy = \frac{1}{2}\{f(x_0+0)+f(x_0-0)\}.$$

Another well known result is the Parseval's equation (1) (see [1]).

THEOREM 2. Let $f(x)$ and $G(y)$ be elements of $L_1(0,\infty)$. If $F(y)$ and $g(x)$ are respectively the direct and inverse μ-th order Hankel transforms of $f(x)$ and $G(y)$, then

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F(y)G(y)dy, \qquad \text{for any } \mu > -\frac{1}{2}. \tag{1.1}$$

Other conditions under which Parseval's equation holds are given by P. Macaulay-Owen [2].

The h_μ-transform has been extended to several spaces of generalized functions. Apparently, A.H. Zemanian [1] was the first to extend the Hankel transform. He introduced the space H_μ of testing functions consisting of all infinitely differentiable complex-valued functions ψ defined on $I=(0,\infty)$ and such that

$$\gamma^{\mu}_{m,n}(\psi) = \sup_{x\in I}\left|x^m(\tfrac{1}{x}D)^n(x^{-\mu-1/2}\,\psi(x))\right| < \infty$$

for every $m,n\in N$. The Hankel transform is an automorphism onto H_μ. For every $f \in H'_\mu$ (the dual space of H_μ), the generalized Hankel transformation $H'_\mu f$ of f was defined by the following generalization of Parseval's equation

$$\langle h'_\mu f,\psi\rangle = \langle f,h_\mu\psi\rangle \ , \quad \text{for every } \psi \in H_\mu .$$

h'_μ is an automorphism onto H'_μ.

Later, E.L. Koh and A.H. Zemanian [3] defined the generalized complex Hankel transformation. For a real number μ and a positive real number α the space $J_{\mu,\alpha}$ was defined as the space of testing functions ψ which are smooth on I and for which

$$\tau^{\mu,\alpha}_{k}(\psi) = \sup_{x\in I}\left| e^{-\alpha x}\, x^{-\mu-1/2}\, S^k_\mu\psi(x)\right|<\infty, \quad \text{for every } k \in N,$$

where $S_\mu = x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$. For each complex number y in the strip $\Omega=\{y \in C:|\mathrm{Im}\, y|<\alpha,\ y \notin (-\infty,0]\}$, $J_{\mu,\alpha}$ contains the function $(xy)^{1/2}J_\mu(xy)$. The h_μ-transform is now defined on the dual space $J'_{\mu,\alpha}$ as follows:

DEFINITION. Let μ be in the interval $-\frac{1}{2} < \mu < \infty$. Then, for every $f \in J'_{\mu,\alpha}$ and $y \in \Omega$,

$$(h'_\mu f)\ (y) = \langle f(x),\ (xy)^{1/2}J_\mu(xy)\rangle .$$

E.L. Koh [4] showed that a distribution $f \in J'_{\mu,\alpha}$ can be written as a finite sum of derivatives of continuous function of exponential descent. More specifically, he established:

THOEREM 3 ([4]). Let f be in $J'_{\mu,\alpha}$. Then f is equal to a finite sum

$$\sum_{i=0}^{k} C_i(\tfrac{d}{dx})^i\ (e^{-\alpha x}\ x^{-\mu-(1/2)-k+1}P_i(x)F_i(x))$$

where the $F_i(x)$ are continuous on $(0,\infty)$ and the $P_i(x)$ are polynomials of degree k.

Other Hankel type transformations have been also extended to certain spaces of generalized functions (see G. Altenburg [5], L.S. Dube and J.N. Pandey [6], J.M. Méndez [7],...).

In this paper we prove a characterization theorem for the generalized functions in H'_μ. Our proof is anaogous to the method employed in structure theorems for Schwartz distributions (see [8] and [4]).

In this paper a function $\psi(x)$ will be called of rapid descent if $x^m D^q \psi(x)$ tends to zero, as $x\to\infty$, for every $m,q\in N$.

2. The space H'_μ of generalized functions. A characterization theorem.

A useful result due to A.H. Zemanian (see [1]) is the following

PROPOSITION 1. Let f be in H'_μ.There exist a positive constant C and nonnegative integers r,k such that

$$|\langle f,\psi\rangle| \leq C \max\{\gamma^\mu_{m,n}(\psi);\ 0\leq m\leq r, 0\leq n\leq k\},\ \text{for every } \psi\in H_\mu.$$

We now present some new properties of the space H_μ of testing functions.

PROPOSITION 2. Let ψ be in H_μ. The function $x^m(\frac{1}{x}D)^n(x^{-\mu-1/2}\psi(x))$ is

a) of rapid descent as $x\to\infty$, and

b) in $L_1(0,\infty)$,

for every $m,n\in N$.

PROOF. It is enough to take into account that

$$|x^m(\tfrac{1}{x}D)^n(x^{-\mu-1/2}\psi(x))| \leq C_{m,n}x^{-2},\ \text{for every } x\in I \text{ and } m,$$

$n\in N$, $C_{m,n}$ being a suitable positive constant.

The main result of this paper is the next.

THEOREM 4. A functional f is in H'_μ if and only if, there exist bounded measurable functions $g_{m,n}(x)$ defined on I, for $m=0,1,\dots,r$ and $n=0,1,\dots,k$, where r and k are nonegative integers depending onf, such that

$$\langle f,\psi\rangle=\langle \sum_{m,n}^{r,k} x^{-\mu-1/2}(-D\tfrac{1}{x})^n\{x^m(-D)g_{m,n}(x)\},\psi(x)\rangle \tag{2.1}$$

for every $\psi\in H_\mu$.

