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A NOTE ON BAZILEVIČ FUNCTIONS

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ABSTRACT. For $\alpha > 0$, let $B_1(\alpha)$ be the class of normalized analytic functions defined in the open unit disc D satisfying $Re(f(z)/z)^{\alpha-1}f'(z) > 0$ for $z \in D$. The sharp lower bound for $Re(f(z)/z)^{\alpha}$ is obtained and the result is generalized to some iterated integral operators.

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INTRODUCTION

For $\alpha > 0$, let $B_1(\alpha)$ denote the class of Bazilevič functions defined in the open unit disc $D = \{z : |z| < 1\}$ normalized so that $f(0) = 0$, $f'(0) = 1$ and such that for $z \in D$,

$$\operatorname{Re} f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} > 0. \tag{1}$$

This class of functions was studied first by Singh [4] and has been considered recently by several authors e.g. [2,3,5]. We note that $B_1(1) = R$, the class of functions whose derivative has positive real part.

For $f \in R$, Hallenbeck [1] showed that for $z = re^{i\theta} \in D$

$$\operatorname{Re}\frac{f(z)}{z} \geq -1 + \frac{2}{r}\log(1+r) > -1 + 2\log 2,$$

with equality for the function $f_1(z) = -z + 2\log(1+z)$ and for $B_1(\alpha)$, the non-sharp estimate $\operatorname{Re}(f(z)/z)^{\alpha} > 1/(1+2\alpha)$ was obtained in [3]. In this note, we give the sharp estimate for the lower bound of $\operatorname{Re}(f(z)/z)^{\alpha}$ when $f \in B_1(\alpha)$ and extend the result to obtain sharp estimates for the real part of some iterated integral operators.

RESULTS

For $z \in D$ and $n = 1, 2, ...$, define

$$I_n(z) = \frac{1}{z}\int_0^z I_{n-1}(t)dt,$$

where $I_0(z) = (f(z)/z)^{\alpha}$.

THEOREM. *Let $f \in B_1(\alpha)$ and $z = re^{i\theta} \in D$. Then for $n \geq 0$,*

$$\operatorname{Re} I_n(z) \geq \gamma_n(r) > \gamma_n(1),$$

where

$$0 < \gamma_n(r) = -1 + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n(k-1+\alpha)} < 1.$$

Equality occurs for the function f_α defined by

$$f_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1}\left(\frac{1-t}{1+t}\right) dt\right)^{1/\alpha}.$$

We note that when $n = 0$,

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^\alpha \geq \frac{\alpha}{r^\alpha}\int_0^r \rho^{\alpha-1}\left(\frac{1-\rho}{1+\rho}\right) d\rho = -1 + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{(k-1+\alpha)},$$

which reduces to $-1 + (2/r)\log(1+r)$ when $\alpha = 1$.

PROOF: From (1) write

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} = h(z),$$

where $h \in P$, i.e., $h(0) = 1$ and $\operatorname{Re} h(z) > 0$ for $z = re^{i\theta} \in D$.

Thus

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^\alpha = \alpha \operatorname{Re}\left(\frac{1}{z^\alpha}\int_0^z t^{\alpha-1} h(t) dt\right).$$

Write $t = \rho e^{i\theta}$, so that

$$\begin{aligned}\operatorname{Re}\left(\frac{f(z)}{z}\right)^\alpha &= \frac{\alpha}{r^\alpha}\int_0^r \rho^{\alpha-1}\operatorname{Re} h(\rho e^{i\theta}) d\rho,\\ &\geq \frac{\alpha}{r^\alpha}\int_0^r \rho^{\alpha-1}\left(\frac{1-\rho}{1+\rho}\right) d\rho,\end{aligned}$$

since $h \in P$.

Hence

$$\operatorname{Re} I_0(z) = \operatorname{Re}\left(\frac{f(z)}{z}\right)^\alpha \geq \frac{\alpha}{r^\alpha}\int_0^r \rho^{\alpha-1}\left(\frac{1-\rho}{1+\rho}\right) d\rho.$$

Next

$$\begin{aligned}\operatorname{Re} I_{n+1}(z) &= \operatorname{Re} \frac{1}{z}\int_0^z I_n(t)dt,\\ &= \frac{1}{r}\int_0^r \operatorname{Re} I_n(\rho e^{i\theta})d\rho,\\ &\geq \frac{1}{r}\int_0^r \left(-1 + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\rho^{k-1}}{k^n(k-1+\alpha)}\right) d\rho,\\ &= \gamma_{n+1}(r),\end{aligned}$$

where the inequality follows by induction.

Now set

$$\phi_n(r) = \alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n(k-1+\alpha)}.$$

This series is absolutely convergent for $n \geq 0$ and $0 < r < 1$. Suitably rearranging pairs of terms in $\phi_n(r)$ shows that $\frac{1}{2} < \phi_n(r) < 1$ and so $0 < \gamma_n(r) < 1$.

Finally we note that since for $n \geq 1$

$$r\phi_n(r) = \int_0^r \phi_{n-1}(\rho)d\rho,$$

induction shows that $\phi_n'(r) < 0$ and so $\gamma_n(r)$ decreases with r as $r \to 1$ for fixed n and increases to 1 as $n \to \infty$ for fixed r.

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