PROOF. Let f be in H'_μ. In veiw of Proposition 1, there exist a constant $C>0$ and nonnegative integers r and k depending on f such that

$$|\langle f,\psi\rangle| \leq C\max\{\gamma^\mu_{m,n}(\psi);\ 0\leq m\leq r, 0\leq n\leq k\}$$

$$=C\max\{\sup_{x\in I}|x^m(\tfrac{1}{x}D)^n(x^{-\mu-1/2}\psi(x))|;\ 0\leq m\leq r, 0\leq n\leq k\},$$

for every $\psi\in H_\mu$.

Since $x^m(\frac{1}{x}D)^n(x^{-\mu-1/2}\psi(x))$ is of rapid descent as $x\to\infty$ (Proposition 2), we get

$$x^m(\frac{1}{x}D)^n(x^{-\mu-1/2}\psi(x)) = \int_\infty^x D_t\{t^m(\frac{1}{t}D)^n(t^{-\mu-1/2}\psi(t))\}dt$$

for every $\lambda \in H_\mu$, $m,n,\in N$.

Hence

$$\sup_{x\in I}\left|x^m(\frac{1}{x}D)^n(x^{-\mu-1/2}\psi(x))\right| \leq \int_0^\infty \left|D_t\{t^m(\frac{1}{t}D)^n(t^{-\mu-1/2}\psi(t))\}\right|dt$$

$$= \left|\left|D_t\{t^m(\frac{1}{t}D)^n(t^{-\mu-1/2}\psi(t)\}\right|\right|_{L_1(0,\infty)}$$

where $|| \ ||_{L_1(0,\infty)}$ denotes the norm on the space $L_1(0,\infty)$. Then we can write

$$|\langle f,\psi\rangle| \leq C\max\{||D_t\{t^m(\frac{1}{t}D)^n(t^{-\mu-1/2}\psi(t))\}||_{L_1(0,\infty)}; 0\leq m\leq r, 0\leq n\leq k\}$$

for every $\psi\in H_\mu$.

We now define the injective map

$$F: H_\mu \longrightarrow FH_\mu$$

$$\psi \longrightarrow (D_t\{t^m(\frac{1}{t}D)^n(t^{-\mu-1/2}\psi(x))\})_{\substack{m=0,\dots,r\\ n=0,\dots,k}}$$

If FH_μ is endowed with the topology induced in it by the product space $A_{r,k}(0,\infty)=(L_1(0,\infty))^{(r+1)(k+1)}$, then

$$G: FH_\mu \longrightarrow C$$

$$F\psi \longrightarrow \langle f,\psi\rangle$$

is continuous linear mapping.

By application of the Hahn-Banach Theorem, G can be extended to $A_{r,k}(0,\infty)$. Therefore, since $A'_{r,k}(0,\infty)$ is isomorhic to $(L_\infty(0,\infty))^{(r+1)(k+1)}$ (see F. Treves [10]), there exist (r+1) (k+1) bounded measurable functions, $g_{m,n}$ $(m=0,\dots,r;n=0,\dots,k)$, such that:

$$G(F\psi) = \langle f,\psi\rangle = \sum_{m=0,n=0}^{r\ \ k} \langle g_{m,n}(x), D\{x^m(\frac{1}{x}D)^n(x^{-\mu-1/2}\psi(x))\}\rangle =$$

$$= \langle \sum_{m=0,n=0}^{r\ \ k} x^{-\mu-1/2}(-D\frac{1}{x})^n\{x^m(-D)g_{m,n}(x)\}, \psi(x)\rangle ,$$

for every $\psi \in H_\mu$.

On the other hand, if f is defined by (2) then $f \in H'_\mu$.

To see this, it is enough to prove that if $\{\psi_\nu\}_{\nu\in N}$ is a sequence in H_μ such that $\psi_\nu \to 0$ as $\nu \to \infty$, then the sequence $\{x^m(\frac{1}{x}D)^n(x^{-\mu-1/2}\psi_\nu(x))\}_{\nu\in N}$ converges to zero as $\nu \to \infty$, in $L_1(0,\infty)$, for every $m,n\in N$. This completes the proof of the theorem.

The Hankel-Schwartz transform defined by the pair

$$F(y)=B_\mu\{f(x)\}(y) = \int_0^\infty x^{2\mu+1} b_\mu(xy)f(x)dx$$

$$f(x)=B_\mu\{F(y)\}(x) = \int_0^\infty y^{2\mu+1} b_\mu(xy)F(y)dy$$

for $\mu > -\frac{1}{2}$, where $b_\mu(z) = z^{-\mu}J_\mu(z)$ and J_μ denotes the Bessel function of the first kind and order μ, was introduced by A.L. Schwartz [9], who estbalished its inversion formula. This integral transformation has been extended by G. Altenburg [5] and J.M. Mendez [7] to the space $H'_{1/2}$ of generalized functions ($H=H_{1/2}$ in their notation) following a procedure analogous to the one employed by A.H. Zemanian [1]. By setting $\mu = \frac{1}{2}$, we can deduce from Theorem 4 the next

COROLLARY. The functional f is in H' if and only if, there exist bounded measurable functions $g_{m,n}(x)$ defined on I, for $m=0,\dots,r,n=0,\dots,k$ where r and k are nonnegative integers depending on f, such that

$$\langle f,\psi\rangle = \langle \sum_{m=0,n=0}^{r\quad k} (-D\tfrac{1}{x})^n\{x^m(-D)g_{m,n}(x)\},\psi(x)\rangle, \quad \psi\in H.$$

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